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ON ENTIRE FUNCTIONS DEFINED BY A DIRICHLET SERIES.*

By S. Mandelbrojt and J. J. Gergen.†

1. *Introduction.* It is a well-known fact that there corresponds to every integral transcendental function $f(z)$ at least one ray L, issuing from the origin, such that in every angle, with vertex at the origin, containing L, $f(z)$ assumes each value, with possibly one exception.‡ Moreover, that there are certain analogies between these half-lines, which are ordinarily called the "lines of Julia," or "lines J," of $f(z)$, and the singularities of a non-integral function has been pointed out by Bloch § and concretely demonstrated by several authors. Biernacki ¶ has proved that if $f(z)$ is an entire function of non-zero order, at least one of its lines J is also a line J with regard to all the derivatives and integrals of $f(z)$. Gontcharoff ‖ has shown that if E is an arbitrary closed set of half-lines issuing from the origin, then there exist entire functions of order $\rho \leq 1/2$ whose lines J coincide with E, and entire functions of order $\rho > 1/2$ whose lines J coincide with the subset of E situated in the angle with vertex at the origin and opening equal to the larger of the two numbers π/ρ, $2\pi - \pi/\rho$. Finally, Pólya has found analogues to several of the theorems on the singularities of a function defined by a lacunary Taylor series. In particular, he established recently the second of the two related propositions:

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* A résumé of the theorems contained in this paper was published in a note, "Sur les fonctions définies par une série de Dirichlet," in *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 189 (1929), pp. 1057-1059.

† National Research Fellow.

‡ This proposition is due to Julia. A proof may be found in Montel, *Leçons sur les familles normales de fonctions analytiques*, Paris (1927), pp. 81-85. Hereafter we shall refer to this book by the letter *M*.

§ Bloch, "Les fonctions holomorphes et méromorphes dans le cercle-unité," *Mémorial des Sciences Mathématiques*, Paris (1926), p. 15.

¶ Biernacki, "Sur les droites de Julia des fonctions entières," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 186 (1928), pp. 1260-1262 and pp. 1410-1412. See also Biernacki, "Sur la theorie des fonctions entières," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A (1929), p. 529.

‖ Gontcharoff, "Sur les fonctions entières et les droits de Julia," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 185 (1927), pp. 1100-1102.

THEOREM A.* *If the maximum density* † *of the non-vanishing coefficients of a power series*

$$\psi(z) = \sum_{n=0}^{\infty} b_{\lambda_n} z^{\lambda_n},$$

with unit radius of convergence, is equal to Δ, *then, on every arc of the unit circle of length* $2\pi\Delta$, $\psi(z)$ *has at least one singular point.*

THEOREM B.‡ *If the maximum density of the non-vanishing coefficients of an entire function*

$$\psi(z) = \sum_{n=0}^{\infty} b_{\lambda_n} z^{\lambda_n},$$

of infinite order, is equal to Δ, *then, in every angle with vertex at the origin and opening* $2\pi\Delta$, $\psi(z)$ *has a line* J.

Theorem A has, of course, been generalized to Dirichlet series,§

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \qquad (s = \sigma + it) \tag{1}$$

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots; \quad \lim_{n=\infty} \lambda_n = \infty; \quad a_{\lambda_n} \neq 0, \quad n > 0.$$

In particular, it has been shown ¶ that, if

$$\lim_{n=\infty} (\lambda_{n+1} - \lambda_n) \geq G > 0$$

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* This proposition is Pólya's generalization of a theorem by Fabry. Fabry's original theorem may be found in his paper "Sur les séries de Taylor qui ont une infinité de points singuliers," *Acta Mathematica*, Vol. 22 (1898), on p. 86. For references to Pólya's generalization and related theorems, see Pólya's paper "Untersuchungen über Lücken und Singularitäten von Potenzreihen," *Mathematische Zeitschrift*, Vol. 29 (1929), p. 626.

† That is, if

$$\lim_{\xi=1-0} \overline{\lim_{r=\infty}} \left[\{N(r) - N(r\xi)\}/(r - r\xi)\right] = \Delta,$$

where $N(r)$ is the number of λ_n's not greater than r. Cf. Pólya, *loc. cit.*, p. 559.

‡ Cf. Pólya, *loc. cit.*, p. 626.

§ The condition that a_{λ_n} be different from zero for n different from zero is imposed for purposes of exposition. It involves no loss of generality except in Theorem VI.

¶ This is a particular case of a theorem of Pólya. For references to this and related theorems see Valiron, "Théorie générale des séries de Dirichlet," *Mémorial des Sciences Mathématiques*, Paris (1926), pp. 21-25. V. Bernstein has recently obtained some results in the same order of ideas. "Sur les points singuliers des fonctions representées par des séries de Dirichlet," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 188 (1929), p. 539.

and the axis c of convergence of (1) is finite, then $f(s)$ has at least one singular point on every segment of c of length $2\pi/G$.

Theorem B has as yet, however, never been generalized to these series, and it is in part the purpose of this paper to consider this problem. With this in mind the lines of interest are clearly horizontal lines parallel to the axis of reals, and the regions some horizontal strips. Rather, however, than confine our attention to regions in which the function in question assumes each value, save possibly one, only once, we shall consider those in which it assumes each value, with the possible exception, infinitely many times. Since the chief weapon in the analysis is the theory of normal families this makes an appreciable difference in the proofs. Our propositions are concerned with lines $\bar{J}$ and J defined as follows:

Definition. A line L, parallel to the axis of reals, is a line $\bar{J}$ of $f(s)$ if, in every horizontal strip containing L, $f(s)$ assumes every value, with possibly one exception, infinitely many times.

Definition. A ray L, issuing from the origin, is a half-line J of $f(s)$ if, in every angle, with vertex at the origin, containing L, $f(s)$ assumes every value, with possibly one exception, infinitely many times.

Theorems I to IV of Section 2 are on the lines $\bar{J}$ of the series (1). In Theorem II we obtain an analogue of the above mentioned theorem on singularities of (1). In Theorem III we suppose that the λ_n's are integers and obtain a generalization of Pólya's result B, weakening the condition on the order of $f(s)$. We employ in the proof a lemma of Pólya. The theorems of Section 3 are on the lines J_1 of the series (1).

2.1. *The Lines $\bar{J}$.* THEOREM I. *Suppose that the series* (1) *is everywhere (absolutely*) convergent and that* $\varliminf_{n=\infty} (\lambda_n/n) \geq G > 0$. *Then, if T, $|t - \bar{t}| \leq \pi/G$, be any horizontal strip of width $2\pi/G$, either $f(s)$ has a line $\bar{J}$ in T, or else*

$$\lim_{m=\infty} |f(s_m)e^{us_m}| = \infty$$

for every u and every sequence of points $\{s_m = \sigma_m + it_m\}$ such that

$$(2) \qquad \lim_{m=\infty} \sigma_m = -\infty, \qquad \varlimsup_{m=\infty} |t_m - \bar{t}| \leq \pi/G.$$

The proof depends partly upon certain results found in the theory of normal families and partly upon the following lemma:

* It is known that for series of this type the axis of simple convergence coincides with the axis of absolute convergence. Cf. Valiron, *loc. cit.*, p. 3.

LEMMA. *If* $0 < \lambda_1 < \lambda_2 \cdots$ *and* $\varliminf_{n=\infty} (\lambda_n/n) \geqq G > 0$, *then*

$$g_j(z) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} z^{2m+1} = z \prod_{n=1}^{\infty}{}' (1 - z^2/\lambda_n^2) \quad (n \neq j),\ j = 1, 2, \cdots$$

is entire, and for every positive number ϵ

$$(2m+1)!\,|\,C^{(j)}{}_{2m+1}\,| \leqq A_1(\epsilon)\{\pi(1+\epsilon)/G\}^{2m+1}, \tag{3}$$

where $A_1(\epsilon)$ *is independent of* m *and* j.

That $g_j(z)$ is entire is well known. Let then n_0 be chosen so large that

$$\lambda_n \geqq nG/(1+\epsilon)$$

for $n \geqq n_0$, and let

$$A_2(\epsilon) = \{n_0(\lambda_1 + 1)(G+1)/\lambda_1\}^{n_0}.$$

We have for any product $\lambda = \lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m}$ of distinct λ_n's

$$A_2\lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m} \geqq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m. \tag{4}$$

In fact, if each factor λ_{k_j} is less than λ_{n_0},

$$\begin{aligned} A_2\lambda &\geqq A_2k_1k_2 \cdots k_m\lambda_1{}^m/n_0{}^m \\ &\geqq A_2k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m\{\lambda_1/(Gn_0)\}^m \\ &\geqq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m, \end{aligned} \tag{5}$$

whereas, if no factor is less than λ_{n_0},

$$\lambda \geqq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m, \tag{6}$$

and (5) and (6) together show that (4) holds in general.

Now for $m \geqq 1$,

$$\begin{aligned} |\,C^{(j)}{}_{2m+1}\,| &= \sum_{k_1=1}^{\infty}{}' \sum_{k_2=k_1+1}^{\infty}{}' \cdots \sum_{k_m=k_{m-1}+1}^{\infty}{}' (\lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m})^{-2} \qquad (k_p \neq j) \\ &\leqq A_2{}^2\{(1+\epsilon)/G\}^{2m} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_m=k_{m-1}+1}^{\infty} (k_1k_2 \cdots k_m)^{-2} \\ &= A_2{}^2\{\pi(1+\epsilon)/G\}^{2m}/(2m+1)!, \end{aligned}$$

the last relation being deduced from the identities

$$\pi ix \prod_1^{\infty} (1 + x^2/n^2) = \sin \pi ix = i \sum_{m=0}^{\infty} (\pi x)^{2m+1}/(2m+1)!.$$

Accordingly, since

$$|\,C_1{}^{(j)}\,| = 1 \leqq A_2{}^2,$$

(3) holds with $A_1 = GA_2{}^2$.

2. 2. Returning now to the proof of Theorem I, we consider the functions[1]

$$(7)\qquad \phi_j(s) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} f^{2m+1}(s),$$

where the $C^{(j)}{}_{2m+1}$ are defined as above. From the inequalities

$$(8)\qquad r^{2m+1} \max_{|s| \leq r} \mid f^{2m+1}(s) \mid \leq 2(2m+1)! \max_{|s| \leq 2r} \mid f(s) \mid \qquad (0 < r),$$

we deduce that the series (7) converges uniformly in every closed region, and hence that $\phi_j(s)$ is entire. Moreover, denoting by $F(s)$ the series

$$F(s) = \sum_{n=0}^{\infty} \mid a_n \mid e^{-\lambda_n s},$$

we have for $\sigma \geq 0$

$$r^{2m+1} \sum_{n=1}^{\infty} \mid a_n(-\lambda_n)^{2m+1} e^{-\lambda_n s} \mid = r^{2m+1} \sum_{n=1}^{\infty} \mid a_n \mid \lambda_n{}^{2m+1} e^{-\lambda_n \sigma}$$
$$\leq r^{2m+1} \mid F^{(2m+1)}(0) \mid$$
$$\leq (2m+1)!\, F(-r),$$

so that in the series (7) it is permissible, when $\sigma \geq 0$, to replace $f^{2m+1}(s)$ by its expansion $\sum_{1}^{\infty} a_n(-\lambda_n)^{2m+1} e^{-\lambda_n s}$ and to change the order of summation. When this is done, we get

$$(9)\qquad \phi_j(s) = -\sum_{n=0}^{\infty} a_n g_j(\lambda_n) e^{-\lambda_n s} = -a_j g_j(\lambda_j) e^{-\lambda_j s},$$

which, being valid for $\sigma \geq 0$, is valid throughout the plane.

From (9) it now follows that for every positive ϵ

$$(10)\qquad \mid a_j g_j(\lambda_j) e^{-\lambda_j s} \mid \leq A_3(\epsilon) \underset{|z-s| = r_\epsilon}{\text{maximum}} \mid f(z) \mid = A_3 N(s),$$

where $r_\epsilon = (\pi + \epsilon)/G$, and A_3 is independent of j and s. In fact,

$$\mid f^{2m+1}(s) \mid \leq (2m+1)!\, \{G/(\pi+\epsilon)\}^{2m+1} N(s),$$

and so

$$\mid a_j g_j(\lambda_j) e^{-\lambda_j s} \mid = \mid \phi_j(s) \mid$$
$$\leq \sum_{m=0}^{\infty} \mid C^{(j)}{}_{2m+1} \mid \, \mid f^{2m+1}(s) \mid$$
$$\leq A_1(\kappa) N(s) \sum_{m=0}^{\infty} \{2\pi/(2\pi+\epsilon)\}^{2m+1}$$
$$= A_3(\epsilon) N(s),$$

where $\kappa = \epsilon/(2\pi + \epsilon)$.

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[1] Functions of this type have been used in similar circumstances by several authors; for references see Valiron, *loc. cit.*, pp. 21-25.

With the aid of (10) we prove that, if for some u the function $|f(s)s^{us}|$ is bounded on a sequence of points $\{s_m = \sigma_m + it_m\}$ satisfying (2), then, no matter how small the positive number ϵ, the family of functions

$$f_n(s) = f(s - 2n\pi/G) \qquad (n = 0, 1, 2, \cdots) \tag{11}$$

is not normal * in the square R_3

$$|\sigma| < (\pi + 3\epsilon)/G, \quad |t - \bar{t}| < (\pi + 3\epsilon)/G.$$

This, of course, is sufficient to establish the theorem. For under these circumstances there is at least one point in R_3 at which the family is not normal,† and thus, since ϵ is arbitrary, there is at least one point $s' = \sigma' + it'$ in the square R_0

$$|\sigma| \leq \pi/G, \quad |t - \bar{t}| \leq \pi/G,$$

at which it is not normal. Accordingly,‡ in every horizontal strip containing the line $t = t'$, $f(s)$ assumes each value, with one possible exception, infinitely many times. Hence the line $t = t'$ is a line $\bar{J}$.

We observe firstly that, if $|f(s)e^{us}|$ is bounded on some sequence of points satisfying (2), then $|f(s)|$ is bounded on a set of points $\{s_m' = \sigma_m' + it_m'\}$ such that

$$\lim_{m=\infty} \sigma_m' = -\infty, \quad |t - \bar{t}| \leq (\pi + 2\epsilon)/G. \tag{12}$$

In fact, if the contrary were true, we should have for all n sufficiently large, say $n \geq n_1$,

$$|f_n(s)| > 1$$

for all s in the square R_2

$$|\sigma| \leq (\pi + 2\epsilon)/G, \quad |t - \bar{t}| \leq (\pi + 2\epsilon)/G;$$

and thus, by a theorem of Mandelbrojt,§ for all $n \geq n_1$ and s and z in the square R_1

$$|\sigma| \leq (\pi + \epsilon)/G, \quad |t - \bar{t}| \leq (\pi + \epsilon)/G,$$

it would follow that

$$\log |f_n(s)| \leq \alpha_{R_1} \log |f_n(z)|, \tag{13}$$

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* For the definition of normality see M, p. 32.

† M, p. 34.

‡ M, p. 61. Evidently the normality or non-normality of a sequence of functions is independent of any finite group of functions.

§ Mandelbrojt, "Sur les suites de fonctions holomorphes," *Journal de Mathématiques (Liouville)*, Vol. 18 (1929), p. 176.

where α_{R_1} is independent of s, z and n. But, on the other hand, because of (10) and our assumption as to the boundedness of $|f(s)e^{us}|$, we can choose a subsequence $\{f_{n_k}(s)\}$ from the sequence (11), and two sets of points $\{z_k\}$, $\{z_k'\}$, all contained in R_1, such that

$$\lim_{k=\infty} (n_k^{-1} \log |f_{n_k}(z_k)|) = \infty,$$
$$0 < \log |f_{n_k}(z_k')| < A_4 n_k,$$

where A_4 is independent of k. We thus have

$$\lim_{k=\infty} \{(\log |f_{n_k}(z_k')|)^{-1} \log |f_{n_k}(z_k)|\} = \infty,$$

which evidently contradicts (13). Hence the boundedness of $|f(s)e^{us}|$ implies that of $|f(s)|$ itself.

To complete the proof we have then only to show that, if $|f(s)|$ is bounded on a set of points satisfying (12), the family (11) is not normal in R_3. For this we observe that under these conditions a subsequence $\{f_{n_k}(s)\}$ can be chosen from the sequence (11), together with two sets of points $\{z_k\}$, $\{z_k'\}$, all contained in R_2, such that

$$\lim_{k=\infty} |f_{n_k}(z_k)| = \infty, \quad |f_{n_k}(z_k')| \leq A_5,$$

where A_5 is independent of k. It is then evident that the sequence $\{f_{n_k}(s)\}$ cannot converge uniformly to a holomorphic function in R_2, and that the sequence $\{|f_{n_k}(s)|\}$ cannot become infinite uniformly there. Thus the family $\{f_n\}$ is not normal in R_3; and the theorem is proved.

2. 3. In our next theorem we employ the notion of order, as defined by Ritt,* of an entire function $f(s)$ represented by an everywhere absolutely convergent Dirichlet series. *This order, which we shall call the R-order of $f(s)$, is, by definition, the* †

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1} \log_2 M_f(\sigma)\},$$

where $M_f(\sigma)$ is the least upper bound of $|f(s)|$ on the vertical line with abscissa σ. It is to be noted, firstly, that the R-order is a natural generalization of ordinary order, inasmuch as the R-order of the entire function

* Ritt, "On Certain Points in the Theory of Dirichlet Series," *American Journal of Mathematics*, Vol. 50 (1928), p. 77.

† By definition

$$\log_2 A = \begin{cases} \log \log A, & A > 1, \\ 0\quad, & A \leq 1. \end{cases}$$

$$\psi(s) = \sum_{n=0}^{\infty} a_n e^{-ns}$$

is equivalent to the ordinary order of $\psi_1(z) = \psi(-\log z)$, and secondly that, as in the case of ordinary order, we have the fundamental theorem,* that the R-order of

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

is ρ when and only when

$$\overline{\lim_{n=\infty}} \ \{(\lambda_n \log \lambda_n)^{-1} \log | a_n | \} = -1/\rho, \tag{14}$$

this conclusion holding if the series is everywhere absolutely convergent and

$$\underline{\lim_{n=\infty}} \ (\lambda_n/\log n) > 0.$$

This formula (14) is important for us. By means of it we are able to eliminate the second of the two possibilities of Theorem I. We prove

2.4. THEOREM II. *If the series* (1) *is everywhere (absolutely) convergent and* $\underline{\lim}_{n=\infty} \ (\lambda_{n+1} - \lambda_n) \geq G > 0$, *and if the R-order of* $f(s)$ *is* $\geq \rho > 0$, *then* $f(s)$ *has a line* $\bar{J}$ *in every horizontal strip of width* $2\pi/\alpha$, *where* α *is the smaller of the two numbers* 2ρ *and* G.

The proof rests on the two following lemmas.

LEMMA 1. *If*

$$\mu_{n+1} - \mu_n \geq G/2 > 0, \quad nG \leq \mu_n < (n+1)G \qquad (n = 1, 2, \cdots),$$

then, for $j \geq 2$,

$$G^3 \leq 16\pi\mu_j^2 \mid g_j(\mu_j) \mid = 16\pi\mu_j^3 \prod_1^{\infty}{}' \mid 1 - \mu_j^2/\mu_n^2 \mid \qquad (n \neq j). \tag{15}$$

We have, writing $\mu_j = (j + \eta) G = \mu G$,

$$\begin{aligned} | g_j(\mu_j) | &= \mu_j (\mu_j^2/\mu^2_{j-1} - 1)(1 - \mu_j^2/\mu^2_{j+1}) \prod_{n=1}^{j-2} (\mu_j^2/\mu_n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu_j^2/\mu_n^2) \\ &\geq jG/(j+2)^2 \prod_{n=2}^{j-1} (\mu^2/n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu^2/n^2) \\ &\geq jG^3/\{\mu_j^2 (j+2)^2\} \prod_{n=1}^{j-1} (\mu^2/n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu^2/n^2). \end{aligned}$$

Hence, if $0 < \eta < 1$,

$$| \mu_j^2 g_j(\mu_j) | \geq j^3 G^3 | \sin \pi\eta | / \{4\pi (j+2)^3 \eta (1 - \eta)\}.$$

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* Ritt, *loc. cit.*, p. 78.

Now

$$|\sin \pi\eta| \geq 2\eta(1-\eta),$$

so that

$$|\mu_j^2 g_j(\mu_j)| \geq j^3 G^3/\{2\pi(j+2)^3\} > G^3/(16\pi)$$

and (15) holds for $0 < \eta < 1$. But $g_j(z)$ is a continuous function of z; hence (15) also holds for $\eta = 0$.

LEMMA 2. *Let ϵ be an arbitrary positive number, and T', $|t - \bar{t}| \leq (\pi + 2\epsilon)/G$, an arbitrary strip of width $2(\pi + 2\epsilon)/G$. Then under the hypotheses of Theorem II,*

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1} \log_2 M_f(\sigma, T')\} \geq \rho,$$

where $M_f(\sigma, T')$ is the maximum of $|f(s)|$ on the segment of the vertical line of abscissa σ contained in T'.

We first choose a positive integer n_0 so large that

$$\lambda_{n+1} - \lambda_n \geq 2\pi G/(2\pi + \epsilon) = G_1$$

when $n \geq n_0 - 1$, and then select a sequence of numbers $\mu_1, \mu_2, \cdots$ containing the sequence $\lambda_{n_0}, \lambda_{n_0+1}, \cdots$ and such that

$$(16) \qquad nG_1 \leq \mu_n < (n+1)G_1, \quad \mu_{n+1} - \mu_n \geq G_1/2$$

for $n = 1, 2, \cdots$.

Now, writing

$$h(s) = \sum_{n=n_0}^{\infty} a_n e^{-\lambda_n s}$$

$$g_j(z) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} z^{2m+1} = z \prod_{n=1}^{\infty}{}' (1 - z^2/\mu_n^2) \qquad (n \neq j),$$

we have, as before,

$$\phi_j(s) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} h^{2m+1}(s) = -\sum_{n=n_0}^{\infty} a_n g_j(\lambda_n) e^{-\lambda_n s};$$

and, in particular, if j_k denotes an integer such that $\mu_{j_k} = \lambda_{n_0+k}$, this reduces to

$$\phi_{j_k}(s) = -a_{n_0+k} g_{j_k}(\mu_{j_k}) e^{\lambda_{n_0+k} s}.$$

Hence, by Lemma 1,

$$0 < A_5(\epsilon) \, | a_{n_0+k} e^{\lambda_{n_0+k} s} | \leq \lambda_{n_0+k}^2 \, |\phi_{j_k}(s)|,$$

where A_5 is independent of s and k. But, by the lemma of Section 2.1,

$$|\phi_{j_k}(s)| \leq N_h(s) \sum_{m=0}^{\infty} (2m+1)! \, C^{(j)}{}_{2m+1} \{G/(\pi + 2\epsilon)\}^{2m+1}$$

$$\leq A_1(\kappa)N_h(s) \sum_{m=0}^{\infty} [2G\pi(\pi+\epsilon)/\{G_1(\pi+2\epsilon)(2\pi+\epsilon)\}]^{2m+1}$$
$$= A_1N_h(s) \sum_{m=0}^{\infty} \{(\pi+\epsilon)/(\pi+2\epsilon)\}^{2m+1}$$
$$= A_6(\epsilon)N_h(s),$$

where $N_h(s)$ is the maximum of $|h(z)|$ on the circle $|s-z| = (\pi+2\epsilon)/G$, $\kappa = \epsilon/(2\pi+\epsilon)$ and A_6 is independent of s and j. Accordingly, we have for $n \geq n_0$ and all s

$$0 < A_7(\epsilon)\,|a_n e^{-\lambda_n s}| \leq \lambda_n^2 N_h(s), \tag{17}$$

where A_7 is independent of s and j.

By employing this inequality, it is not difficult to show that

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1}\log_2 M_h(\sigma, T')\} \geq \rho, \tag{18}$$

where $M_h(\sigma, T')$ has the same meaning with regard to h as $M_f(\sigma, T')$ has with regard to f. Suppose that δ is arbitrarily small but positive. Let

$$\sigma_n = -\{(1-\delta)\rho\}^{-1}\log\lambda_n.$$

Let s_n be the point $\sigma_n + i\bar{t}$ and let $s_n' = \sigma_n' + it_n'$ be the point on the circle $|s - s_n| = (\pi+2\epsilon)/G$ at which $|h(s)|$ is a maximum. Writing

$$P = \exp[-\rho(1-\delta)(\pi+2\epsilon)/G]$$

we have, from (17),

$$\begin{aligned} \text{(19)}\quad \exp[\rho(1-\delta)\sigma_n']\log M_h(\sigma_n', T') &\geq \rho\exp[\sigma_n\rho(1-\delta)]\log N_h(s_n) \\ &\geq P\lambda_n^{-1}[\log(A_7\lambda_n^{-2}) + \log|a_n| + \{\rho(1-\delta)\}^{-1}\lambda_n\log\lambda_n] \\ &\geq P\log\lambda_n\,[(\lambda_n\log\lambda_n)^{-1}\log(A_7\lambda_n^{-2}) \\ &\qquad + (\lambda_n\log\lambda_n)^{-1}|a_n| + \{\rho(1-\delta)\}^{-1}]. \end{aligned}$$

Thus, from (14),

$$\overline{\lim_{n=-\infty}} \{e^{\sigma_n'\rho(1-\delta)}\log M_h(\sigma_n', T')\} = \infty;$$

and this, of course, proves (18).

To conclude the proof of the lemma it is necessary now only to observe that

$$M_f(\sigma, T') \geq M_h(\sigma, T') - Be^{-\lambda_{n_0}\sigma},$$

where B is independent of σ, for this implies that

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1}\log_2 M_f(\sigma, T')\} = \overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1}\log_2 M_h(\sigma, T')\} \geq \rho.$$

2. 5. As a consequence of Lemma 2, Theorem I, and a theorem due to Valiron, the proof of II is almost immediate. Let us suppose that T, $|t-\bar{t}| \leq \pi/\alpha$ is an arbitrary horizontal strip of the plane. Then if ϵ is

[illegible] arbitrary but positive, $\alpha_\epsilon = \alpha(1-\epsilon)$, and T_ϵ is the horizontal strip $|t - \bar{t}\,| \leq 2\pi/\alpha_\epsilon$, we have

$$\overline{\lim_{\sigma=-\infty}} \; \{(-\sigma)^{-1} \log_2 M_f(\sigma, T_\epsilon)\} > \alpha_\epsilon/2,$$

where $M_f(\sigma, T_\epsilon)$ has the same meaning with regard to T_ϵ as $M_f(\sigma, T')$ has in regard to T'. In fact,

$$2\pi/\alpha_\epsilon \geq 2\pi/\{G(1-\epsilon)\} > 2(\pi + 2\epsilon)/G,$$

[illegible]

$$\rho \geq \alpha_\epsilon/\{2(1-\epsilon)\} > \alpha_\epsilon/2.$$

Now, according to Valiron,* if $E(z)$ is holomorphic in an angle [illegible] containing an angle S, the latter having its vertex at the origin and an opening [illegible] γ and if

$$\overline{\lim_{r=0}} \; \{(-\log r)^{-1} \log_2 M_E'(r, S)\} > \gamma/2,$$

where $M_E'(r, S)$ is the maximum of $E(z)$ on the arc of the circle [illegible] contained in S, then $E(z)$ assumes each value, save possibly one, on an infinite sequence of points in S' admitting the origin as a limit point. By making then the transformation $z = e^s$, we find that $|f(s)|$ is bounded on no sequence of points $\{s_m = \sigma_m + it_m\}$ such that

$$\overline{\lim_{m=\infty}} \; \sigma_m = -\infty, \quad |t_m - \bar{t}\,| \leq \pi(1+\epsilon)/\alpha_\epsilon.$$

This together with Theorem I is sufficient to establish II, for

$$\underline{\lim_{n=\infty}} \; (\lambda_n/n) \geq G > \alpha_\epsilon/(1+\epsilon),$$

and therefore in the strip

$$|t - \bar{t}\,| \leq 2\pi(1+\epsilon)/\alpha_\epsilon$$

$f(s)$ has a line J. But ϵ is arbitrarily small and $\alpha_\epsilon - \alpha$ tends to zero with ϵ; hence there is a line J in T.

§ 6. When the λ_n's are integers the conditions on the gaps in Theorem [illegible] can be made more general. Pólya † has shown that, if an entire function [illegible] of order ρ is represented by a series

$$\psi(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n},$$

in which the maximum density of the sequence $\{\lambda_n\}$ is Δ, then $\psi(z)$ i[illegible]

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* Valiron, "Fonctions entières et fonctions méromorphes d'une variable," *Mémorial des Sciences Mathématiques*, Paris (1925), p. 15.

† Pólya, *loc. cit.*, p. 622.

effectively of the order ρ in every sector S with vertex at the origin and opening $2\pi\Delta$. That is to say,

$$\overline{\lim_{r=\infty}} \{(\log r)^{-1} \log_2 M\psi'(r, S)\} = \rho,$$

where $M\psi'(r, S)$ is the maximum of $|\psi|$ in the arc of the circle $|z| = r$ contained in S. By observing that * $\underline{\lim}_{n=\infty} (\lambda_n/n) \geq 1/\Delta$ and employing this result rather than Lemma 2 in Section 2.5, we deduce

THEOREM III. *Suppose that in the series* (1) *the* λ_n*'s are integers, that their maximum density is* Δ, *and that* $f(s)$ *is an entire function of R-order* $\geq \rho > 0$. *Then* $f(s)$ *has a line* $\bar{J}$ *in every horizontal strip of width* $2\pi/\alpha$, *where* α *is the smaller of the two numbers* 2ρ *and* $1/\Delta$.

2.7. We conclude our theorems on the lines $\bar{J}$ with the following:

THEOREM IV. *If the series* (1) *is everywhere (absolutely) convergent and* $\underline{\lim}_{n=\infty} (\lambda_{n+1} - \lambda_n) = \infty$, *and if* $f(s)$ *is of unbounded R-order, then every horizontal line is a line* $\bar{J}$ *of* $f(s)$.

This is evidently an immediate corollary of Theorem II.

3.1. *The Lines* J_1. Turning now to the lines J_1 of a function represented by a Dirichlet series, we prove

THEOREM V. *Suppose that the series* (1) *is everywhere (absolutely) convergent and that* $\underline{\lim}_{n=\infty} (\lambda_n/n) \geq G > 0$. *Then if* L, $s = re^{i\bar{\theta}}$ $(r \geq 0)$, *be any ray issuing from the origin and lying in the left-hand half plane,* $\sigma \leq 0$, *either* L *is a half-line* J_1 *of* $f(s)$, *or else*

$$\lim_{m=\infty} |f(s_m)| = \infty$$

on every sequence of points $\{s_m = r_m e^{i\theta_m}\}$ *such that*

$$\lim_{m=\infty} r_m = \infty, \quad \lim_{m=\infty} \theta_m = \bar{\theta}. \tag{20}$$

The proof rests on the inequality (10) of Section 2.1. We show that if $|f(s)|$ is bounded on a sequence of points $\{s_m\}$ satisfying (20), then, no matter how small the positive number ζ, the family of functions

$$f_n(s) = f(2^n s) \qquad (n = 1, 2, \cdots)$$

is not normal in the region S,

$$1/2 < r < 1, \quad |\theta - \bar{\theta}| < \zeta \qquad (s = re^{i\theta}).$$

* Cf. Pólya, *loc. cit.*, p. 559.

T is as in Theorem I, is sufficient to prove V, for under these circumstances there must be a point on L at which the family is not normal; thus L is a line J_1.

We observe that if the point $\bar{s} = \bar{\sigma} + i\bar{t}$ and the number η are so chosen that the circle C,

$$| s - \bar{s} | = \eta,$$

is entirely contained in S as well as in the left-hand half-plane, then

(21) $$\overline{\lim_{n \to \infty}}\ N_{f_n}(\bar{s}) = \infty,$$

where $N_{f_n}(\bar{s})$ is the maximum of $| f_n(s) |$ on C. In fact, we have

(22) $$N_{f_n}(\bar{s}) = N_f(s_n),$$

where $N_f(s_n)$ is the maximum of $| f |$ on the circle C_n,

$$| s - s_n | = 2^n \eta, \quad s_n = 2^n \bar{s}.$$

Now the circles C_n are, for n great enough, of arbitrarily large radii; thus, by the inequality (10) of Theorem I, we have, for all large values of n,

$$N_{f_n}(s_n) \geq A e^{-2^n \bar{\sigma}},$$

where A is independent of n. Thus, by (22), we get (21).

From (21) and the boundedness of $| f(s) |$ on the points $\{s_n\}$ it follows immediately that the family is not normal in S; and this completes the proof.

3 2. The second of the two possibilities in the preceding theorem is eliminated in certain cases. In particular, since

$$| f(it) | \leq \sum_{n=0}^{\infty} | a_n |,$$

and the series on the right converges, we have

THEOREM VI.* *Under the hypotheses of Theorem V, the upper and lower halves of the imaginary axis are half-lines J_1 of $f(s)$.*

. . 3. Another way of eliminating this possibility is to make certain hypotheses as to the order of $f(s)$. Thus:

THEOREM VII. *If the series (1) is everywhere (absolutely) convergent and $\lim (\lambda_{n+1} - \lambda_n) > 0$, and if $f(s)$ is of non zero R-order, then every half-line issuing from the origin, and lying in the left-hand half-plane, is a J_1 of $f(s)$.*

This theorem is a particular case of VI when the line coincides with the imaginary axis. Let us then consider a line L, $s = re^{i\bar{\theta}}$ $(r \geq 0)$, with $\pi/2 < \bar{\theta} < 3\pi/2$, and the values of $| f(s) |$ in the angle S,

* It is readily verified that this conclusion holds for any exponential polynomial

$$0 < r, \quad | \theta - \bar{\theta} | < \zeta,$$

where ζ is any arbitrarily small positive number. We shall show that

$$| f(s) | \leq 1$$

on a sequence of points $\{z_m\}$ lying in S and such that $| z_m |$ becomes infinite with m. Because ζ is arbitrary, this is, by Theorem V, sufficient to prove VII.

We prove firstly that

$$\overline{\lim_{r=\infty}} \, \{ (\log r)^{-1} \log_2 M'(r, s) \} = \infty, \tag{23}$$

where $M'(r, s)$ is the maximum of $| f(s) |$ on the segment of the circle $| s | = r$ contained in S. For this we choose $0 < G \leq \underline{\lim}_{r=\infty} (\lambda_{n+1} - \lambda_n)$ and employ the results and notation of the Lemma 2, Section 2.4, with the exception that in this case we denote by s_n the point whose real part is σ_n and which lies on L. We then have for large values of r, rather than (19),

$$\begin{aligned} e^{\rho(1-\delta)\sigma_n'} \log | h(s_n') | &\geq P e^{\rho(1-\delta)\sigma_n} \log | h(s_n) | \\ &\geq P \log \lambda_n \left[(\lambda_n \log \lambda_n)^{-1} \log (A_7 \lambda_n)^{-2} \right. \\ &\quad \left. + (\lambda_n \log \lambda_n)^{-1} | a_n | + \{ \rho (1 - \delta) \}^{-1} \right], \end{aligned}$$

where ρ is finite, positive and less than the R-order of $f(s)$. Hence

$$\lim_{n=\infty} \left[e^{\sigma_n' \rho (1-\delta)} \log | h(s_n') | \right] = \infty.$$

But

$$\sigma_n' \leq - | s_n' | \, b,$$

where b is the smaller of the two numbers $| \cos (\bar{\theta} - \zeta) |$, $| \cos (\bar{\theta} + \zeta) |$. Hence, since

$$| f(s_n') | \geq | h(s_n') | - B e^{\lambda_{n_0} | s_n' |},$$

where B is independent of n, we get

$$\overline{\lim_{n=\infty}} \, \{ (\log | s_n' |)^{-1} \log_2 | f(s_n') | \} = \infty,$$

which justifies (23).

To complete the proof we now need only apply the theorem of Valiron for an angle, as stated in Section 2.5 for we are thereby assured of the existence of the sequence of points $\{z_m\}$ lying in S on which $| f(s) |$ is bounded.

4. In conclusion we remark that we are not prepared to state whether it is possible for the second of the alternatives of Theorem I to be fulfilled. The same may be said for Theorem V. Another problem which suggests itself is that of determining under what conditions a Dirichlet series has a line $\bar{J}$.

ON SETS OF POLYNOMIALS AND ASSOCIATED LINEAR FUNCTIONAL OPERATORS AND EQUATIONS.*

By I. M. SHEFFER.†

I. INTRODUCTION.

Let $\{P_n(x)\}$, $(n = 0, 1, \cdots)$ be an infinitive sequence of polynomials such that $P_n(x)$ is of degree not exceeding n. As we shall be constantly dealing with sequences having this property, we shall refer to such a sequence simply as a *set*. We consider in § II an algebra of sets of polynomials, each set being an element P. Associated with a set is a triangular infinite matrix of the type used in summability theory for series; and the properties of sets are obtained from consideration of these matrices.

We consider integral and non-integral powers of a set, and apply the results to solving linear equations in sets. Then power series in sets are considered, and applied to implicit functions of sets and to the characteristic equation. Characteristic polynomials corresponding to a set are introduced, and in terms of them a canonical form is obtained for sets. Finally, we take up as a particular case the set M which corresponds to the Cesàro matrix (in divergent series theory), and determine conditions that a set P be commutative with M.

§ III is devoted to the analytic aspect of the theory of sets of polynomials. Each set P gives rise to a linear functional operator L (and hence to an equation), which can be expressed as a linear differential operator of infinite order. It possesses characteristic functions which are precisely the characteristic polynomials already introduced. Associated with L is a second operator, $\mathfrak{L}$, which possesses the same characteristic numbers as L, but whose characteristic functions are formal power series. *The treatment is formal.* In terms of the characteristic functions of L and $\mathfrak{L}$ we can obtain formal expansions of functions and solve formally the linear functional equations that arise.

In § IV we take up the systems of linear equations in infinitely many unknowns which arise from sets. By replacing characteristic polynomials by characteristic vectors, much of the preceding theory can be applied.

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† National Research Fellow.

It may be of some value to indicate the position of the present paper with respect to related publications. The algebraic aspect of sets is an attempt to generalize the permutability property enjoyed by the so-called Appell Polynomials.*

On the subject of linear differential equations of infinite order, it is necessary to mak mention of the fundamental memoir by Bourlet,† where it is shown that linear operators are formally equivalent (in the field of analytic functions) to differential equations of infinite order, and where an equation for the "resolving" operator is obtained. Subsequent works on special equations have been published by von Koch, Perron, Hilb, H. T. Davis, the present writer, and others.‡

The use of the *adjoint equation* (that is made in § III) was suggested by the equation for the resolving kernel derived in a previous paper § for quite a different equation. This concept we believe to be useful in even more general linear differential equations than are considered here.

II. The Algebra of Sets.

1. *Integral Powers and Permutability.* Let $\{P_n(x)\}$ be a set of polynomials. We denote the set by P, and consider P as an element in an algebra subject to the two rules of operation:

(i) Addition: $P+Q$ is the set $\{P_n(x)+Q_n(x)\}$, $(n=0,1,\cdots)$.

(ii) Multiplication: Let

$$P_n(x)=p_{n0}+p_{n1}x+\cdots+p_{nn}x^n,\quad Q_n(x)=q_{n0}+q_{n1}x+\cdots+q_{nn}x^n.$$

Then PQ is the set $\{PQ_n(x)\}$, $(n=0,1,\cdots)$, where

$$PQ_n(x)=p_{n0}Q_0(x)+p_{n1}Q_1(x)+\cdots+p_{nn}Q_n(x). \tag{1}$$

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* Appell, "Sur une classe de polynomes," *Annales Scientifiques de l'École Normale Supérieure*, (2), Vol. 9 (1880), pp. 119-144. In a later footnote we give the definition of these polynomials.

† C. Bourlet, "Sur les opérations en général et les équations différentielles linéaires d'ordre infini," *Annales de l'École Normale Supérieure*, (3), Vol. 14 (1897), pp. 133-190.

‡ References can be found in H. T. Davis, "Differential Equations of Infinite Order with Constant Coefficients," this Journal, Vol. 52 (1930), pp. 97-108; I. M. Sheffer, "Linear Differential Equations of Infinite Order, with Polynomial Coefficients of Degree One," *Annals of Mathematics*, Vol. 30 (1929), pp. 345-372.

§ I. M. Sheffer, "Expansions in Generalized Appell Polynomials, and a Class of Related Linear Functional Equations," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 261-280.

It is easily seen that the usual laws of association and distribution hold, as does the commutative law for addition; and that in general multiplication is *not* commutative.

Definition. If $PQ = QP$ we shall say that the sets P and Q are *permutable.** A necessary and sufficient condition for permutability is given, in terms of the coefficients p_{ij}, q_{ij}, by expanding PQ and Q ' and equating like powers of x.

Given a set $P : P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n$, $(n = 0, 1, \cdots)$. With P we associate the triangular infinite matrix $M_P : \| m_{nk} \|$, where $m_{nk} = p_{nk}$, $(k = 0, 1, \cdots, n;\ m_{nk} = 0,\ k > n)$; and we observe that

$P + Q$ in sets becomes $M_{P+Q} = M_P + M_Q$ in matrices;
PQ in sets becomes $M_{PQ} = M_P \cdot M_Q$ in matrices.

The algebra of sets is, then, essentially the algebra of the associated infinite matrices; and we can rewrite the properties of matrices in terms of sets.

Notation. $I : I_n(x) = x^n$ is the *identity* set; $D : D_n(x) = d_n x^n$ is a *diagonal* set;

$S : S_n(x) = sx^n$ is a *scalar* set.

I, S, and 0 are the only sets permutable with every set.

Definition. For every positive † integer i, $P^i = PP^{i-1}$ is the i-th power of P, and we write $P^i : \{P_n{}^i(x)\}$, $(n = 0, 1, \cdots)$.

P^i is uniquely determined from P, and is permutable with P.

Definition. $P^{-1} : \{P_n{}^{-1}(x)\}$ is defined by $I = P^0 = P(P^{-1})$; and $P^{-k} : \{P_n{}^{-k}(x)\}$ by $P^{-(k-1)} = P(P^{-k})$.

Definition. P is *non-singular* if $P_n(x)$ is of degree exactly n, $(n = 0, 1, \cdots)$; otherwise *singular.*

$P^{-1}, P^{-2}, \cdots$ exist if and only if P is non-singular; and when this is so, $P^{-1}, P^{-2}, \cdots$ are uniquely determined, and $P_n{}^{-k}(x)$ is of degree exactly n. Further, P^i, P^j are permutable for all integers i, j. Some immediate corollaries are: ‡

* An interesting system of permutable sets is furnished by the Appell Polynomials: If $P_n(x)$, $Q_n(x)$ are defined by

$$A(t)e^{tx} = \sum_0^\infty P_n(x)t^n/n!, \qquad B(t)e^{tx} = \sum_0^\infty Q_n(x)t^n/n!,$$

then P and Q are permutable, and, in fact, the set PQ is given by

$$A(t)B(t)e^{tx} = \sum_0^\infty PQ_n(x)t^n/n!$$

† $P^0 \equiv I$.

‡ When negative powers are used it is assumed that P is non-singular.

(i) $\sum_{i=-l}^{m} \lambda_i P^i$ and $\sum_{j=-s}^{t} \mu_j P^j$ are permutable.*

(ii) $P^i P^j = P^{i+j}$, i, j any integers; and $(P^i)^j = (P^j)^i = P^{ij}$.

(iii) If P is non-singular then the only set Q satisfying the equation $PQ = 0$ is $Q = 0$.

(iv) If P is non-singular then a necessary and sufficient condition that Q and R be permutable is that $PQR = PRQ$.

(v) If P and Q are non-singular then $(PQ)^{-1} = Q^{-1}P^{-1}$.

(vi) If P and Q are permutable, so are P^i and Q^j, i, j any integers; and so are † $\sum_{i=-l}^{m} \lambda_i P^i$, $\sum_{j=-s}^{t} \mu_j Q^j$; and $\sum_{i,j=-l,-s}^{m,t} \lambda_{ij} P^i Q^j$ is permutable to P (and to Q).

(vii) If p_{nn} is the coefficient of x^n in $P_n(x)$, then $p_{nn}{}^i$ is the coefficient of x^n in $P_n{}^i(x)$, i any integer.

2. *Non-Integral Powers.* Till now we have considered only positive and negative integral powers. It is however desirable to have the complete continuum (real or complex) of powers.

Definition. The numbers $\{p_n = p_{nn}\}$, $(n = 0, 1, \cdots)$ in $P_n(x) = p_{n0} + \cdots + p_{nn}x^n$ are the *characteristic numbers* ‡ for P.

COROLLARY. *The characteristic numbers for P^i are $\{p_n{}^i\}$, i any integer; and if $s(\lambda)$ is any polynomial, the set $s(P)$ has the characteristic numbers $s(p_n)$. Moreover, if P, Q have the characteristic numbers $\{p_n\}$, $\{q_n\}$, then PQ and QP have the characteristic numbers $\{p_n q_n\}$.*

Definition. P is *complete* § if $p_m \neq p_n$, $(m \neq n)$.

Definition. Let σ be any number (real or complex). Then P^σ is any set which satisfies the two following conditions:

(i) P^σ is permutable with P;

(ii) the characteristic numbers $\{p_n{}^{(\sigma)}\}$ for P^σ are given by ¶ $p_n{}^{(\sigma)} = p_n{}^\sigma$.

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* These series may extend to infinity in both directions, provided convergence takes place. We shall consider this point presently.

† See preceding footnote.

‡ The reason for this name will appear. It should not be overlooked that the characteristic numbers form an ordered set of numbers.

§ Our use of the property of "completeness" is such that its definition can be widened to include a more general class of sets. We shall return to this remark in a later footnote.

¶ We may agree to choose the same determination (for all n) of the logarithm in $P_n{}^\sigma = e^{\sigma \log p_n}$.

We observe that for $\sigma = i =$ integer, this definition coincides with that already given for P^i.

LEMMA. *Let P be complete, and let $\{q_n\}$ be an arbitrary set of numbers. There exists a unique set Q with characteristic numbers $\{q_n\}$ which is permutable with P.*

The lemma follows from a consideration of the equations resulting when one equates PQ with QP.

THEOREM. *If P is complete there exists at least one set P^σ for every σ whose real part is positive; and if in addition, P is non-singular there is at least one P^σ for all σ.*

Both parts of the theorem follow from the preceding lemma. That there may exist more than one P^σ is a consequence of the many-valuedness of p_n^σ for $\sigma \neq$ integer.

LEMMA. *The properties $P^\alpha P^\beta = P^{\alpha+\beta}$, $(P^\alpha)^\beta = P^{\alpha\beta}$* hold for all [illegible]

[illegible] *Application to Equations in Sets.*† Let $s(\lambda)$ be the polynomial

$$s(\lambda) = s_1\lambda + s_2\lambda^2 + \cdots + s_k\lambda^k;$$

and let P be a givn set. We seek a set X which satisfies the equation ‡

$$(a) \qquad s_1X + s_2X^2 + \cdots + s_kX^k = P.$$

Let the characteristic numbers of P be $\{p_n\}$. If X exists then P, being a polynomial in X, is permutable with X. Also, $\{x_n\}$ being the characteristic numbers of X,

$$(b) \qquad s_1x_n + s_2x_n^2 + \cdots + s_kx_n^k = p_n, \qquad (n = 0, 1, \cdots).$$

Conversely, any set X, which is permutable with P and whose characteristic numbers x_n satisfy (b), is a solution of (a). By a preceding lemma, if P is complete such an X can be found. Hence

THEOREM. *X is a solution of* (a) *if and only if it is permutable with P and its characteristic numbers $\{x_n\}$ satisfy* (b). *If P is complete, there exists at least one solution X.*

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* It is to be understood that when any determination of the many-valued function [illegible] has been made on one side of the equation, then a suitably chosen determination on the other side will make the equation a true one.

† The methods here used apply equally well to equations in square matrices (see [illegible]), when the given matrix has n linearly independent invariant directions.

‡ If $s(\lambda)$ contains a constant term s_0, we can reduce the case to (a) by transposing s_0I to the right.

In a similar manner we can treat the set-equation

$$L_1 X + L_2 X^2 + \cdots + L_k X^k = P, \tag{c}$$

where $L_1, \cdots, L_k$ are permutable with P. Letting $l_n^{(i)}$, p_n, x_n be the characteristic numbers of L_i, P, X, we obtain the

THEOREM. *If X is a solution of* (c) *then the x_n satisfy*

$$l_n^{(1)} x_n + l_n^{(2)} x_n^2 + \cdots + l_n^{(k)} x_n^k = P_n, \qquad (n = 0, 1, \cdots). \tag{d}$$

Moreover, if P is complete, there exists at least one solution X which is permutable with P.

And we can equally well consider the set-equation

$$\sum_{i_1, \ldots, i_k = 0}^{r} L_{i_1 \ldots i_k} X_1^{i_1} X_2^{i_2} \cdots X_k^{i_k} = P \tag{e}$$

in the k unknown sets $X_1, \cdots, X_k$, where there is no term independent of the X_i, and where the $L_{i_1 \ldots i_k}$ are sets all permutable with P.

THEOREM. *If a solution $X_1, \cdots, X_k$ exists, then the characteristic numbers $x_n^{(l)}$, $(l = 1, \cdots, k)$ satisfy the equations*

$$\sum_{i_1, \ldots, i_k = 0}^{r} l_n^{(i_1 \ldots i_k)} x_n^{(1) i_1} x_n^{(2) i_2} \cdots x_n^{(k) i_k} = p_n, \qquad (n = 0, 1, \cdots); \tag{f}$$

and if P is complete, there exists at least one solution $X_1, \cdots, X_k$, each X_i permutable with P.

4. *Power Series in Sets; Characteristic Equation.* Let $Q^{(0)}, Q^{(1)}, \cdots$ be an infinite sequence of sets. We may then form the series

$$Q = \sum_{k=0}^{\infty} Q^{(k)}, \tag{1}$$

and ask when this symbol is itself a set.

Definition. The series (1) converges if

$$q_{ni} = \lim_{s=\infty} \sum_{k=0}^{s} q_{ni}^{(k)} \quad \text{exists,} \quad \begin{cases} n = 0, 1, 2, \cdots \\ i = 0, 1, \cdots, n \end{cases};$$

and the *sum* is the set

$$Q : \{Q_n(x) = q_{n0} + q_{n1} x + \cdots + q_{nn} x^n\}.$$

If $Q^{(k)} = f_k P^k$, where the f_k's are scalars, we have a *power series* in the set P:

$$f(P) = \sum_{k=0}^{\infty} f_k P^k.$$

Let M be a square matrix of order n, and let its n characteristic numbers be $\lambda_1, \cdots, \lambda_n$. Then it is known that

LEMMA. *If $\lambda_1, \cdots, \lambda_n$ are distinct, a necessary and sufficient condition that $\sum_0^{\infty} f_k M^k$ converge is that the series $f(z) = \sum_0^{\infty} f_k z^k$ converge for $\lambda_1, \cdots, \lambda_n$; and if $\lambda_1, \cdots, \lambda_n$ are not necessarily distinct a sufficient condition for convergence is that for each distinct λ (of order s, say), the series for $f(\lambda)$, $f'(\lambda), \cdots, f^{(s-1)}(\lambda)$ shall converge.*

To apply this lemma we make the following observations:

If P is any set, its associated matrix M_P has the characteristic numbers $\rho_0, \rho_1, \cdots$. If we denote by M_{P_n} the matrix consisting of the first $n+1$ rows and columns of M_P, then M_{P_n} has the characteristic numbers $\rho_0, \cdots, \rho_n$ and to it the lemma applies. Furthermore, the operations $P + Q$, PQ or their equivalents $M_{P+Q} = M_P + M_Q$, $M_{PQ} = M_P \cdot M_Q$ give rise to the operations $M_{(P+Q)_n} = M_{P_n} + M_{Q_n}$, $M_{(PQ)_n} = M_{P_n} \cdot M_{Q_n}$. As we let n tend to infinity, then, there results the

THEOREM. *If P is complete, a necessary and sufficient condition that the series $f(P) = \sum_0^{\infty} f_k P^k$* converge is that $\sum_0^{\infty} f_k z^k$ converge for $z = \rho_i$ $(i = 0, 1, \cdots)$; *and for any set P a sufficient condition for convergence is that for each distinct ρ_i (say of order* s_i), the series $f(\rho_i)$, $f'(\rho_i), \cdots, f^{(s_i-1)}(\rho_i)$ shall converge.*

COROLLARY. *If $f(P) = \sum_0^{\infty} f_k P^k$ converges, its characteristic numbers are $\{f(\rho_i)\}$.*

COROLLARY. *$f(P)$ is permutable with P.*

Let $f(z) = \sum_0^{\infty} r_n(z) = \sum_0^{\infty} s_n(z)$ be two expansions for the analytic function $f(z)$. If $\sum_0^{\infty} r_n(A)$ and $\sum_0^{\infty} s_n(A)$ both converge one may ask if they have the same sum-set. But first: what are we to mean by $r_n(A)$, $s_n(A)$? If $r_n(z)$ and $s_n(z)$ are polynomials there is no difficulty. But if they are general analytic functions, $r_n(A)$ and $s_n(A)$ must be defined. We make the definition: Let $f(z)$ be an analytic function. Then $f(P)$ is any set satisfying the conditions

* ... is permissible.

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(i) $f(P)$ is permutable with P;
(ii) the characteristic numbers f_n of $f(P)$ are related to the p_n of P by $f(p_n) = f_n$, $(n = 0, 1, \cdots)$.

It is to be noted that given $f(z)$, $f(P)$ need not exist for every P, and for two reasons: (1) some p_n may be outside the region of existence of $f(z)$; (2) even if the $f(p_n)$ exist, it does not follow that a set with these characteristic numbers will exist and be permutable with P. Further, if $f(z)$ is multiple-valued, $f(P)$ if it exists, will not be unique.

With this definition we can now establish the

THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} r_k(z)$ *converge at* $z = p_n$ $(n = 0, 1, \cdots)$, *where* $f(z)$ *and the* $r_k(z)$ *are analytic functions. Further, let* $\sum_{k=0}^{\infty} r_k(P)$ *converge. Then its sum-set is* $f(P)$.

Proof. Let Q be the sum-set. $r_k(P)$ is, by definition, permutable with P. We then find by direct multiplication that $QP = PQ$. Again, the characteristic numbers of $r_k(P)$ are $\{r_k(p_n)\}$, so that $q_n = \sum_{k=0}^{\infty} r_k(p_n)$. But this is $f(p_n)$. Hence $Q = f(P)$.

Two interesting examples of power series are e^P and $\log(1 + P)$:

$$e^P = I + P/1! + P^2/2! + \cdots; \quad \log(I + P) = P - P^2/2 + P^3/3 - \cdots.$$

If P and Q are permutable then we have the functional equation $e^P \cdot e^Q = e^{P+Q}$.

Let $f(x, y)$ be analytic in (x, y) in a region about $(0, 0)$, and suppose $f(0, 0) = 0$, $\partial f(0, 0)/\partial y \neq 0$. By the implicit function theorem there exists a function $y(x) = \sum_{n=0}^{\infty} y_n x^n$ analytic about $x = 0$, with $y(0) = 0$, which makes $f(x, y) \equiv 0$ in x. Now let P be a variable set, and consider the set-equation

$$f(P, Q) = 0, \tag{a}$$

in the unknown set Q. There exists a solution in the form $Q = \sum_{n=0}^{\infty} y_n P^n$ provided the characteristic numbers of P are sufficiently small; and Q is permutable with P.

Given a set P. To its components $P_n(x)$ there correspond finite characteristic equations; and to P itself, under suitable convergence conditions, there corresponds a transcendental characteristic equation. The precise relations are given by the two following theorems.

THEOREM. *Let*

$$\Delta_n(\lambda) = (\lambda - p_0)(\lambda - p_1) \cdots (\lambda - p_n), \qquad (n = 0, 1, \cdots); \tag{1}$$

[illegible]

$$\Delta_n(P_n) = 0. \tag{2}$$

[illegible] is [illegible] if

$$\Delta_n(\lambda) = \delta_{n0} + \delta_{n1}\lambda + \cdots + \delta_{n,n+1}\lambda^{n+1},$$

[illegible]

$$\delta_{n0}P_n^0(x) + \delta_{n1}P_n^1(x) + \cdots + \delta_{n,n+1}P_n^{n+1}(x) = 0, \qquad (n = 0, 1, \cdots \tag{2'}$$

[illegible] M_P denote again the matrix of the first $n+1$ rows and [illegible] P. Then (2′) is a restatement of the fact that M_{P_n} satisfies its [illegible] equation.

COROLLARY. *We also have*

$$\delta_{n0}P_i^0(x) + \delta_{n1}P_i^1(x) + \cdots + \delta_{n,n+1}P_i^{n+1}(x) = 0, \qquad (i = 0, 1, \cdots$$

THEOREM. *Let the complete set P have the characteristic numbers* $\{p_n\}$. [illegible] *analytic function* $\Delta(z) = \sum_0^\infty \delta_k z^k$ *exists such that the series* [illegible] $\Delta($[illegible] [illegible] $1, \cdots)$ *all converge, and such that $\Delta(z)$ vanishes at* [illegible] *P satisfies the "characteristic" equation*

$$\Delta(P) = 0: \quad \sum_0^\infty \delta_k P^k = 0.$$

Proof. By a previous theorem $\Delta(P) = \sum_0^\infty \delta_k P^k$ converges, and it [illegible] [illegible] with P. Further, its characteristic numbers are $\Delta(p_n) = 0$, [illegible] from the fact that P is complete follows that $\Delta(P) = 0$.

It may happen that $\Delta(z)$ vanishes at $z = p_n$ but that the power series [illegible] $\Delta(z)$ diverges at $z = p_n$. We cannot then use the power series. But if, [illegible] example, the set of numbers $\{p_n\}$ lies in a simply-connected region in [illegible] $\Delta(z)$ is analytic, we can expand $\Delta(z)$ in a series of polynomials, [illegible] ent in this region:

$$\Delta(z) = \sum_0^\infty G_n(z);$$

[illegible] P satisfies the characteristic equation $\sum_0^\infty G_n(P) = 0$.

5. *Characteristic Polynomials; Canonical Form.* Associated with [illegible] P [illegible] a set Θ which is of importance in questions of permutability, and [illegible] of which a canonical form for P is possible. It has, moreover, an [illegible] ive character under an operation soon to be considered, and we term

[illegible] is to be recalled that $P_n^i(x)$ is the $(n+1)$st polynomial in the set P^i.

it the *characteristic set* for P; its polynomial components $\Theta_n(x)$ are the *characteristic polynomials.*

Let $\{p_n\}$ be the characteristic numbers for the set P, and define P^* as the diagonal set $\{P_n^*(x) = p_n x^n\}$. Then

Definition. Θ is a *characteristic set* for P if it satisfies the two conditions:

(i) $\Theta P = P^* \Theta$;

(ii) $\Theta_n(x)$ is of degree n, so that Θ is non-singular.

It is to be observed that if Θ is a characteristic set, so is $C\Theta$ where C is any diagonal set all of whose characteristic numbers are different from zero. But this is the maximum of arbitrariness, and is for our purpose unessential:

THEOREM.† *If P is complete there exists a characteristic set Θ which is unique to within an arbitrary diagonal-set multiplier on the left with non-zero characteristic numbers.*

Proof. If we equate the n-th polynomials on both sides of (i) we obtain a set of equations for the coefficients of $\Theta_n(x)$. These equations permit θ_{nn} to be arbitrary, after which $\theta_{n,n-1}, \cdots, \theta_{n0}$ are uniquely determined. The arbitrariness of θ_{nn} ($\neq 0$) allows for multiplication of Θ by a diagonal set on the left.

By its definition, Θ is non-singular, and Θ^{-1} exists. Hence

THEOREM. *If P is complete it possesses the canonical form*

$$P = \Theta^{-1} P^* \Theta. \tag{1}$$

We come now to a theorem on permutability:

THEOREM. *Let P be complete and Θ its characteristic set. A necessary and sufficient condition that Q be permutable with P is that Q possesses Θ as a characteristic set; and Q has then the canonical form $Q = \Theta^{-1} Q^* \Theta$.*

Necessity: Given $PQ = QP$. Then $\Theta PQ = \Theta QP$; that is, $P^* \Theta Q = \Theta QP$, or

$$(\Theta Q) P = P^* (\Theta Q). \tag{a}$$

In other words, the set ΘQ satisfies the first condition for a characteristic set.

† It was remarked in a previous footnote that the definition of completeness can be enlarged. Thus, to have the set Θ exist it is not necessary that $p_m \neq p_n$, $m \neq n$. What is necessary is that for each n, the matrix M_{P_n} (of order $n + 1$) have $n + 1$ linearly independent invariant directions. The definition of completeness could be given to be equivalent to this property, and most of the subsequent results would continue to hold.

we rewrite (a) in terms of the polynomial components, we find that $(\Theta Q)_n(x)$ must be a multiple of $\Theta_n(x)$, so that (b) $\Theta Q = D\Theta$, D being a diagonal set. Now $\theta_{nn} \neq 0$ so that on equating the coefficients of x^n in the n-th polynomial of (b) we have $d_n = q_n$, or $D = Q^*$.

Sufficiency: Given $Q = \Theta^{-1}Q^*\Theta$. Then

$$\Theta PQ = P^*\Theta Q = P^*Q^*\Theta, \qquad \Theta QP = Q^*\Theta P = Q^*P^*\Theta.$$

But† $P^*Q^* = Q^*P^*$, whence by operating on both equations with Θ^{-1} on the left we obtain $PQ = QP$.

THEOREM. *If P possesses a characteristic set Θ, and $s(\lambda)$ is a polynomial, then $s(P)$ has the canonical form $s(P) = \Theta^{-1}s(P^*)\Theta$. Further if $f(P) = \sum_{k=0}^{\infty} f_k P^k$ converges, then $f(P) = \Theta^{-1}f(P^*)\Theta$.*

6. *The Cesaro Set, and Permutability.*‡ Let M_P be the triangular matrix associated with P:

$$M_P : \left\| \begin{matrix} p_{00} & & & & & \\ p_{10}p_{11} & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ p_{n0}p_{n1} & \cdot & \cdot & \cdot & p_{nn} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\| .$$

Such matrices are of some interest in the summability theory of divergent series: they give rise to the Silverman-Toeplitz methods of summation. Perhaps the most important method both historically and practically is the Cesaro method of arithmetic means. Its matrix is $\| m_{nk} \|$, $m_{nk} = 1/(n+1)$, $(k = 0, 1, \cdots, n;\ m_{nk} = 0,\ k > n)$, and the corresponding set of polynomials $M: \{M_n(x)\}$ is given by

$$M_n(x) = (1 + x + \cdots + x^n)/(n+1).$$

We shall obtain the canonical form for M, and shall consider the conditions of permutability § with M.

---

† We use the LEMMA. *All diagonal sets are permissible.*

‡ In this section we usually omit proofs; they are easily provided.

§ In divergent series theory, permutability is of interest, for if two "regular" methods of summation are permutable, they give the same sum to any sequence which they sum.

From the standpoint of matrices, a characterization of sets (that is, matrices) permutable with M has been given by Hausdorff, *Mathematische Zeitschrift*, Vol. 9 (1921), pp. 74-109; pp. 280-299.

Let P be a set, and $\{p_n\}$ its characteristic numbers. On equating PM with MP we obtain the

THEOREM. *A necessary and sufficient condition that P be permutable with M is that*

$$\text{(A)}\quad P_n(x) = p_n x^n - \binom{n}{1}\Delta p_n x^{n-1} + \binom{n}{2}\Delta^2 p_n x^{n-2} - \cdots + (-1)^n \binom{n}{n}\Delta^n p_n,$$
where $\Delta p_n = p_n - p_{n-1}$, $\Delta^2 p_n = \Delta(\Delta p_n)$, *etc.*

Further results on permutability are:

THEOREM. *P is permutable with M if and only if*

$$\text{(B)}\qquad P_n(x) = p_n(x-1)^n + \binom{n}{1}p_{n-1}(x-1)^{n-1} + \cdots + \binom{n}{n}p_0.$$

COROLLARY. *P^s is permutable with M, and*

$$P_n{}^s(x) = p_n{}^s(x-1)^n + \binom{n}{1}p^s{}_{n-1}(x-1)^{n-1} + \cdots + \binom{n}{n}p_0{}^s.$$

THEOREM. *Let $P(t)$ be the formal power series $P(t) \sim \sum_0^\infty p_n t^n/n!$ A necessary and sufficient condition that P be permutable with M is that*

$$\text{(C)}\qquad e^t P\,[t(x-1)] \sim \sum_0^\infty P_n(x)t^n/n!$$

Definition. $P(t) \sim \sum_0^\infty p_n t^n/n!$ is the *generating function* for the set $P: \{P_n(x)\}$.

COROLLARY. *If ${}_kP(t)$ is the generating function for M^{-k}, then*

$$ {}_kP(t) = (d/dt)\{t \cdot {}_{(k-1)}P(t)\},$$

and

$$\text{(1)}\qquad {}_kP(t) = e^t S_k(t)$$

where $S_k(t)$ is a polynomial of degree k defined by

$$\text{(2)}\qquad S_0(t) = 1, \qquad S_k(t) = (t+1)S_{k-1}(t) + tS'_{k-1}(t).$$

Furthermore,

$$\text{(3)}\qquad t^n = b_{n0}S_0(t) + b_{n1}S_1(t) + \cdots + b_{nn}S_n(t)$$

where the b_{ni} are given by

$$\text{(4)}\quad R_n(t) \equiv 1(t-1)(t-2)\cdots(t-n) = b_{n0} + b_{n1}t + \cdots + b_{nn}t^n;$$

and

$$\text{(5)}\qquad S_n(t) = a_{n0} + a_{n1}t + \cdots + a_{nn}t^n$$

where the a_{ni} satisfy

$$t^n = a_{n0}R_0(t) + a_{n1}R_1(t) + \cdots + a_{nn}R_n(t).$$

L... ing R, S denote the sets $\{R_n(x)\}$, $\{S_n(x)\}$, we see that

$$(RS)_n(t) = (SR)_n(t) = t^n = I_n(t).$$

C... OLLARY. *R and S are inverse sets: $R = S^{-1}$, $S = R^{-1}$.*

F... n the recurrence relation (2) for $S_n(t)$ we can establish the

L... MA. *The zeros of $S_n(t)$ are real, negative, and simple, and ... the zeros of $S_{n-1}(t)$.*

T... OREM. *If P and Q are each permutable with M, then P and ...*

C... litions (A), (B) and (C) for permutability with M involve ... istic numbers $\{p_n\}$ of P. From (C) we obtain a further necessary ... sufficient condition:

$$P_0(x) \quad \binom{n}{1}P_1(x) + \cdots + (-1)^n\binom{n}{n}P_n(x) = (-1)^n p_n(x \quad 1)^n.$$

... this we get a condition *independent* of the characteristic num...

T... OREM. *All sets P permutable with M (and only those) sa...*

$$\begin{aligned}(x-1)\{P_0'(x) \quad \binom{n}{1}P_1'(x) + \cdots + (-1)^n\binom{n}{n}P_n'(x)\} \\ -n\{P_0(x) - \binom{n}{1}P_1(x) + \cdots + (-1)^n\binom{n}{n}P_n(x)\} = 0. \\ (n = 0, 1, \cdots).\end{aligned}$$

In divergent series theory, M and its positive integral powers (the C... of positive integral order) are very important, whereas the nega... are useless since they give rise to summation methods which do not ... convergent series. It is therefore rather remarkable that every ... permutable with M (and this includes M, M^2, $M^3, \cdots$) can ... as a series of polynomials in M^{-1}:

T... OREM. *Let P be a set permutable with M, and $\{p_n\}$ its character... numbers. Let $\{q_n\}$ be defined by*

$$q_n = \frac{p_0}{0!}\,\frac{(-1)^n}{n!} + \frac{p_1}{1!}\,\frac{(-1)^{n-1}}{(n-1)!} + \cdots + \frac{p_n}{n!}\,\frac{1}{0!}.$$

... P is given by the following convergent series of polynomials in M^{-1}:

... (E), primes denote differentiation.

$$P=\sum_{n=0}^{\infty} q_n I(M^{-1}-1)(M^{-1}-2)\cdots(M^{-1}-n).$$

Proof. Set $T^{(n)}=I(M^{-1}-1)\cdots(M^{-1}-n)$. On expanding we have

$$T^{(n)}=b_{n0}I+b_{n1}M^{-1}+\cdots+b_{nn}M^{-n},$$

so that the generating function for $T^{(n)}$ is

$$T^{(n)}(t)=b_{n0}\cdot P_0(t)+\cdots+b_{nn}\cdot {}_nP(t)=t^ne^t.$$

Now let $L_i(t)\sim\sum_{n=0}^{\infty} l_{in}t^n$, $(i=0,1,\cdots)$, and suppose $\sum_{i=0}^{\infty} l_{in}\doteq l_n$ exists, $(n=0,1,\cdots)$; and set $L(t)\sim\sum_0^{\infty} l_nt^n$. Further, let

$$e^tL_i[t(x-1)]\sim\sum_{n=0}^{\infty} L_{i,n}(x)t^n/n!, \qquad e^tL[t(x-1)]\sim\sum_{n=0}^{\infty} L_n(x)t^n/n!.$$

Then

$$L_n(x)=\sum_{i=0}^{\infty} L_{in}(x), \qquad (n=0,1,\cdots),$$

and each series is a *convergent* series.

Applying this, we see that $K=\sum_{n=0}^{\infty} q_nT^{(n)}$ will converge if in the formal sum $\sum_{n=0}^{\infty} q_nT^{(n)}(t)$, the coefficients of $1, t, t^2,\cdots$ all exist. But this is true since

$$\sum_0^{\infty} q_nT^{(n)}(t)\sim\sum_{n=0}^{\infty} t^n\,[q_0/n!+q_1/(n-1)!+\cdots+q_n/0!]\sim\sum_{n=0}^{\infty} p_nt^n/n!.$$

Hence the sum-set K exists and its generating function is $\sum_0^{\infty} p_nt^n/n!$. Since this is also the generating function for P, we must have $K=P$.

The characteristic set for M is a very simple set:

THEOREM. *M has the characteristic set*

$$\Theta:\quad \Theta_n(x)=(x-1)^n.$$

COROLLARY. *The set P, with characteristic numbers $\{p_n\}$ is permutable with M if and only if*

$$P=\Theta^{-1}P^*\Theta,$$

where $P^*:\{P_n^*(x)=p_nx^n\}$ and Θ is given above.

We can now give a new proof of the second preceding theorem: M^{-1} has the characteristic numbers $\{n+1\}$, $(n=0,1,\cdots)$, whence $T^{(n)}$ has the

characteristic numbers $\{1 \cdot k(k-1) \cdots (k-n+1)\}$, $(k=0,1,\cdots)$. Therefore $K=\sum_0^\infty q_n T^{(n)}$ has the characteristic numbers

$$\{\sum_{n=0}^{\infty} q_n \cdot 1(k-1) \cdots (k-n)\},$$
$$=\{k!\,[q_0/k! + q_1/(k-1)! + \cdots + q_k/0!\,]\} = \{p_k\}, \qquad (k=0,1,\cdots).$$

That is, the K-series converges, and $K=P$.

III. Linear Operators and Equations Associated with Sets.

1. *Introduction.* We have considered in § II the algebraic aspects of sets. Going farther, it is possible to treat sets of polynomials analytically. In so doing we are led to associate with each set a definite linear operator and this pair: the set of polynomials and the operator in turn give rise to a second operator and set of functions (no longer polynomials). We examine the relations existing between these pairs, and in general consider the formulation of the functional equations on the two operators. *The treatment is formal.*[1]

2. *The Linear Operator L.* Let P be a given set:

$$P: P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n, \qquad (n=0,1,\cdots).$$

We wish to determine a linear operator L which will transform polynomials into polynomials and in such manner that the identity set I is carried into P:

$$(1) \qquad L[x^n] = P_n(x), \qquad (n=0,1,\cdots).$$

Such an operator may be put in various forms; for our purpose it will be convenient to express L as a differential expression of infinite order.

THEOREM. *The linear operator*

$$(2) \qquad L \equiv L[y(x)] = \sum_{n=0}^{\infty} L_n(x)\, y^{(n)}(x),$$

where $L_n(x)$ is the polynomial

$$(3) \qquad L_n(x) = (1/n!)\,[P_n(x) - \binom{n}{1} x P_{n-1}(x) + \binom{n}{2} x^2 P_{n-2}(x) - \cdots + (-1)^n x^n P_0(x)\,],$$

carries I into P.

[1] Conditions under which the formal processes can be shown to hold rigorously are reserved for another occasion.

Proof. The method of induction can be used. Or better: *define* the formal power series

$$P(t;x) \sim \sum_{n=0}^{\infty} P_n(x)t^n/n!, \tag{4}$$

$$L(t;x) \sim \sum_{n=0}^{\infty} L_n(x)t^n. \tag{5}$$

The right hand member of (3) is the coefficient of t^n in the expansion of $e^{-tx}P(t;x)$, so that on multiplying (3) by t^n and summing we obtain

$$L(t;x) \sim e^{-tx}P(t;x). \tag{6}$$

Again, from (2),

$$L[e^{tx}] \sim e^{tx}L(t;x) \sim P(t;x); \tag{7}$$

and also

$$L[e^{tx}] \sim \sum_{0}^{\infty} (t^n/n!)L[x^n].$$

On comparing this with (7) we have $L[x^n] = P_n(x)$.

LEMMA. *If L is the operator* (2), *and Q is any set, then*

$$L[Q] = QP; \text{ i.e., } L[Q_n(x)] = (QP)_n(x), \qquad (n = 0, 1, \cdots).$$

COROLLARY. *If L_P, L_Q are the operators for P, Q, then L_P, L_Q are commutative if and only if P and Q are permutable.*

In the study of linear operators one usually seeks the characteristic functions; that is, those functions which repeat themselves (with a constant multiplier) under the operation. For L we have the

THEOREM. *The linear differential equation of infinite order*

$$L[u(x)] - \lambda u(x) = 0 \tag{8}$$

has a polynomial solution if and only if λ is one of the characteristic numbers $\{p_n\}$ of P; *and if $p_n \neq p_i$ $(i \neq n)$, then to $\lambda = p_n$ corresponds a unique polynomial,* which is of degree exactly n.*

Proof. On substituting $u(x) = u_0 + u_1x + \cdots + u_sx^s$ into (8) one finds for $u_0, \cdots, u_s$ a set of $s + 1$ linear homogeneous equations, the determinant of which vanishes only when $\lambda = p_0, p_1, \cdots, p_s$. When λ has one of these values there is at least one solution. If for all i different from n, $p_i \neq p_n$, then to $\lambda = p_n$ corresponds just one solution, and it is of degree $s = n$.

If some of the characteristic numbers $\{p_n\}$ are equal there need not

* There is however an arbitrary constant multiplier.

. . . polynomial solution of (8) for every degree $n = 0, 1, \cdots$. This . . . able, so we limit ourselves to the case $p_m \neq p_n$, $m \neq n$.

COROLLARY. *If P is complete there exists one and only one polynomial solution of (8) for every degree $n = 0, 1, \cdots$.*

Moreover,

THEOREM. *If P is complete the polynomial solutions of (8) form the characteristic set for P.*

Proof. Let Θ denote the characteristic set for P. Then $\Theta P = P^* \Theta$. . . $L[\Theta] = \Theta P$ and $(P^*\Theta)_n(x) = p_n \Theta_n(x)$, so that $L[\Theta_n(x)] = p_n \Theta_n(x)$.

Let P be non-singular. Then P^{-1} exists, and if L^{-1} is the linear operator of P^{-1},

$$L^{-1}[x^n] = P_n^{-1}(x),$$

. . . have

LEMMA. *$L^{-1}L = LL^{-1}$ is the identity operator; that is,*

$$L^{-1}[P_n(x)] = x^n, \qquad L[P_n^{-1}(x)] = x^n.$$

COROLLARY. *A formal solution of the functional equation*

$$L[y(x)] = f(x)$$

. . .

$$y(x) = L^{-1}[f(x)].$$

In general, however, this expression does not converge.

If $P = D$ is a diagonal set: $D_n(x) = d_n x^n$, L assumes a simple form:

$$L[y(x)] = \sum_{n=0}^{\infty} \sigma_n x^n y^{(n)}(x),$$

where

$$\sigma_n = (1/n!)[d_n - \binom{n}{1} d_{n-1} + \cdots + (-1)^n d_0].$$

THEOREM. *If P is complete, and L_P, L_Θ, L_{P^*}, $L_{\Theta^{-1}}$ are the operators for P, Θ, P^*, Θ^{-1}, then*

$$L_P = L_\Theta L_{P^*} L_{\Theta^{-1}},$$

the operations being performed from left to right.

8. *The Linear Operator $\mathfrak{L}$.* We have defined $P(t; x)$ and $L(t; x)$ in . . . (5) as formal power series in t. If we rewrite them as power series . . . we have

$$(4') \qquad P(t;x) \sim \sum_{n=0}^{\infty} \mathcal{P}_n(t) x^n/n!,$$

$$(5') \qquad L(t;x) \sim \sum_{n=0}^{\infty} \mathcal{L}_n(t) x^n,$$

where $\mathcal{P}_n(t)$, $\mathcal{L}_n(t)$ are themselves formal power series *beginning with* t^n:

$$\mathcal{P}_n(t) \sim p_{nn} t^n + \frac{p_{n+1,n}}{n+1} t^{n+1} + \frac{p_{n+2,n}}{(n+2)(n+1)} t^{n+2} + \cdots,$$

$$\mathcal{L}_n(t) \sim l_{nn} t^n + l_{n+1,n} t^{n+1} + l_{n+2,n} t^{n+2} + \cdots.$$

Definition. A sequence $\mathcal{S} : \{\mathcal{S}_n(t)\}$, $(n = 0, 1, \cdots)$ is a *triangular function set* if $\mathcal{S}_n(t)$ is a formal power series beginning with t^n or higher power; and the *characteristic numbers* of $\mathcal{S}$ are the coefficients of $1, t, t^2, \cdots$ in $\mathcal{S}_0(t)$, $\mathcal{S}_1(t)$, $\mathcal{S}_2(t), \cdots$ respectively

COROLLARY. *The characteristic numbers for $\mathcal{P}$ are the same as those for P, namely $\{p_n\}$.*

In addition to L we consider a second operator, $\mathcal{L}$, the *adjoint of L*:

$$(9) \qquad \mathcal{L} \equiv \mathcal{L}[u(t)] \equiv \sum_{n=0}^{\infty} \mathcal{L}_n(t) u^{(n)}(t).$$

THEOREM. *$\mathcal{L}$ carries triangular sets into triangular sets, and in particular, the identity $\mathcal{I} : \{\mathcal{I}_n(t) = t^n\}$ goes over into the triangular set $\mathcal{P} : \{\mathcal{P}_n(t)\}$:*

$$(10) \qquad \mathcal{L}[t^n] = \mathcal{P}_n(t).$$

The first part of the theorem is immediate. (10) can be proved in the way that (1) was established.

Consider now the equation

$$(11) \qquad \mathcal{L}[\mathcal{D}(t)] - \lambda \mathcal{D}(t) = 0.$$

THEOREM. *Equation* (11) *has a formal power series solution if and only if $\lambda = p_n$ $(n = 0, 1, \cdots)$; and if p_n is such that $p_i \neq p_n$, $(i \neq n)$, then there exists a unique formal series $\mathcal{D}(t)$* for $\lambda = p_n$, *and the coefficient of t^n in $\mathcal{D}(t)$ is not zero, but all preceding coefficients are zero.*

The result follows on substituting the series $\mathcal{D}(t) = d_0 + d_1 t + \cdots$ into (11), and equating coefficients of like powers of t.

Definition. The triangular set $\mathcal{P}$ is complete if P is complete; and it is non-singular if P is non-singular.

THEOREM. *If $\mathcal{P}$ is complete there exists for each n one and only one power series $\mathcal{D}(t)$ whose first non-zero coefficient is that for t^n.*

Let $\mathcal{D}_n(t)$ be the unique power series corresponding to n. Then

(12) $$\mathcal{L}[\mathfrak{D}_n(t)] - p_n\mathfrak{D}_n(t) = 0.$$

We term the power series $\{\mathfrak{D}_n(t)\}$ the *characteristic functions* for the operator $\mathcal{L}$.

Since $\mathfrak{D}_n(t)$ begins with t^n (the coefficient being different from zero), we have the

COROLLARY. *The set $\mathfrak{D}$ of characteristic functions is non-singular.*

Definition. Let $\mathcal{S} : \mathcal{S}_n(t) \sim \sigma_{nn}t^n + \sigma_{n,n+1}t^{n+1} + \cdots$ be a triangular set. If $\mathcal{T}$ is another such set, then $\mathcal{S}\mathcal{T}$ is the set

$$\mathcal{S}\mathcal{T} : (\mathcal{S}\mathcal{T})_n(t) \sim \sigma_{nn}\mathcal{T}_n(t) + \sigma_{n,n+1}\mathcal{T}_{n+1}(t) + \cdots.$$

THEOREM. *If $\mathfrak{P}$ is complete, $\mathfrak{D} : \{\mathfrak{D}_n(t)\}$ is a characteristic set for $\mathfrak{P}$; i. e.,*

(13) $$\mathfrak{D}\mathfrak{P} = \mathfrak{P}^*\mathfrak{D},$$

where

$$\mathfrak{P}^* : \mathfrak{P}_n^*(t) = p_n t^n.$$

Proof. For every set $\mathfrak{Q}$,

(14) $$\mathfrak{Q}\mathfrak{P} = \mathcal{L}[\mathfrak{Q}].$$

Taking $\mathfrak{Q} = \mathfrak{D} : \mathfrak{D}\mathfrak{P} = \mathcal{L}[\mathfrak{D}] = \mathfrak{P}^*\mathfrak{D}$.†

LEMMA. *If $\mathcal{S}$ is non-singular then $\mathcal{S}^{-1}$ exists such that*

$$\mathcal{S}^{-1}\mathcal{S} = \mathcal{S}\mathcal{S}^{-1} = \mathcal{I} \quad (identity).$$

COROLLARY. *If $\mathfrak{P}$ is complete it has the canonical form*

(15) $$\mathfrak{P} = \mathfrak{D}^{-1}\mathfrak{P}^*\mathfrak{D}.$$

THEOREM. *If $\mathfrak{P}$ is complete, a necessary and sufficient condition that $\mathfrak{Q}$ be permutable with $\mathfrak{P}$ is that $\mathfrak{Q} = \mathfrak{D}^{-1}\mathfrak{Q}^*\mathfrak{D}$.*

4. *Associated Functional Equations; Formal Expansions.* With regard to L and $\mathcal{L}$ we make the convention that L operates in the variable x and $\mathcal{L}$ in t.

Let P be non-singular so that P^{-1} and L^{-1} exist.

THEOREM. *Formally we have* ‡

(17) $$L[P^{-1}(t;x)] \sim e^{tx} \sim \mathcal{L}[P^{-1}(t;x)].$$

This follows from the relations

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† $\mathfrak{P}^*\mathfrak{D}$ is the set $(\mathfrak{P}^*\mathfrak{D})_n(t) \sim p_n\mathfrak{D}_n(t)$.

‡ $P^{-1}(t;\, x)$ is *not* $\{P(t;\, x)\}^{-1}$.

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$$L[P^{-1}(t;x)] \sim \sum_0^\infty (t^n/n!) L[P_n^{-1}(x)] \sim \sum_0^\infty t^n x^n/n!,$$

$$\mathcal{L}[P^{-1}(t;x)] \sim \sum_0^\infty (x^n/n!)\ \mathcal{L}[\mathcal{P}_n^{-1}(t)] \sim \sum_0^\infty x^n t^n/n!.$$

This result enables us to obtain a formal solution to the equations

(18) $L[y(x)] = f(x)$, (19) $\mathcal{L}[u(t)] = s(t)$.

Let $f(x) = \sum_0^\infty f_n x^n$ be an entire function with $\limsup |f^{(n)}(0)|^{1/n} = \sigma$, so that $F(x) = \sum_0^\infty n! f_n x^n$ has a non-zero radius of convergence ($= 1/\sigma$). Then

$$f(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t} e^{x/t}dt,$$

C being a contour sufficiently close to, and surrounding, the origin.

THEOREM. (18) *has the formal solution*

$$y(x) \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t} P^{-1}\left(\frac{1}{t}; x\right) dt, \tag{20}$$

Γ *surrounding the origin.*

Proof. From (20),

$$L[y] \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t} L[P^{-1}\left(\frac{1}{t}; x\right)]dt \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t} e^{x/t}dt = f(x).$$

Likewise we have the

THEOREM. (19) *has the formal solution*

$$u(t) \sim \frac{1}{2\pi i}\int_\Gamma \frac{S(x)}{x} P^{-1}\left(t; \frac{1}{x}\right)dx. \tag{21}$$

Formal solutions can be obtained in yet another way, namely in $\Theta_n(x)$-expansions and $\mathcal{D}_n(t)$-expansions. Consider the equation with a parameter:

$$y(x) = f(x) + \lambda L[y(x)]. \tag{22}$$

The corresponding homogeneous equation is satisfied by $\Theta_n(x)$ for $\lambda = \lambda_n = 1/p_n$.

THEOREM. *If*

$$f(x) \sim \sum_0^\infty f_n \Theta_n(x) \tag{23}$$

then

$$(24) \qquad y(x) \sim \sum_0^\infty \lambda_n f_n \Theta_n(x)/(\lambda_n - \lambda)$$

is a formal solution of (22) *for every* $\lambda \neq \lambda_n$.

Proof. $L[y] \sim \sum_0^\infty f_n \Theta_n(x)/(\lambda_n - \lambda)$, so that both members of (22) agree. Likewise

THEOREM. *If*

$$(25) \qquad s(t) \sim \sum_0^\infty s_n \mathcal{D}_n(t),$$

then

$$(26) \qquad u(t) \sim \sum_0^\infty \lambda_n s_n \mathcal{D}_n(t)/(\lambda_n - \lambda)$$

is a formal solution of

$$(27) \qquad u(t) = s(t) + \lambda \mathcal{L}[u(t)]$$

for every $\lambda \neq \lambda_n$.

THEOREM. *The function* e^{tx} *has the expansion* *

$$(28) \qquad e^{tx} \sim \sum_{n=0}^\infty \Theta_n(x) \mathcal{D}_n(t).$$

Proof. Set $e^{tx} \sim \sum_0^\infty \Theta_n(x) \mathcal{H}_n(t)$. From

$$(29) \qquad L[e^{tx}] = \mathcal{L}[e^{tx}]$$

we obtain

$$\sum_0^\infty p_n \Theta_n(x) \mathcal{H}_n(t) \sim \sum_0^\infty \Theta_n(x) \mathcal{L}[\mathcal{H}_n(t)],$$

so that

$$R(t;x) \sim \sum_0^\infty \{\mathcal{L}[\mathcal{H}_n(t)] - p_n \mathcal{H}_n(t)\} \Theta_n(x) \underset{t,x}{\equiv} 0.$$

Now let $M_{(k)}$ be an operator commutative with L and having the characteristic numbers

$$(a) \qquad \{\mu_n^{(k)}\}: \qquad \mu_n^{(k)} = p_n, \quad n \neq k, \quad \mu_k^{(k)} \neq p_k.$$

Having the same characteristic polynomials $\Theta_n(x)$ as has L, then

$$(M - L)[R(t;x)] \sim \sum_{n=0}^\infty (\mu_n^{(k)} - p_n) \{\mathcal{L}[\mathcal{H}_n(t)] - p_n \mathcal{H}_n(t)\} \Theta_n(x) \underset{t,x}{\equiv} 0;$$

and on using (a) this reduces to

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* Since $\Theta_n(x)$ and $\mathcal{D}_n(t)$ are unique only to within arbitrary constant multipliers, it is assumed in (28) that these multipliers are suitably chosen. It suffices that the product of the coefficient of x^n in $\Theta_n(x)$ and the coefficient of t^n in $\mathcal{D}_n(t)$ be $1/n!$.

$$\mathcal{L}[\mathcal{H}_k(t)] - p_k\mathcal{H}_k(t) = 0.$$

P being complete, $\mathcal{H}_k(t)$ must then coincide* with $\mathcal{D}_k(t)$; which establishes (28).

The expansion (28) is in terms of $\Theta_n(x)$, $\mathcal{D}_n(t)$. Recalling how $P(t;x)$ is related to P, we see that e^{tx} bears the same relation to the identity set I. Now I has the characteristic numbers $\{i_n = 1\}$. This suggests an expansion for $P(t;x)$:

THEOREM. *$P(t;x)$ has the expansion*

$$(30) \qquad P(t;x) \sim \sum_{n=0}^{\infty} p_n\Theta_n(x)\,\mathcal{D}_n(t).$$

(30) follows from (7) and (28).

(30) may be regarded as a *canonical form* for $P(t;x)$. It puts in evidence the *characteristic polynomials* $\Theta_n(x)$, *functions* $\mathcal{D}_n(t)$, and *numbers* p_n.

COROLLARY. *If P is complete, a necessary and sufficient condition that Q be permutable with P is that*

$$Q(t;x) \sim \sum_{0}^{\infty} q_n\Theta_n(x)\,\mathcal{D}_n(t),$$

where $\{q_n\}$ are the characteristic numbers of Q.

If in (28) we expand in powers of t or of x we find:

$$(31) \qquad x^n = \mathcal{D}_0^{(n)}(0)\Theta_0(x) + \mathcal{D}_1^{(n)}(0)\Theta_1(x) + \cdots + \mathcal{D}_n^{(n)}(0)\Theta_n(x),$$

$$(32) \qquad t^n = \Theta_n^{(n)}(0)\,\mathcal{D}_n(t) + \Theta_{n+1}^{(n)}(0)\,\mathcal{D}_{n+1}(t) + \Theta_{n+2}^{(n)}(0)\,\mathcal{D}_{n+2}(t) + \cdots.$$

From these relations can be determined the formal expansions of functions of x or of t in $\Theta_n(x)$-series or $\mathcal{D}_n(t)$-series respectively.

IV. SYSTEMS OF LINEAR EQUATIONS ASSOCIATED WITH SETS.

The theory of the operators L and $\mathcal{L}$ has a counterpart in the related field of systems of linear equations in infinitely many unknowns. This is the subject of the present section.

Consider again the operator

$$(1) \qquad L\colon \quad L[y(x)] = \sum_{0}^{\infty} L_n(x)y^{(n)}(x),$$

where

$$L_n(x) = l_{n0} + l_{n1}x + \cdots + l_{nn}x^n.$$

* The other alternative: $\mathcal{H}_k(t) \equiv 0$ is not possible, for it would mean that on the right hand side of (28) there was no term in x^nt^n.

On operating with L on the power series $y(x) = \sum_0^\infty y_n x^n$ we arrive at a system of linear expressions to which (1) is formally equivalent:

$$(2) \qquad L^*: \quad L^*\{y\} \equiv a_{ii}y_i + a_{i,i+1}y_{i+1} + a_{i,i+2}y_{i+2} + \cdots, \qquad (i = 0, 1, \cdots).$$

The quantities a_{ij}, l_{ij} are related by

$$(3) \qquad a_{nk} = k!\,[l_{k-n,0}/n! + l_{k-n+1,1}/(n-1)! + \cdots + l_{k,n}/0!],$$

$$(4) \qquad l_{kn} = (1/k!)\,[a_{nk} - \binom{k}{1}a_{n-1,k-1} + \cdots + (-1)^n a_{0,k-n}].$$

But the right hand member of (3) is seen to be precisely p_{kn}, so that

LEMMA. *a_{nk} is given by*

$$(5) \qquad a_{nk} = p_{kn}.$$

In other words, if we write out the infinite matrix representing the operator L^* of (2), then the elements in the successive columns are the coefficients of the polynomials $P_0(x), P_1(x), \cdots$.

The operator L^* carries vectors into vectors in space of infinitely many dimensions. In particular, if all components of a vector beyond the i-th are zero, the transformed vector has the same property. We term a sequence of vectors q^*: $q_n^* = (q_{n0}, q_{n1}, \cdots)$ a *set* if in q_n^* all components after the one of index n are zero. L^* carries sets into sets.

Let i^* be the identity set: $i_n^* = (0, \cdots, 0, 1, 0, \cdots)$, the 1 appearing as the component of index n.

THEOREM. *L^* carries set i^* into set p^*: $L^*\{i_n^*\} = p_n^*$.*

With (2) we consider the system of homogeneous equations

$$(6) \qquad a_{ii}y_i + a_{i,i+1}y_{i+1} + \cdots = \lambda y_i, \qquad (i = 0, 1, \cdots).$$

THEOREM. *A vector solution of a finite number of components exists if and only if $\lambda = p_s$, $(s = 0, 1, \cdots)$; and if p_n is such that $p_i \neq p_n$, $i \neq n$, then to $\lambda = p_n$ corresponds only one vector solution with all components zero after that of index n (and this last component is not zero).*

THEOREM. *If $p_m \neq p_n$, $m \neq n$* for all *m, n, to each n exists one and only one solution of degree n; and the set of these solutions is the set* †

$$\theta^*: \ \theta_n^* = (\theta_{n0}, \theta_{n1}, \cdots, \theta_{nn}, 0, 0, \cdots): \quad L^*\{\theta_n^*\} = p_n\theta_n^*.$$

This is a consequence of the relation $(\Theta P)_n(x) = p_n\Theta_n(x)$.

Corresponding to $\mathfrak{L}$ there is a second system of linear expressions. In

† It is to be recalled that $\Theta_n(x) = \theta_{n0} + \theta_{n1}x + \cdots + \theta_{nn}x^n$ is the characteristic polynomial of degree n for P.

(7) $$\mathscr{L}: \mathscr{L}[u(t)] = \sum_0^\infty \mathscr{L}_n(t)u^{(n)}(t)$$

substitute $u(t) = \sum_0^\infty u_n t^n$. There results

(8) $$\mathscr{L}^*: \mathscr{L}^*\{u\} \equiv b_{i0}u_0 + b_{i1}u_1 + \cdots + b_{ii}u_i, \qquad (i = 0, 1, \cdots).$$

$\mathscr{L}^*$ carries a vector, whose first k components are zero, into another such vector. For this operator we define a *set* of vectors as a sequence $^*q: \{^*q_n\}$ in which all components of $^*q_n$ of index less than n are zero. $\mathscr{L}^*$ carries sets into sets.

LEMMA. *We have* $b_{ij} = j!p_{ij}/i!$.

COROLLARY. *In the matrix* $\| b_{ij} \|$ *which defines* $\mathscr{L}^*$, *the elements in the successive columns are precisely the coefficients of* $\mathcal{P}_0(t), \mathcal{P}_1(t), \cdots$.

THEOREM. $\mathscr{L}^*$ *carries the identity set* $^*i$ *into* $^*p$: $\mathscr{L}^*\{^*i_n\} = {}^*p_n$, *where the components of* $^*p_n$ *are the coefficients of* $1, t, t^2, \cdots$ *in* $\mathcal{P}_n(t)$.

THEOREM. *If* $p_m \neq p_n$, $m \neq n$, *the homogeneous equation*

$$\mathscr{L}^*\{u\} = \lambda u$$

has a solution if and only if $\lambda = p_n$, $(n = 0, 1, \cdots)$; *and to each* n *corresponds one solution* $^*d_n$, *and its components of index less than* n *are zero. Moreover, the set* $^*d: \{^*d_n\}$ *of solutions is precisely the set*

$$^*d_n = (0, \cdots, 0, d_{nn}, d_{n,n+1}, \cdots),$$

where

$$\mathcal{D}_n(t) \sim d_{nn}t^n + d_{n,n+1}t^{n+1} + \cdots.$$

Definition. If q^*, r^* are the sets

$$q_n^* = (q_{n0}, q_{n1}, \cdots, q_{nn}, 0, 0, \cdots), \quad r_n^* = (r_{n0}, \cdots, r_{nn}, 0, 0, \cdots),$$

then $(qr)^*$ is the set

$$(qr)_n^* = q_{n0}r_0^* + q_{n1}r_1^* + \cdots + q_{nn}r_n^*;$$

and if

$$^*q: {}^*q_n = (q_{nn}, q_{n,n+1}, \cdots), \qquad ^*r: {}^*r_n = (r_{nn}, r_{n,n+1}, \cdots),$$

then

$$^*(qr): {}^*(qr)_n = q_{nn}{}^*r_n + q_{n,n+1}{}^*r_{n+1} + \cdots.$$

THEOREM. *For* $p_m \neq p_n$, $m \neq n$, *the vector sets* p^*, $^*p$ *have the canonical forms*

$$p^* = \theta^{*-1}(p^{**})\theta^*, \qquad ^*p = {}^*d^{-1}(^*p^*)d^*,$$

where† p^{**}, $^*p^*$ *are the (same) diagonal sets*

$$p_n^{**} = {}^*p_n^* = (0, 0, \cdots, 0, p_n, 0, \cdots).$$

PRINCETON UNIVERSITY,
PENNSYLVANIA STATE COLLEGE.

† The different notation p^{**}, $^*p^*$ for the same vectors is to accord with previous notation.

CONCERNING SIMPLE CONTINUOUS CURVES AND RELATED POINT SETS.

By R. L. Wilder.

In his paper "Concerning Simple Continuous Curves,"* R. L. Moore applied the term *simple continuous curve* to any point set which is either an arc, a simple closed curve, an open curve or a ray, and gave various topological characterizations for a point set that falls within the general class of such curves, as well as for the individual types of curves.†

The intent of the present paper is to supplement the work of Moore just cited, extending two of his results to more general spaces, and giving certain new definitions which the author believes may be of value.‡ As Moore made clear in his paper, the requirement of *boundedness* in a definition of arc may introduce great difficulties in certain problems. It will be noted that none of the definitions of arc, simple closed curve, etc., given in the present instance makes use of this condition. Furthermore, the condition that the set in question be *closed* is eliminated in several cases—a feature that would seem to be of importance in problems concerning non-closed sets.

To summarize briefly, before proceeding to the demonstrations: In § 1, Moore's characterization of simple continuous curves as a class is extended to any euclidean space, and is then applied to establish a new characterization based on a certain type of set which the author has found useful in other connections. In § 2 definitions are given of arc, the more general class of sets known as *irreducible connexes,*§ and of simple closed and *quasi-closed* curves.

In this connection it is indicated how a simple proof may be obtained of the theorem¶ that every interval of an open curve is an arc; also a

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* *Transactions of the American Mathematical Society*, Vol. 21 (1920), pp. 313-320. This paper will be referred to hereinafter as C. S. C.

† Other definitions have been given by various authors, but no attempt will be made here to give a complete bibliography of these. Cf. C. S. C., however.

‡ As indicated in following footnotes, the results of this paper were submitted to the American Mathematical Society several years ago, and as now published the details differ in some particulars from the original, due chiefly to the fact that by unification of treatment it is possible to present the various results in a single paper.

§ Cf. B. Knaster and C. Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 206-255, §§ 2 and 3.

¶ Cf. R. L. Moore, "On the Fundations of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), pp. 131-164, Theorem 49, and C. S. C., Theorem 3. Both of the proofs given by Moore are for the plane.

simple proof of Moore's first definition of arc in C. S. C. is outlined, which establishes it for general spaces. In § 3 the notion of quasi-closed curve is used to obtain further definitions of simple closed curve, three of which require neither that the set in question be bounded nor that it be closed.

§ 1

*A property which characterizes simple continuous curves.**

It is well known † that if, in a plane S, C_1 and C_2 are two closed, mutually exclusive point sets and M is a bounded continuum having at least one point in common with each of the sets C_1, C_2, then there exists a point set H, a subset of M, such that H is connected and contains no point of either C_1 or C_2, but such that C_1 and C_2 each contains at least one limit point of H. In case C_1 and C_2 are subsets of M, we shall say that H, together with its limit points in C_1 and C_2, is a set $K(C_1, C_2)M$. In general, if M is not bounded, there may not exist any set $K(C_1, C_2)M$ for certain subsets C_1, C_2, of M, but it is easy to see that every continuum, whether bounded or not, contains some sets $K(C_1, C_2)M$ for the proper selections of C_1 and C_2.‡ We shall show that a continuum in which every set $K(C_1, C_2)M$ is an arc § must be a simple continuous curve.

* The content of this section was presented to the American Mathematical Society April 2, 1926.

† Cf. Anna M. Mullikin, "Certain Theorems Relating to Plane Connected Point Sets," *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 144-162.

‡ For instance, if A and B are distinct points of M, let K_1 and K_2 denote circles with centers at A and radii d_1 and d_2, respectively, such that $d_1 < d_2 <$ distance AB. Let $K_i \cdot M = C_i$ $(i = 1, 2)$. Then M, whether bounded or not, contains a set $K(C_1, C_2)M$.

Since we shall not restrict ourselves to the plane in the present paper, it should be pointed out that Miss Mullikin's theorem cited above holds in euclidean space of any number of dimensions.

So far as the existence of sets $K(C_1, C_2)M$ is concerned, the condition of *connectedness im kleinen*, or *local connectedness*, as applied to connected sets in general, seems to imply more than *closed*. For in the case of a connected set M which is connected im kleinen, if C_1 and C_2 are mutually exclusive sets that are closed with respect to M, then M, whether bounded or not, contains a *bounded* set $K(C_1, C_2)M$. Cf. my paper "The Non-Existence of a Certain Type of Regular Point Set," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 439-446, Theorem 4. For other applications of the notion of a set $K(C_1, C_2)M$, or of related sets, see papers of mine in *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), pp. 725-728, and Vol. 16 (1930), pp. 233-240.

§ As our basic definition of arc, we take the following: An *arc* is a closed, connected point set which is irreducibly connected between two points. Cf. N. J. Lennes, "Curves in Non-Metrical Analysis Situs with an Application in the Calculus

LEMMA 1. *A continuum M which has the property that every set $K(C_1, C_2)M$ is an arc is a continuous curve.*

Proof. Suppose there exists a continuum M such that every set $K(C_1, C_2)M$ is an arc and which is not a continuous curve. Then by a theorem due to R. L. Moore and the present author [*] there exist two concentric spheres K_1 and K_2 and a sequence of subcontinua of M, viz., M_∞, M_1, M_2, $M_3, \cdots$ such that (1) each of these continua contains at least one point of K_1 and K_2, respectively, but no point exterior to K_1 or interior to K_2, (2) no two of these sub-continua have a point in common, and no two of them contain points of any connected subset of M which lies wholly in the set $K_1 + K_2 + I$ (where I consists of all points whose distance from the common center of K_1 and K_2 is a number whose magnitude lies between the radii of K_1 and K_2 on the real number continuum), (3) M_∞ is the sequential limiting set of the sequence $M_1, M_2, M_3, \cdots$, (4) if K is that component

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of Variations," *American Journal of Mathematics*, Vol. 33 (1911), pp. 285-326; G. H. Hallett, Jr., "Concerning the Definition of a Simple Continuous Arc," *Bulletin of the American Mathematical Society*, Vol. 25 (1919), pp. 325-326; B. Knaster and C. Kuratowski, *loc. cit.*, Theorem 27. The last two papers just referred to establish the fact that the word "bounded" may be omitted from Lennes' definition. That a set satisfying the above definition is compact, in very general spaces, may be shown by a method of argument similar to that used by R. L. Moore in the proof of Theorem 49 of his paper "On the Foundations of Plane Analysis Situs," *loc. cit.* (cf. footnote in Hallett's paper in this connection). From the argument of Knaster and Kuratowski, it is evident that such a set of points, when imbedded in a locally compact metric space, is homeomorphic with the linear interval [0, 1]. For their Lemma XXVI is true in such a space (the word "compact" being substituted for the word "bounded"), and the proof of their Theorem XXIII holds in such a space if one recalls a theorem of Alexandroff to the effect that a connected and locally compact metric space is separable. Cf. P. Alexandroff, "Über die Metrisation der im Kleinen kompakten topologischen Raume," *Mathematische Annalen*, Vol. 92 (1924), pp. 294-301. Accordingly, unless specifically stated otherwise, we shall assume throughout the present paper that the point sets considered are imbedded in such a space, although it will be clear that many of the proofs given below hold in more general spaces if an *arc* is defined as above for those spaces.

As several of the proofs given below depend upon the theorem that every two points of a continuous curve (i. e., a connected and connected im kleinen continuum) are the end-points of an arc of that curve, we point out that this theorem may be proved, on the basis of the above definition of arc, by a type of argument introduced by R. L. Moore, in "A Theorem Concerning Continuous Curves," *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

[*] Cf. R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377, Lemma 1. This lemma is true in any locally compact metric space whether the continuum in question is compact or not. Cf. G. T. Whyburn, *Bulletin of the American Mathematical Society*, Vol. 34, p. 409, abstract no. 18.

of $M \cdot (K_1 + K_2 + I)$ which contains M_∞, then all of the continua M_1, M_2, $M_3, \cdots$ lie in a single component U of $M - K$. If we let C_1 and C_2 be two distinct points of the set M_∞, it is clear that the set $U + C_1 + C_2$ is a set $K(C_1, C_2)M$ which is not an arc, and consequently the supposition that M is not a continuous curve leads to a contradiction.

The same proof evidently establishes the following lemma:

LEMMA 2. *If a continuum M has the property that every set $K(C_1, C_2)M$ is an arc, then every subcontinuum of M is a continuous curve.*

For the purposes of the present paper, we shall define an *end-point* of a continuous curve M to be a point of M that is not an interior point of any arc of M.*

LEMMA 3. *Let E be any set of end-points of a continuous curve M. Then $M - E$ is connected.*

For if $M - E$ were the sum of two mutually separated sets M_1 and M_2, an arc of M joining a point of M_1 to a point of M_2 would contain a point of E as an interior point.

LEMMA 4. *In any euclidean space, if no continuous subset of a continuum M has more than two boundary points with respect to M, then M is an arc, a simple closed curve, an open curve or a ray.*†

Proof. It is clear from the characterization of continua that are not continuous curves, quoted in the proof of Lemma 1 above, that M must be a continuous curve.

Suppose M contains a simple closed curve ‡ J. If $M - J$ is not vacuous,

* Since this paper was originally written, it has been shown by G. T. Whyburn (for the plane) and W. L. Ayres (for any euclidean space and certain types of metric spaces), that an end-point in this sense is equivalent to an end-point as defined in my paper "Concerning Continuous Curves" (*loc. cit.*, p. 358). Cf. G. T. Whyburn, "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 369-400, Theorem 12, and W. L. Ayres, "Concerning Continuous Curves in Metric Space," *American Journal of Mathematics*, Vol. 51 (1929), pp. 577-594, Theorem 5.

† Lemma 4 is of course an extension of Moore's Theorem 6 of C. S. C., which was proved for the euclidean plane. It is important to observe that we are using the definition of boundary point just as given by Moore in this connection, viz., if M is a subset of N, the *boundary* of M with respect to N is the set of all points $[X]$ such that X is either a point or a limit point of M and also either a point or a limit point of $N - M$.

‡ By a simple closed curve we mean a sum of two arcs that have common end-points, but have no other points in common. By *open curve* and *ray* we denote point sets as defined by Moore, C. S. C., p. 341.

let P be a point of M that is not on J. Then M contains an arc PQ whose end-points are P and a point Q on J, and such that $PQ - Q$ is a subset of $M - J$. Let A be a point of $PQ - (P + Q)$, and let B and C be points of J neither of which is identical with Q. Let QB and QC denote arcs of J that do not contain C and B, respectively. Then it is clear that the set $AQ + QB + QC$, where AQ is that arc of PQ whose end-points are A and Q, is a continuous subset of M that has at least three boundary points with respect to M. Consequently, $M \equiv J$.

Suppose M contains no simple closed curve, and that it has at least two end-points, A and B. Then it is clear, from a method of argument similar to that used in the preceding paragraph, that M cannot contain any points other than those that lie on an arc of M from A to B.

Suppose M contains no simple closed curve and has one and only one end-point, A. Since every bounded continuous curve contains at least two points which do not cut it,* and since every non-cut point of an acyclic continuous curve is an end-point of that curve,† it is clear that M cannot be bounded. Then, by a theorem of Kuratowski,‡ A is the end-point of a ray in M. It is easy to see that M can contain no points that do not lie on this ray.

If M contains no simple closed curve and has no end-point, it follows as in the preceding paragraph that M is unbounded. Let P be any point of M. Then $M - P$ is the sum of two mutually separated sets, M_1 and M_2.§ Consider the continuous curve $M_1 + P$.¶ It follows easily from our hypothesis that P is a non-cut point of $M_1 + P$, and hence an end-point of $M_1 + P$. Consequently, as shown in the preceding paragraph, $M_1 + P$ contains a ray r_1

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* Cf. S. Mazurkiewicz, "Un théorème sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 119-130.

† Cf. R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, Theorem 7. Although the sufficiency proof of this theorem (the part needed in the present connection) makes use of an accessibility theorem true in general only for the plane, the proof is easily modified so as to avoid this theorem. However, since the present paper was originally written, W. L. Ayres has shown (*loc. cit.*) that any point of a continuous curve M which is both a non-cut point of M and lies on no simple closed curve of M is an end-point of M. We have already noted above the validity of these theorems for end-points as defined in the present paper.

‡ C. Kuratowski, "Quelques propriétés topologiques de la demi-droite," *Fundamenta Mathematicae*, Vol. 3 (1922), pp. 59-64.

§ Cf. my paper "Concerning Continuous Curves," *loc. cit.*, and W. L. Ayres, *loc. cit.*

¶ That $M_1 + P$ is connected follows from a theorem of Knaster and Kuratowski (*loc. cit.*, Theorem VI).

with P as end-point. Similarly, $M_2 + P$ contains a ray r_2 with P as end-point. Evidently $r_1 + r_2$ is an open curve which is identical with M.

THEOREM 1. *In order that a continuum M in a euclidean space should be a simple continuous curve, it is necessary and sufficient that every set $K(C_1, C_2)M$ be an arc.*

Proof. The condition stated in the theorem is necessary. For let K denote some set $K(C_1, C_2)M$ of a simple continuous curve M, and let A and B be points of C_1 and C_2, respectively.

In case M is a simple closed curve, $M - (A + B)$ is the sum of two mutually separated sets, M_1 and M_2. Since $K - (A + B)$ is connected, it is a subset of M_1, say. But $M_1 + A + B$ is an arc and therefore a set which is irreducibly connected from A to B. Consequently $K \equiv M_1 + A + B$.

In case M is an arc, there exists only one arc, t, from A to B in M, and every connected subset of M which contains both A and B contains t. Hence (1) K contains t. On the other hand, since $M - (A + B)$ is the sum of the set $t - (A + B)$, and a set (vacuous or non-vacuous) which is separated from $t - (A + B)$, the subset $K - (A + B)$ of $M - (A + B)$ must lie wholly in $t - (A + B)$, a set with which (as already shown) it has points in common. Consequently, (2) t contains K. From (1) and (2) it follows that $t \equiv K$.

The proofs for the cases where M is a ray and an open curve are similar to the proof of the preceding paragraph.

The condition stated in the theorem is sufficient. For suppose M is not a simple continuous curve. Then it has a continuous subset, N, which has at least three distinct boundary points, A, B and C, with respect to M (Lemma 4). We note that by Lemma 2, N is a continuous curve.

The points A, B and C are end-points of N. Suppose C, for instance, is an interior point of an arc t of N, and let a and b be the end-points of t. Let R be a sphere with center at C and not enclosing a or b. Since, by Lemma 1, M is itself a continuous curve, there exists a sphere R_1 concentric with R such that all points of M interior to R_1 are joined to C by an arc of M which lies wholly interior to R. As C is a boundary point of N, there exists, interior to R_1, a point x of $M - N$. Let s be an arc of M with x and C as end-points and lying wholly interior to R. On s, in the order from x to C, let y be the first point of t. It is clear that y is distinct from a and b. Let that portion of s from x to y be denoted by u. Then the set $(u + t) - (a + b + x)$ is connected and $u + t$ is a set $K(a + b, x)M$. But $u + t$ is not an arc. Hence the supposition that C is not an end-point of N leads to a contradiction.

By Lemma 3, $N - (A + B + C)$ is connected. Hence N is a set

$K(A+B, C)M$ and is therefore an arc. This is absurd, since an arc is disconnected by the omission of any three distinct points. Hence the supposition that N has more than two boundary points with respect to M leads to a contradiction, and M is a simple continuous curve.

§ 2

*On connected sets which cut the plane.**

In a recent paper † I investigated some of the properties of a connected set M which contains more than one point and which remains connected on the omission of any connected subset. I found that M is a point set having properties very similar to those of a simple closed curve. Thus, if A and B are any two points of M, M is the sum of two sets K and N which are irreducibly connected from A to B and such that $K-(A+B)$ and $N-(A+B)$ are mutually separated. Furthermore, if M lies in a plane S, then S is cut by M in the sense that there exist at least two points, x and y, of $S-M$ which do not lie in any subcontinuum of $S-M$.

For the purposes of the present paper I shall call a point set which has the internal properties of the set M described in the last paragraph a *quasi-closed curve.*

LEMMA 5. *In order that a connected set M should be irreducibly connected between two of its points, A and B, it is necessary and sufficient that, if P be any point of M distinct from A and B, $M-P$ should be the sum of two mutually separated sets, K and N, neither of which contains both A and B.*

Proof. That the condition stated in this lemma is necessary follows at once from a theorem proved by Knaster and Kuratowski.‡

That the condition is sufficient is easily shown as follows: Suppose M is not irreducibly connected from A to B. Then it contains a proper connected subset, m, which contains both A and B. Let Q be a point of $M-m$. By hypothesis, $M-Q$ is the sum of two mutually separated sets, K and N, neither of which contains both A and B. As m is connected, and K and N are mutually separated, m is a subset of one of the latter sets, say K. But

* The content of this section was presented to the American Mathematical Society, December 30, 1924.

† "On a Certain Type of Connected Set Which Cuts the Plane," *Proceedings of the International Mathematical Congress, Toronto, 1924*, University of Toronto Press, pp. 423-437.

‡ Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Theorem XVI.

m contains both A and B and therefore K contains both A and B contradictory to the hypothesis.

As a consequence of Lemma 5 an arc may be defined as follows:

THEOREM 2. *If A and B are distinct points, an arc from A to B is a closed and connected set of points M containing A and B such that if P is any point of M distinct from A and B, $M - P$ is the sum of two mutually separated sets neither of which contains both A and B.*

It is perhaps of interest to point out here that the definition of arc embodied in Theorem 2 is sufficient to prove Theorem 3 of C. S. C., to the effect that if A and B are two points of an open curve M the interval AB of M is an arc from A to B. The proof as given in C. S. C. shows that the interval AB satisfies the above definition. I mention this, since the proof of Def. 1 in C. S. C., upon which depends the proof of the fact that AB is an arc, is rather long and assumes the Zermelo Postulate.

It may be of interest, however, to give a simple proof of Moore's Def. 1, as well as to show that it holds in more general spaces than the euclidean plane:

THEOREM 3. *In a locally compact metric space, let A and B be distinct points, and M a closed and connected set containing A and B, such that (1) $M - A$ and $M - B$ are connected, (2) if P is any point of M distinct from A and B, then $M - P$ is the sum of two mutually separated connected sets. Then M is an arc from A to B.*

Proof. The set M is a continuous curve. For if not, there exist two concentric spheres K_1 and K_2, a sequence of sub-continua of M, viz., M_∞, M_1, M_2, $M_3, \cdots$, and sets K and U satisfying the conditions (1)–(4) outlined in the proof of Lemma 1 above. If we let $\bar{U}$ denote U together with its limit points, it is clear that $\bar{U}$ contains M_∞.

By hypothesis, all points of M_∞, except possibly two, disconnect M. But clearly none of these points disconnects $\bar{U}$. The set of all these points being non-denumerable, a violation is obtained of a theorem of R. L. Moore * to the effect that no continuum M contains a continuum $\bar{U}$ which contains a non-denumerable set of points that disconnect M but not $\bar{U}$. Thus the supposition that M is not a continuous curve leads to a contradiction.

* "Concerning the Cut-Points of Continuous Curves and of Other Closed and Connected Sets," *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106, Theorem B^*. Alexandroff has shown (*loc. cit.*) that every connected and locally compact space is perfectly separable, and hence satisfies the Lindelöf theorem, upon which the proof of Moore's theorem depends.

The curve M contains no simple closed curve, for if it does, another violation of the theorem of Moore just quoted results.

The curve M contains an arc from A to B. Denote this arc by t, and consider the set $M - t$. If $M - t$ is vacuous, the theorem is proved. If $M - t$ is non-vacuous, let C be one of its components, and let P be the boundary point of C in t (it is easy to see that C can have no other boundary point in t, since M contains no simple closed curve). It follows from condition (1) of the theorem that P is distinct from A and B. But then $M - P$ is the sum of *three* mutually separated sets, viz., C, C_A and C_B, where C_A is the component of $M - P$ that contains A, and C_B the set $M - (P + C + C_A)$. That C_B is not vacuous follows from the fact that it must contain the point B. But this is a violation of condition (2) of the theorem, and consequently $M - t$ is vacuous.

As another application of Lemma 5 we have the following theorem.

THEOREM 4. *In order that a connected set M should be a quasi-closed curve, it is necessary and sufficient that it contain no cut-points and be disconnected by the omission of any two of its points.*

Proof. That the conditions stated in the theorem are necessary follows from the properties stated in the first paragraph of this section, and the theorem of Knaster and Kuratowski referred to in the proof of Lemma 5.

The conditions stated in the theorem are sufficient. Let A and B be any two points of M. Then

$$M - (A + B) = K + N,$$

where K and N are mutually separated sets.

The set $M - A$ is connected by hypothesis. Hence $M - A$ is a connected set which is disconnected by the omission of a point B, and therefore $K + B$ and $N + B$ are connected. Similarly, $K + A$ and $N + A$ are connected. Let

$$k = K + A + B,$$
$$\text{and} \quad n = N + A + B.$$

Clearly k and n are connected sets.

Either K contains non-cut points of k or it does not. Suppose it does, and that P is such a point. Similarly, suppose that N contains a non-cut point, Q, of n. Then

$$M - (P + Q) = (k - P) + (n - Q).$$

That is, $M - (P + Q)$ is the sum of two connected sets which have a point,

A, in common and is therefore connected. This is a contradiction of the hypothesis. Hence either K contains no non-cut points of k or N contains no non-cut points of n. Suppose the latter to be the case. Then every point of n, except A and B, is a cut-point of n. Hence, if x is a point of N,

$$n - x = n_1 + n_2,$$

where n_1 and n_2 are mutually separated sets.

Neither of the sets n_1, n_2, contains both A and B. For suppose n_1 contains both A and B. Now

$$M - x = (K + n - x) = K + n_1 + n_2.$$

The sets K and n_2 are mutually separated, since n_2 is a subset of N and K and N are mutually separated. Hence $M - x$ is the sum of two mutually separated sets, $K + n_1$ and n_2, and M is disconnected by the omission of x, contrary to hypothesis. Thus neither of the sets n_1, n_2, contains both A and B.

Hence n is a connected set containing A and B such that if x is any point of n distinct from A and B, $n - x$ is the sum of two mutually separated sets neither of which contains both A and B. By Lemma 5, n is irreducibly connected from A to B.

Suppose K contains non-cut points of k, and let P be such a point. Let x be a point of N. Then

$$n - x = n_1 + n_2,$$

where n_1 and n_2 are connected sets such that n_1 contains A and n_2 contains B, and

$$M - P - x = (k - P) + n_1 + n_2.$$

But each of the connected sets n_1 and n_2 has a point in common with $k - P$, and since the latter set is connected, $M - P - x$ is connected contrary to hypothesis. Hence K cannot contain any non-cut points of k. That is, every point of k distinct from A and B disconnects k. That k is irreducibly connected from A to B can now be shown as in the case of n. It follows that M is a quasi-closed curve.

COROLLARY. *In order that a closed and connected set M should be a simple closed curve it is necessary and sufficient that M should contain no cut-points and be disconnected by the omission of any two of its points.**

* Since I presented this paper to the Society it has come to my attention that J. R. Kline has announced the result contained in this corollary for the case where the set M lies in euclidean space of two dimensions. Cf. his paper in *Proceedings of the*

§ 3

*On the definition of simple closed curve.**

LEMMA 6. *Let M be a set irreducibly connected from A to B, and let K be a subset of M which consists of all points of some arc, t, except an end-point, P, of that arc. Then P is a point of M.*

Proof. Suppose P is not a point of M. Let Q be the other end-point of t, and let x be a point of t distinct from P and Q, lying between † Q and B on M. Denote by m that portion of M from Q to B. The set m is irreducibly connected from Q to B.‡

The set $t-(P+Q)$ is a subset of m. For suppose y were a point of $t-(P+Q)$ lying between A and Q on M. Now $M-Q$ is the sum of two mutually separated connected sets, M_1 and M_2, neither of which contains both x and y.§ Let K_i denote the set of those points of $K-Q$ that lie in M_i $(i=1,2)$. Then $t-(P+Q)=K-Q=K_1+K_2$, and as K_1 and K_2 are mutually separated sets, $t-(P+Q)$ is not connected. As this is impossible, it follows that no points of $t-(P+Q)$ lie between A and Q on M.

The point B does not belong to t. For suppose it does. Then all of m belongs to t; for if y were a point of m not in t, it would follow from the fact that $m-y$ is the sum of two mutually separated sets containing Q and B, respectively, that $t-(P+Q)$ is not connected. Consider the set $t-B$. Since B cannot be either P or Q, $t-B$ is the sum of two mutually separated sets, t_1 and t_2, which contain P and Q, respectively. Since K is clearly identical with m, in view of what has been shown above, we must have $t_1 \equiv P$ and $t_2 \equiv m-B$. But then P is not a limit point of m, and consequently not of $t-P$. As this is impossible, the supposition that B is a point of t leads to a contradiction.

National Academy of Sciences, Vol. 9 (1923), pp. 7-12 (Theorem 4). Kline refers, in this connection, to an earlier result of Tietze, who stated a similar result but imposed the unnecessary condition of connectedness im kleinen on the set M, as well as certain restrictive conditions upon the space in which M is imbedded which do not hold when the space is an euclidean n-space $(n>1)$. Cf. *Mathematische Zeitschrift*, Vol. 5 (1919), p. 289. In the same connection Tietze has a result (for the same type of imbedding space) similar to Lemma 7 of the present paper.

* The content of this section was presented to the American Mathematical Society January 1 and September 8, 1926.

† We are of course referring here to the linear order on M. Cf. Knaster and Kuratowski, *loc. cit.*, Theorem XX.

‡ Knaster and Kuratowski, *loc. cit.*, Corollary XXIV.

§ Cf. Knaster and Kuratowski, *loc. cit.*, pp. 219-220.

Denote the set $t - P$ by m_1 and the set $m - m_1$ by m_2. As m_2 contains B, m_2 is non-vacuous. Since we have supposed P not a point of M, m_2 can contain no limit point of m_1. Then since m is connected, m_1 contains a limit point, C, of m_2. Let D be a point of t between C and P. It is easy to see that D is between C and B on m. Denote the portion of t from Q to C by QC. The set m_2 being connected,* it follows that $QC + M_2$ is a connected set containing Q and B. But this set does not contain D, and hence is a proper connected subset of m containing Q and B, contradicting the fact that m is irreducibly connected from Q to B. Thus the supposition that P is not a point of M leads to a contradiction.

THEOREM 5. *Let M be a connected im kleinen continuum which is the sum of two sets N_1 and N_2, each irreducibly connected from A to B, such that $N_1 \cdot N_2 = A + B$. Then M is a simple closed curve.*

Proof. Let

$$N_1 - (A + B) = n_1,$$
$$N_2 - (A + B) = n_2.$$

If n_1 and n_2 are mutually separated, it follows at once that N_1 and N_2 are closed sets and therefore arcs with end-points A and B. In this case M is a simple closed curve.

Suppose that n_1 and n_2 are not mutually separated, and that, for the sake of definiteness, n_2 contains a limit point, P, of n_1. I shall show that in this case M cannot be connected im kleinen.

The point P is a sequential limit point of a set of distinct points, $P_1, P_2, P_3, \cdots$ all belonging to n_1. There exists, between the points of n_1 and the points of the linear continuum $0 \leq x \leq 1$, a one-to-one correspondence T in which order is preserved. For every positive integer n, let the point of the linear continuum corresponding to P_n under the correspondence T be denoted by x_n; suppose the point whose abscissa is zero corresponds to A, and is denoted by a, and that the point whose abscissa is 1 corresponds to B and is denoted by b. The set of points $\{x_n\}$ has at least one limit point, x, and x is a limit point of some subset of this set which is contained, say, in the interval ax. Call this subset X. Then x is a sequential limit point of a sequence of distinct points of X, viz., $a_1, a_2, a_3, \cdots$, where for every positive integer $n > 1$, a_n lies on the interval $a_{n-1}x$.

For every positive integer n, let the point of the sequence $P_1, P_2, P_3, \cdots$

* Knaster and Kuratowski, *loc. cit.*, Theorem XV.

which corresponds to a_n under the correspondence T be denoted by y_n. Clearly P is a sequential limit point of the sequence $y_1, y_2, y_3, \cdots$. Let that point of N_1 which corresponds to x under the correspondence T be denoted by Y.

The point Y is a limit point of that portion of N_1 from A to Y, and there exists a sequence of distinct points of N_1, viz., $z_1, z_2, z_3, \cdots$ having Y as a sequential limit point, and such that for every positive integer $n \geqq 1$, the point z_n lies between y_n and y_{n+1} on N_1. Let that portion of N_1 from y_n to z_n be denoted by t_n. The limit set* of the sequence of sets $t_1, t_2, t_3, \cdots$ is a closed and connected † point set t, containing P and Y.

Let K be a spherical neighborhood with center P and such that Y is exterior to K. Denote the point set consisting of K and its frontier, F, by k. There exists a continuum N which is a subset of t and of k, and contains P and at least one point of F.‡ The set N contains no points of N_1. For suppose Q is a point of N_1 belonging to N. Then Q is a point of that portion of N_1 from Y to B, and, since it is a point of t, a limit point of that portion of N_1 from A to Y. The point Q is therefore identical with Y. But this is absurd, since Y is not a point of k. Hence N contains no points of N_1.

As M is closed, N is a subset of M, and therefore of n_2. Then N is an arc, since every closed and connected subset of an irreducible connexe is an arc.§ Let C be a point of this arc distinct from its end-points. There exists a spherical neighborhood T_1 with center at C such that if t_1 denotes the point set consisting of T_1, plus its frontier, then t_1 contains neither Y, A, B, nor any points of N_2 that are not also points of N. As M is a continuous curve, there exists, concentric with and lying interior to T_1, a spherical neighborhood T_2, such that if a and b are points of M lying interior to T_2, there exists an arc ab whose end-points are a and b, is a subset of M, and lies wholly interior to T_1.

There exists a positive integer j such that y_j and y_{j+1} lie interior to T_2 and z_j lies exterior to T_1. There exists an arc s which is a subset of M, has y_j and y_{j+1} as end-points, and lies wholly interior to T_1. No points of s belong to N_2. For suppose such points exist. Call the set of such points r. All points of r are obviously points of N, and r is closed, being the set of points common to two closed sets. Hence, as y_j is not a point of r, there

* By the limit set of a sequence of sets $M_1, M_2, M_3, \cdots$ is meant the set of all points $\{x\}$, such that x is a limit point of some set of points $m_1, m_2, m_3, \cdots$ where for every positive integer n, m_n is a point of M_n.

† Cf. S. Janiszewski, "Sur les continus irreductibles entre deux points," *Journal de L'Ecole Polytechnique* (2), Vol. 16 (1912), p. 98, Theorem 1.

‡ Cf. Anna M. Mullikin, *loc. cit.*, Theorem 1.

§ Cf. Knaster and Kuratowski, *loc. cit.*, Corollary XXVIII.

exists on s, in the order from y_j to y_{j+1}, a first point, D of r, and D is distinct from y_j. That portion of s from y_j to D is an arc e. All points of e except D belong to N_1. Hence D is a point of N_1, by the above Lemma 6. But as D is obviously distinct from A and B, it must be a point common to n_1 and n_2. This is a contradiction of the hypothesis. Hence the supposition that s contains points of N_2 leads to a contradiction. It follows that s is a subset of N_1, and is identical with that portion of N_1 from y_j to y_{j+1}. But as z_j is a point of this portion, z_j is therefore a point of s. This is impossible, as s contains no points exterior to T_1.

Thus the supposition that n_1 and n_2 are not mutually separated leads to a contradiction, and the theorem is proved.

LEMMA 7. *If M is a connected im kleinen set which is irreducibly connected from A to B, then M is a simple continuous arc having A and B as end-points.**

Proof. It is necessary to prove only that M is closed.

Suppose M is not closed. Then there exists a point P which is a limit point of M and does not belong to M. There exists a correspondence, T, preserving order, between the points of M and the set of points of the linear continuum $0 \leq x \leq 1$, in which A and B correspond to the points whose abscissas are 0 and 1, respectively. Denote the latter points by a and b, respectively. As in the proof of Theorem 5, it can be shown that there exist, on M, two sequences of points, $y_1, y_2, y_3, \cdots$ and $z_1, z_2, z_3, \cdots$, such that (1) the set of all points on ab corresponding, under the correspondence T, to points of these sequences has a sequential limit point, x, whose transform in M is a point Y, (2) all points of these sequences lie between A and Y (or Y and B) on M, and for every positive integer $n \geq 1$, z_n lies between y_n and y_{n+1} on M, (3) Y is the sequential limit point of the sequence $z_1, z_2, z_3, \cdots$, and P is the sequential limit point of the sequence $y_1, y_2, y_3, \cdots$.

That M is not connected im kleinen at Y is shown very easily by considering a spherical neighborhood T with center Y and such that P is exterior to T.

THEOREM 6. *If M is a quasi-closed curve which is connected im kleinen, then M is a simple closed curve.*

* Since this paper was originally written, the result stated in this lemma has been given with a different proof by G. T. Whyburn, in his paper "Concerning Regular and Connected Point Sets," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 685-689; also, a different proof of the lemma has been given by the present author in his paper "On Connected and Regular Point Sets," *ibid.*, Vol. 34 (1928), pp. 649-655.

Proof. As M is a quasi-closed curve, it is the sum of two sets M_1 and M_2 which are irreducibly connected between two points, A and B, of M, and such that $M_1 - (A + B)$ and $M_2 - (A + B)$ are mutually separated. That M_1, say is connected im kleinen at all points distinct from A and B is evident. To show that it is connected im kleinen at the latter points, consider the point A in particular. If M_1 is not connected im kleinen at A, there exists a spherical neighborhood K_1 with center at A, not enclosing B, such that if K_2 is any neighborhood concentric with K_1 and lying interior to K_1, then K_2 encloses a point x of M_1 such that the portion of M_1 from x to A contains points exterior to K_1. Since M is connected im kleinen at A, there exists a neighborhood C with center at A, such that if P is any point of M interior to C, P is joined to A by a connected subset of M which contains no points of M exterior to K_1. Now there exists a point x of M_1 interior to C, such that the portion of M_1 from x to A contains points exterior to K_1. But there does exist a connected subset, N of M, containing both x and A and lying interior to K_1. Obviously N is not a subset of M_1. Let

$$N \cdot (M_i - A) = N_i, \qquad (i = 1, 2).$$

Then $N - A$ is the sum of two sets N_1 and N_2, where N_1 and N_2 are mutually separated sets. The set $N_1 + A$ is a connected subset[*] of M_1 containing x and A and lying wholly interior to K_1, which is clearly impossible. Thus M_1 is connected im kleinen at A and similarly at B. Likewise M_2 is connected im kleinen at both A and B. The theorem now follows as a consequence of Lemma 7.

LEMMA 8. *Let M be a connected point set such that if A and B are any two distinct points of M, then $M - (A + B)$ is the sum of two mutually separated connected sets. Then M is a quasi-closed curve.*

Proof. The set M contains no cut-points. For, suppose there exists in M a point A such that $M - A$ is the sum of two mutually separated sets, K and N. Then K and N are connected. For if K, say, is the sum of two mutually separated sets K_1 and K_2, then, if B is a point of N,

$$M - (A + B) = K_1 + K_2 + (N - B),$$

and the sets K_1, K_2 and $N - B$ are mutually separated, thus contradicting the condition of the theorem.

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[*] Knaster and Kuratowski, *loc. cit.*, Theorem VI.

The sets K and N contain no cut-points. For if K, say, contains a point B such that $K - B$ is the sum of two mutually separated sets K_1 and K_2,

$$M - (A + B) = K_1 + K_2 + N,$$

where K_1, K_2 and N are mutually separated sets, thus contradicting the condition stated in the theorem.

Then, if P is a point of K, and Q is a point of N, $K - P$ and $N - Q$ are connected sets which have a common limit point, A. Hence

$$M - (P + Q) = (K - P) + (N - Q) + A,$$

and therefore $M - (P + Q)$ is the sum of two connected sets $(K - P) + A$ and $(N - Q) + A$ having in common the point A, and is therefore connected, contradicting the condition stated in the theorem.

Thus the supposition that M contains a cut-point leads to a contradiction, and the fact that M is a quasi-closed curve follows from Theorem 4.

THEOREM 7. *In order that a continuum M should be a simple closed curve, it is necessary and sufficient that if A and B are any two distinct points of M, then $M - (A + B)$ is the sum of two mutually separated connected sets.*

This theorem is a consequence of Lemma 8.

THEOREM 8. *In order that a connected and connected im kleinen set M should be a simple closed curve, it is necessary and sufficient that if A and B are any two distinct points of M, then $M - (A + B)$ is the sum of two mutually separated connected sets.*

Theorem 8 is a consequence of Lemma 8 and Theorem 6. It will be noted that the definition of simple closed curve embodied herein does not require that the set M be either closed or bounded. This is also characteristic of the following two definitions.

THEOREM 9. *In order that a point set M should be a simple closed curve, it is necessary and sufficient that it be connected and connected im kleinen, and that it should contain no cut-points and be disconnected by the omission of any two of its points.*

Theorem 9 is a consequence of Theorems 4 and 6.

THEOREM 10. *In order that a connected and connected im kleinen point*

set M should be a simple closed curve, it is necessary and sufficient that it remain connected upon the omission of any connected subset.

Theorem 10 is a consequence of the result referred to at the beginning of § 2, and of Theorem 6.

Theorem 10 is perhaps more striking than any of the others, in view of the fact that there exist* connected and connected im kleinen point sets which contain no continua whatsoever and in that the addition of the simple condition that the set remain connected upon the omission of any connected subset is sufficient to render the set into such a simple continuum as a simple closed curve.

It seems to me probable that the following theorem is true: *Let M be a connected im kleinen point set which is the sum of two sets M_1 and M_2 which are irreducibly connected between two points A and B and which have only A and B in common. Then M is a simple closed curve.* I have not yet been able to establish it, if true, but it obviously includes Theorems 5 and 6, and is more general than either.

University of Michigan,
Ann Arbor, Michigan.

* Cf. R. L. Wilder, "A Connected and Regular Point Set Which Has No Subcontinuum," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 332-340; also B. Knaster and C. Kuratowski, "A Connected and Connected im Kleinen Point Set Which Contains No Perfect Subset," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 106-109.

NOTE ON FUNCTIONS OF R-TH DIVISORS.

By E. T. Bell.

1. The term r-th divisors of n (r, n integers, $r \geq 0$, $n > 0$) is used by D. H. Lehmer * to designate the set of positive integers defined as follows: $\delta_0 = n$; $\delta_r \equiv$ the set of all divisors of all the integers in the set δ_{r-1} $(r \geq 1)$. The set d_r of r-th proper divisors is defined similarly with the restriction that a proper divisor of an integer $m > 0$ is a divisor $< m$ of m. If r is not an integer ≥ 0, the definition can easily be extended, and in fact r may be any real or complex number provided that u^r, which is defined presently, is interpreted in the irregular field IF of all numerical functions as demanded by the postulates of the field; namely as $\exp(r \log u)$. The theory of transcendental analytic functions in IF is developed in a paper to appear shortly.†

2. The entire theory of functions of the numbers δ_r, d_r is implicit in the equations in IF.

2.1 $$ {}_rf = u^r f, \qquad {}_rf' = (u-1)^r f, $$

in which f is an arbitrary numerical function (or arbitrary element of IF), u is defined by $u(n) = 1$ for all integers $n > 0$, and the definitions of ${}_rf$, ${}_rf'$ are (for all integers $n > 0$), ${}_rf(n) \equiv \sum f(\delta_r)$, ${}_rf'(n) \equiv \sum f(d_r)$, ${}_0f = {}_0f' = f$. The notation in 2.1 is as in any of the papers cited in § 5; the $\sum$'s refer to all numbers of the respective sets δ_r, d_r. To prove 2.1, it is sufficient to translate the definitions of ${}_rf(n)$, ${}_rf'(n)$ into the notation of IF; namely

$$ {}_rf = u_{r-1}f, \qquad {}_rf' = (u-1)_{r-1}f', $$

for all integers $r > 0$. As in IF, the equations 2.1 then define ${}_rf$, ${}_rf'$ for all real or complex numbers r.

Since u is regular in IF it has a unique inverse, $u^{-1} = \mu$, (Möbius' or Mertens function). To develop the properties of ${}_rf$, ${}_rf'$ ad libitum it is sufficient merely to manipulate the right-hand members of 2.1 according to the rules of elementary algebra, not dividing by f if f is irregular (f is irregular in IF if and only if $f(1) = 0$), as IF is an instance of an abstract irregular field. If at any stage it be required to interpret the results in

* *American Journal of Mathematics*, Vol. 52 (1930), pp. 293-304.

† *Transactions of the American Mathematical Society.* Further references are collected in Section 5.

terms of ${}_rf$, ${}_rf'$, this may be done (for all r) by the substitutions, equivalent to 2.1),

2.2 $$f = {}_rf/u^r = \mu^r{}_rf = {}_rf'/(u-1)^r.$$

3. A few examples will suffice to show how the properties of ${}_rf$, ${}_rf'$ follow by elementary manipulations of 2.1. First may be noted the alternative definitions for all integers r: u^rf is the r th so-called numerical integral, and $(u-1)^rf$ the r-th proper numerical integral of f. If ${}_rf$, ${}_rf'$ be regarded as having been obtained by this process, instead of by summations over δ_r, d_r, we shall write f_r, f_r'. Thus

3.1 $${}_rf = f_r = u^rf, \qquad {}_rf' = f_r' = (u-1)^rf,$$

and as in 2.1 we take for all real or complex numbers r as definitions of f_r, f_r' the elements u^rf, $(u-1)^rf$ of IF. In passing it may be repeated (as pointed out in the papers cited in Section 5) that the numerical integral uf of Bugaieff is in no way distinguished from any other simple (binary) product in IF, and similarly for the Liouville-Dedekind inversion; namely, if $uf = F$, then $f = F/u = \mu F$, which is on the same footing as $gf = h$, $f = h/g = g^{-1}h$, where g is *any* regular element of IF and g^{-1} is its reciprocal in IF .

Let r be an integer. If in IF a product gf is factorable(see Section 5, references), and one of its factors, say g, is factorable, then f is factorable. Since u^r is factorable, a necessary and sufficient condition that $f_r(={}_rf)$ be factorable is that f be factorable. Hence if for any particular integer r, f_r is factorable, it is factorable for all integers r.

As a second example, since $u^{r+s} = u^ru^s = u^su^r$, and similarly for $u-1$, we see from 2.1 that

$${}_{r+s}h = h_{r+s} = {}_rh_s = {}_sh_r \qquad (h = f \text{ or } f').$$

Again, if $F = uf$, and it be required to express the inversion formula $f = F/u$ in terms of functions F_r, we write $u-1 = v$ for convenience, so that

$$f = F/u = F/(1+v) = F(1-v+v^2-\cdot\cdot\cdot) = \sum_{j=0} (-1)^jF_j.$$

The last series obviously terminates when f has the argument n (an arbitrary finite integer), as $F_j(n) = 0$ for $j =$ a certain finite integer, from the definition of d_j.

The connection between ${}_rf$, ${}_rf'$ is obvious from 2.1. For simplicity let r be an integer ≥ 0. Then, $(v \equiv u-1)$,

$$(-1)^r{}_rf' = (1-u)^r f = \sum_{j=0}^{r} (-1)^j u^j f = \Sigma\, (-1)^j \binom{r}{j}\, {}_jf;$$

$$ {}_rf = u^r f = (1+v)^r f = \sum_{j=0}^{r} \binom{r}{j} v^j f = \Sigma \binom{r}{j}\, {}_jf'.$$

As another identity suggested by 2. 1 we have

$$f + (-1)^r{}_rf = f\,[1-(1-u)^r] = \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} u^{j-1}uf,$$

and hence, if $F = uf$,

$$f + (-1)^r{}_rf = \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} F_{j-1}.$$

An alternative form has F_i instead of ${}_iF$, since

$$u^{j-1}uf = u^{j-1}F = {}_{j-1}F = F_{j-1}.$$

The binomial theorem for a positive integral exponent r has a variety of interpretations. To take only one, $u - v = 1$; hence

$$f = (u-v)^r f = \sum_{j=0}^{r} (-1)^j \binom{r}{j} u^j v^{r-j} f,$$

and we write $u^j v^{r-j} f$ in either of the forms (among others) $u^j(v^{r-j}f)$, $v^{r-j}(u^j f)$, and hence as any one of

$$u^j{}_{r-j}f', \quad u^j f'_{r-j}, \quad v^{r-j}{}_jf, \quad v^{r-j}f_j,$$

that is as any one of the eight

$$({}_{r-j}f')_j, \quad {}_j({}_{r-j}f'), \quad (f'_{r-j})_j, \quad {}_j(f'_{r-j}),$$
$$({}_jf')_{r-j}, \quad {}_{r-j}({}_jf)', \quad (f_j)'_{r-j}, \quad {}_{r-j}(f_j)'.$$

Hence f has (among many more) the eight binomial expressions

$$f = \sum_{j=0}^{r} (-1)^j \binom{r}{j} F^{(j)},$$

where $F^{(j)}$ denotes any one of the above functions. The forms of the $F^{(j)}(n)$ are immediately written out from the notation: a prefix s denotes a sum over s-th divisors, a suffix s an s-iterated numerical integration, an accent indicates proper, lack of an accent all divisors of the set, and operations within the ( ) are to be performed before those outside. This example illustrates the commutative, associative and distributive laws in IF, which are abstractly identical with the like in a field.

Finally, by the simple properties of what were called functional powers in IF, all formulas for f_r, $_rf'$, $\cdots$ can be transposed from sum to product forms without computations. In the papers cited in Section 5 the extension to functions of elements in any commutative semigroup having a unique composition theorem (as for example in any finite abelian group) was noted and developed.

4. The explicit forms of $_rf$, $_rf'$ (and hence of f_r, f_r') are written down from the generator (see papers 5.3 and that of 1915) of f. If f is factorable, and $F(x, z)$ is its generator, the generator of $_rf$ is $F(x, z)$ [illegible]. If $h_a(x)$ is the coefficient of z^a in the formal development of this as a power series in z, the explicit form of $f_r(n)$ is

$$f_r(n) = \prod h_a(p), \tag{4.1}$$

where $n = \prod p^a$ is the prime decomposition of n. If f is not factorable, 4.1 is the essentially simplest form of $_rf$; the form corresponding to 4.1 is obtained as in the papers cited. To find the explicit form of $_rf'$ we apply to the explicit form of f_r the theorem $(-1)^r {}_rf' = \sum (-1)^j \binom{r}{j} {}_jf$.

The simple, general method for obtaining the generator of a given f from its arithmetical definition was given in the previous papers, and in that of 1915 a list of the generators of practically all of the factorable numerical functions in the literature, with many more, was written out. Some of the simplest are: the generator of u is $1/(1-z)$; that of Euler's ϕ is $(1-z)/(1-xz)$; that of σ_k (sum of k-th powers of all divisors) is $1/(1-z)(1-x^kz)$; that of μ is $1-z$; that of the unit function η in IF is 1; that of the zero function ω in IF is 0.

The definitions of $_rf$, $_rf'$ connect an arbitrary numerical function with the very special ones u, $u-1$. If for the latter any elements a, b, $\cdots$ in IF be substituted, the treatment is precisely the same. From the standpoint of IF, which is the irregular field of *all* numerical functions, there is no particular reason for distinguishing u, $u-1$, $\cdots$ from any other elements of IF.

5. A concise account of the theory is given in *Algebraic Arithmetic* (*American Mathematical Society Colloquium Publications No. 7,* 1927), and an adequate summary in *The Journal of the Indian Mathematical Society*, Vol. 17 (1928). It was however developed first in *University of Washington Publications in Science,* Vol. 1, No. 1 (1915), pp. 1-44, where there are numerous applications. For the specific parts used in this paper we may refer to

5.1 *Tohoku Mathematical Journal*, Vol. 17 (1920), pp. 221-231.

5.2 *Bulletin of the American Mathematical Society*, Vol. 28 (1922), pp. 111-122.

5.3 *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 135-154.

5.4 *L'Enseignement Mathématique*, t. 23 (1923), pp. 305-308.

The last contains everything necessary for the present paper except the theory of generators, which will be found in 5. 3.

It may be pointed out that the generalizations to any systems having unique-decomposition theorems are not true generalizations in the sense of modern algebra; they are merely other solutions of a certain set of postulates. To obtain true generalizations, the set of postulates must be modified in the direction of weakness.

For the theory of irregular fields, see either *Algebraic Arithmetic* above, or

5.5 *Annals of Mathematics*, Vol. 27 (1926), pp. 511-536.

THE UNIFORM APPROXIMATION OF A SEQUENCE OF INTEGRALS.

By R. L. Jeffery.

Let $g_1, g_2, \cdots, g_p$ be p functions of x summable on (a, b). It has been shown by Lebesgue.* that there exists a finite sub-division α of (a, b), and on each interval of this sub-division with length α_i a point ξ_i of x such that

$$(1) \qquad \left| \int_a^b g_h dx - \sum_i g_h(\xi_i)\alpha_i \right| < \epsilon \quad (h = 1, 2, \cdots, p).$$

Lebesgue remarks † that such an approximation can evidently be made for an infinite sequence of functions. Our investigation seems to show that this remark needs some qualification, even when the sequence is bounded in x and n. In Lebesgue's results, for a given ϵ, both α_i and ξ_i are fixed. We show that it is always possible to find ξ_i on each interval with length α_i so that (1) holds, for any α with norm sufficiently small. Conditions are also determined under which the approximation can be made, independent of n, for an infinite sequence of functions. The paper concludes with some applications to functions of two variables.

It will make for brevity if we agree once for all that α shall represent a finite sub-division of (a, b), and β a sub-set of the intervals of α; α_i and ξ_i, β_j and ξ_j shall denote the length of and a point on an interval of α and β respectively.

Theorem I. *Let $g_1, g_2, \cdots, g_p$ be p functions of x summable on the measurable set E contained on (a, b). Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to choose from α with norm sufficiently small a sub-set of intervals β, and on each interval of β a point ξ_j of E, such that*

$$\left| \int_E g_h dx - \sum_j g_h(\xi_j)\beta_j \right| < \epsilon \quad (h = 1, 2, \cdots, p),$$

and at the same time $| mE - m\beta | < \epsilon'$. If E is the interval (a, b) then the approximating sums can be taken over all the intervals of α.

We shall first establish the theorem for two bounded functions, g_1, and g_2. Let l and L be the bounds of the set defined by g_h $(h = 1, 2)$. Divide (l, L)

* *Annales de Toulouse*, (3), Vol. I, p. 33.
† *loc. cit.*, p. 34.

into n equal parts each of length η, where n is large enough to insure the following results:

(a) $$\eta(b-a) < \epsilon/3.$$

(b) $$\left| \int_E g_h dx - \sum_i \{l+(i-1)\eta\} me_i^h \right| < \epsilon/3 \qquad (h=1,2),$$

where e_i^h has the usual significance. Let e_{ij} be the set of points common to e_i^1 and e_j^2. Let t be greater than zero but otherwise arbitrary. Following the method used by Lebesgue, we can associate with each set e_{ij} a finite set of non-overlapping intervals a_{ij} with the following properties:

(1) Each interval of a_{ij} contains at least one point of e_{ij}, and the measure of the part of e_{ij} not on a_{ij} is less than t.

(2) The measure of the part of e_{ij} on a_{ij} differs from the measure of a_{ij} by not more than t.

(3) There are no points common to any of the n^2 sets a_{ij} $(i, j = 1, 2, \cdots, n)$.

(4) The measure of a_{ij} and e_{ij} differ by not more than t. This follows from (1) and (2).

If U is the larger of the two numbers $|L|$ and $|l|$, and

$$S_1^h = \sum_i \{l+(i-1)\eta\} me_i^h \qquad (h=1,2),$$
$$S_2^1 = \sum_{i=1}^n \{l+(i-1)\eta\} \sum_{j=1}^n ma_{ij},$$
$$S_2^2 = \sum_{j=1}^n \{l+(i-1)\eta\} \sum_{i=1}^n ma_{ij},$$

then from (4) and the fact that

$$e_i^1 = \sum_{j=1}^n e_{ij}, \quad \text{and} \quad e_j^2 = \sum_{i=1}^n e_{ij},$$

we get

(5) $$|S_1^h - S_2^h| < n^2 U t \qquad (h=1,2).$$

Let M be the largest number of intervals in any of the n^2 sets a_{ij}. Let $\delta > 0$ be less than the length of the smallest interval in any of these n^2 sets, but otherwise arbitrary. Let α be any finite sub-division of (a, b) for which $\alpha_i < \delta$, and let u_{ij} be the intervals of α which have a part in common with a_{ij}, and which contain points of e_{ij}. Then mu_{ij} can neither be greater than ma_{ij} by more than $2M\delta$, nor, on account of (2), can it be less than ma_{ij} by more than t. Hence in any case we have

(6) $$| mu_{ij} - ma_{ij} | < 2M\delta + t.$$

Consequently, if we write

$$S_3^1 = \sum_{i=1}^{n} \{l + (i-1)\eta\} \sum_{j=1}^{n} mu_{ij},$$

and

$$S_3^2 = \sum_{j=1}^{n} \{l + (j-1)\eta\} \sum_{i=1}^{n} mu_{ij},$$

it is easily seen that

(7) $$| S_2^h - S_3^h | < n^2 U (2M\delta + t) \qquad (h = 1, 2).$$

If now for all combinations of i, j we fix in each interval of the set u_{ij} a point of e_{ij}, and in the expansion of S_3^1 and S_3^2 we replace $l + (i-1)\eta$ and $l + (j-1)\eta$ by the value of the corresponding function at the point so fixed, we arrive at

$$\sum g_h(\xi_j)\beta_j \qquad (h = 1, 2),$$

where β includes all the intervals of the n^2 sets u_{ij}. From (4) and (6) we get

(8) $$| mE - m\beta | < n^2 t + n^2 (2M\delta + t) = \lambda.$$

It is also easy to verify that

(9) $$| S_3^h - S_4^h | < \eta (b - a) = \rho.$$

Combining (5), (7), and (9), we get

(10) $$| S_1^h - S_4^h | < U\lambda + \rho \qquad (h = 1, 2).$$

Then from (a), (b), and (10), we have

$$\left| \int_E g_h dx - \sum_j g_h(\xi_j)\beta_j \right| < \epsilon \qquad (h = 1, 2),$$

and at the same time the right hand side of (8) is less than ϵ', provided first t, and then δ has been fixed sufficiently small.

In the case of p bounded functions the procedure would be the same, except that in the place of the n^2 sets e_{ij} there would be n^p sets similarly defined.

Now suppose the functions $g_1, \cdots, g_p$ unbounded but summable on E. For N a positive integer let E_N be the part of E at which $-N \leqq g_h \leqq N$ $(h = 1, 2, \cdots, p)$. It is clear that E_N is measurable, and that the limit as N becomes infinite of E_N is all the points of E. Hence for N sufficiently large, we have

$$(1) \qquad | mE - mE_N | < \epsilon'/2,$$

and at the same time

$$(2) \qquad \left| \int_E g_h dx - \int_{E_N} g_h dx \right| < \epsilon/2 \qquad (h = 1, 2, \cdots, p).$$

The functions g_h are bounded over E_N, and we have seen that if the norm of α is sufficiently small, then from α we can pick a sub-set of intervals β and points ξ_j of E_N, for which

$$(3) \qquad \left| \int_{E_N} g_h dx - \sum_j g_h(\xi_j)\beta_j \right| < \epsilon'/2 \quad (h = 1, 2, \cdots, p),$$

and at the same time

$$(4) \qquad | ME_N - m\beta | < \epsilon'/2.$$

Combining (2) with (3), and (1) with (4), we get the first part of the theorem.

Now let E be the interval (a, b). It is evident that there exists $\delta > 0$ such that if e is any measurable set on (a, b) with $me < \delta$, then

$$\int_e \{ | g_1 | - \cdots - | g_p | \} dx < \epsilon/3.$$

Also, on account of the first part of the theorem, if the norm of α is sufficiently small, we have β a sub-set of α such that

$$(1) \qquad \left| \int_a^b g_h dx - \sum_j 2 g_h(\xi_j)\beta_j \right| < \epsilon/2 \quad (h = 1, 2, \cdots, p),$$

and at the same time $| b - a - m\beta | < \delta$. If $\gamma = \alpha - \beta$, then $m\gamma < \delta$, and it is easy to show that there exists ξ_k on each interval of γ with length γ_k such that

$$\sum_k \{ | g_1(\xi_k) | + \cdots + | g_p(\xi_k) | \} \gamma_k \leqq \int_\gamma \{ | g_1 | + \cdots + | g_p | \} dx.$$

Hence

$$\left| \sum_k g_h(\xi_k)\gamma_k \right| < \epsilon/2 \qquad (h = 1, 2, \cdots, p),$$

and this with (1) gives

$$\left| \int_a^b g_h dx - \sum_i g_h(\xi_i)\alpha_i \right/ < \epsilon \quad (h = 1, 2, \cdots, p).$$

None of the foregoing results hold, in general, for an infinite sequence of functions. We give three examples which throw light on this point from various angles.

Let $g_n = n(1 - nx)$ on $0 < x \leqq 1/n$, and $g_n = 0$ elsewhere on $(0, 1)$. Then for every n we have

$$\int_0^1 g_n dx = 1/2.$$

It is easily seen, however, that for any sub-division α whatever of $(0, 1)$, and for any choice of ξ_i, we have

$$\sum_i g_n(\xi_i) = 0$$

for all n sufficiently large.

In this example the sequence converges for each x, but it is not bounded in x and n, nor is it integrable.*

Let $g_{nk} = \log n$ on $k - 1/n \leqq x \leqq k/n$, and $g_{nk} = 0$ elsewhere on $(0, 1)$ $(k = 1, 2, \cdots, n,\ n = 1, 2, \cdots)$. In this case

$$\int_0^1 g_{nk} dx = \log n/n \leqq 1/e.$$

But if M is arbitrarily large, then for any sub-division whatever of $(0, 1)$ and any choice of ξ_i, it is possible to find g_{nk} so that

$$\sum_i g_{nk}(\xi_i)\alpha_i > M.$$

In this example the sequence is neither bounded in x and n, nor is it convergent. We conclude with an example of a sequence which is bounded in x and n.

Divide the interval $(0, 1)$ into n parts and bisect each of these parts. Let g_{nk} be zero at the irrational points of one-half of each sub-division, and unity at the irrational points of the other half. Doing this in all possible ways gives rise to 2^n functions g_{nk} $(k = 1, 2, \cdots, 2^n)$ on the irrational points of $(0, 1)$. For x rational on $(0, 1)$ let $g_{nk} = 0$ for all n and $k = 1, 2, \cdots, 2^n$. We thus arrive at an infinite sequence of functions bounded in x and n, and such that

$$\int_0^1 g_{nk} dx = \tfrac{1}{2} \qquad (k = 1, 2, \cdots, 2^n,\ n = 1, 2, \cdots).$$

But it is not difficult to show that if $(\xi_1, \xi_2, \cdots, \xi_l)$ are any l irrational numbers on $(0, 1)$ then there exists at least one function g_{nk} such that $g_{nk}(\xi_i) = 0$ $(i = 1, 2, \cdots, l)$. From this it would readily follow that for any sub-division α of $(0, 1)$ and any choice of ξ_i there exists at least one function g_{nk} for which

$$\sum g_n(\xi_i)\alpha_i = 0.$$

We now prove

* Hobson, *Functions of a Real Variable*, second ed., Vol. 2, § 201.

THEOREM II. *Let $g_1, g_2, \cdots$ be a sequence of functions, measurable on (a, b), bounded in x and n, and such that as n increases $g_n(x)$ converges to $g(x)$. Then there exists $\delta > 0$ such that if $\alpha_i < \delta$, it is possible to choose ξ_i so that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

For an arbitrary $\eta > 0$ let $S(\eta, l)$ be the set of x-points for which $| g_n(x) - g(x) | < \eta$ for $n \geqq l$. It can be shown that this set is measurable, and that as l increases the measure of $S(\eta, l)$ approaches $b - a$. This, and the fact that g_n is bounded in x and n allows us to fix $l = l'$ so that,

$$(1) \qquad \left| \int_a^b g_n dx - \int_{S(\eta, l')} g_n dx \right| < \eta \qquad (n = 1, 2, \cdots),$$

and at the same time

$$(2) \qquad b - a - mS(\eta, l') < \eta.$$

Also, we can find $n' \geqq l'$ such that,

$$(3) \qquad \left| \int_{S(\eta, l')} g_n dx - \int_{S(\eta, l')} g_{n'} dx \right| < \eta \qquad (n \geqq n').$$

Then, on account of Theorem I, from α with norm sufficiently small we can choose a sub-set of intervals β, and on each interval of this set a point ξ_j of $S(\eta, l)$ such that both the following inequalities hold:

$$(4) \qquad \left| \int_{S(\eta, l)} g_n dx - \sum_j g_n(\xi_j)\beta_j \right| < \eta \quad (n = 1, 2, \cdots, n').$$

$$(5) \qquad | mS(\eta, l) - m\beta | < \eta.$$

Since $n' \geqq l'$ and ξ_j belongs to $S(\eta, l)$ we have,

$$(6) \qquad \left| \sum_j g_{n'}(\xi_j)\beta_j - \sum_j g_n(\xi_j)\beta_j \right| < \eta(b - a) \qquad (n \geqq n').$$

Since (4) holds for $n = n'$, then by taking into consideration (3) and (6), we have

$$(7) \qquad \left| \int_{S(\eta, l')} g_n dx + \sum_j g_n(\xi_j)\beta_j \right| < 2\eta + \eta(b - a) \qquad (n = 1, 2, \cdots),$$

and this with (1) gives

$$(8) \qquad \left| \int_{S(\eta, l')} g_n dx - \sum_j g_n(\xi_j)\beta_j \right| < 3\eta + \eta(b - a) \qquad (n = 1, 2, \cdots).$$

Let $\gamma = \alpha - \beta$, and let M be the least upper bound of $| g_n |$ for all n and x. Then, on account of (2) and (5), for any choice of ξ_i we have

$$| \sum_k g_n(\xi_k)\gamma_k | < 2\eta M \qquad (n = 1, 2, \cdots),$$

and since η is arbitrary, this with (8) gives the desired result.

If the sequence $g_1, g_2, \cdots$ is not bounded in x and n, but is such that the sequence of integrals is equi-convergent,* then g is summable,† and the sequence of functions is completely integrable.‡ With these facts established, it is not difficult to obtain inequalities (1) and (2) above. We can then proceed as above to inequality (8), thus getting

Corollary I. *Let $g_1, g_2, \cdots$ be a sequence of functions, summable on (a, b), convergent to a summable function, and such that the sequence of integrals is equi-convergent. Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to choose from α with norm sufficiently small, a sub-set of intervals β and on each interval of β a point ξ_j such that*

$$| \int_a^b g_n dx - \sum_j g_n(\xi_j)\beta_j | < \epsilon \qquad (n = 1, 2, \cdots),$$

and at the same time $| b - a - m\beta | < \epsilon'$.

The question now arises as to whether or not the conditions of Corollary I are sufficient to permit the approximating sums to be taken over all the intervals of any set α with norm sufficiently small. An example answers this question in the negative. This example also brings to light a class of functions, satisfying the conditions of Corollary I, for which there exists no $\delta > 0$ such that for every α with $\alpha_i < \delta$ it is possible to find ξ_i for which

$$(1) \qquad | \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i | < \epsilon,$$

but for every $\delta > 0$ it is possible to find at least one α with $\alpha_i < \delta$, and a choice of ξ_i so that (1) holds independent of n.

Divide the interval $[1/(k+1), 1/k]$ $(k = 2, 3, \cdots)$ at the points $t_{1k}, t_{2k}, \cdots$ where $t_{1k} = 1/k$, and where $t_{(n-1)k} - t_{nk} = 1/k^5$ so long as the point t_{nk} falls to the right of or on $1/(k+1) + 1/k^5$. If by making $t_{(n-1)k} - t_{nk} = 1/k^5$ the point t_{nk} would fall to the left of $1/(k+1) + 1/k^5$ then place the

* Hobson, *Functions of a Real Variable*, second ed., Vol. 2, § 208.

† de la Vallee Poussin, *Transactions of the American Mathematical Society*, Vol. 15, Theorem I.

‡ Hobson, *loc. cit.*, § 209.

point t_{nk} half way between $t_{(n-1)k}$ and $1/(k+1)$. Let $g_n = k^3$ on $t_{nk}, t_{(n-1)k})$ $(k = 2, 3, \cdots)$, and $g_n = 0$ elsewhere on $(0, 1)$.

It is easily shown that g_n converges to a summable function g, that for all n

$$\int_0^1 g_n dx < \sum_k (1/k^2),$$

and that the sequence of integrals is equi-convergent. But if M is arbitrarily large and δ is any positive number, it is possible to find a sub-division α of $(0, 1)$ with norm less than δ, and a value of n such that

$$\sum g_n(\xi_i)\alpha_i > M. \tag{1}$$

Fix $k = k_1$ so that

$$k^3/k_1(k_1+1) > M, \text{ and } 1/k_1(k_1+1) < \delta.$$

Take any sub-division α of $(0, 1)$ with norm less than δ and such that one of its intervals is $(1/k_1 + 1, 1/k_1)$. Denote this interval of the set α by α_j, and fix ξ_j any point on α_j. Then for some n we have

$$g_n(\xi_j)\alpha_j \geqq k_1{}^3\alpha_j \geqq k_1{}^3/k_1(k_1+1) \geqq M,$$

and inequality (1) follows from this.

Nevertheless, for ϵ and δ any two positive numbers, it is possible to find a sub-division of $(0, 1)$ with norm less than δ and points ξ_i such that for this particular sub-division and choice of ξ_i we have

$$\left| \int_0^1 g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon.$$

It is easily verified that

$$\int_0^{1/k'} g_n dx \leqq \sum_{k'}^{\infty} (k^3/k^5) \leqq \sum_{k'}^{\infty} (1/k^2) < \epsilon/2 \tag{1}$$

for all n, provided k' is sufficiently large.

On $(1/k', 1)$ g_n is bounded in x and n, and consequently satisfies the conditions of Theorem II. Hence there exists $\delta' < \delta$ such that for any sub-division α' of $(1/k', 1)$ with norm less than δ' it is possible to find ξ_i for which

$$\left| \int_{1/k'}^1 g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon/2 \qquad (n = 1, 2, \cdots). \tag{2}$$

To any such sub-division of $(1/k', 1)$ adjoin the interval $(0, 1/k')$, thus getting a sub-division α of $(0, 1)$. If on this adjoined interval of α we fix

$\xi_1 = 0$, the desired result readily follows from (1), (2), and the fact that $g_n(0) = 0$ for all n.

We now state

THEOREM III. *Let the sequence of functions $g_1, g_2, \cdots$ be summable on (a, b), and converge uniformly to the summable function g. Then on each interval of any finite sub-division α of (a, b) with norm sufficiently small, it is possible to find ξ_i so that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

This readily follows from the uniform convergence of the sequence, and the fact that Theorem I applies to any finite number of functions of the sequence.

A study of the sequence $g_n = \log n/n$ for $0 \leqq x \leqq 1/n$, and $g_n = 0$ elsewhere on $(0, 1)$ shows that the uniform convergence requirement of Theorem III is not necessary.

THEOREM IV. *Let $g_1, g_2, \cdots$ be a sequence of functions summable on (a, b) which converges to the summable function g. Let $\bar{x}$ be the points of non-uniform convergence of the sequence. If $\bar{x}$ has zero content, if g_n is bounded on $\bar{x}$ and n, and if the sequence of integrals is equi-convergent, then there exists α and ξ_i such that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

Under the conditions, of the theorem the set $\bar{x}$ can be put in a finite set of intervals β such that if ξ_i is a point of $\bar{x}$ then

$$\left| \sum_j g_n(\xi_j)\beta_j \right| < \epsilon/3, \tag{1}$$

and at the same time

$$\left| \int_\beta g_n dx \right| < \epsilon/3. \tag{2}$$

If (a_j, b_j) is one of the p closed intervals of the set complementary to β, then on (a_j, b_j) the sequence converges uniformly to g. Hence by Theorem III, for any sub-division α^j of (a_j, b_j) with norm sufficiently small, we have

$$\left| \int_{a_j}^{b_j} g_n dx - \sum_i g_n(\xi_i^j)\alpha_i^j \right| < \epsilon/3p \qquad (j = 1, 2, \cdots, p). \tag{3}$$

We can now combine (1), (2), and (3), to obtain the desired result.

Applications to a Function of Two Variables. Let $f(x,y)$ $(a \leqq x \leqq b, c \leqq y \leqq d)$ be bounded, continuous in y for each x, and measurable in x for each y. For δ and η, two arbitrary positive numbers, let $G_{\delta\eta}$ be the set of x-points for which $| f(x,y') - f(x,y'') | < \eta$ for every pair of values y', y'' for which $| \bar{y}' - \bar{y}'' | < \delta$. The set $G_{\delta\eta}$ is measurable.*

To show this let $\bar{y}$ be an everywhere dense countable set on (a,b), and let $\bar{G}_{\delta\eta}$ be the set of x-points for which $| f(x,\bar{y}') - f(x,\bar{y}'') | < \eta$ for any two values $\bar{y}'$, $\bar{y}''$ of $\bar{y}$ for which $| \bar{y}' - \bar{y}'' | < \eta$. That $\bar{G}_{\delta\eta}$ is measurable readily follows from the countability of $\bar{y}$ and the fact that for a fixed y, $f(x,y)$ is measurable. The continuity of $f(x,y)$ in y for a fixed x can then be used to show that $G_{\delta\eta}$ and $\bar{G}_{\delta\eta}$ are identical. As δ approaches zero, $G_{\delta\eta}$ tends to include all the points of (a,b); this and the boundedness of $f(x,y)$ give, for δ sufficiently small, the following two inequalities:

$$(1) \qquad \left| \int_a^b f(x,y)\,dx - \int_{G_{\delta\eta}} f(x,y)\,dx \right| < \eta$$

$$(2) \qquad b - a - mG_{\delta\eta} < \eta.$$

The function $F(y) = \int_b^a f(x,y)\,dx$ is continuous.†

Hence there exists $\delta' < \delta$ such that

$$(3) \qquad | F(y') - F(y'') | < \eta$$

for any two values y', y'' of y which satisfy $| y' - y'' | < \delta'$. Let (c,d) be divided at the points $y_0 = c$, y_1, $y_2, \cdots, y_p = d$ where $0 < y_k - y_{k-1} < \delta$ $(k = 1, 2, \cdots, p)$. Applying Theorem I to the function $f(x,y_k)$ $(k = 0, 1, \cdots, p)$ and x on $G_{\delta\eta}$, we can select from a finite sub-division of (a,b) with norm sufficiently small, a sub-set β, and ξ_j a point of $G_{\delta\eta}$ such that

$$(4) \qquad \left| \int_{G_{\delta\eta}} f(x,y_k)\,dx - \sum_j f(\xi_j, y_k)\beta_j \right| < \eta \qquad (k = 1, 2, \cdots, p),$$

and at the same time $| m\beta - mG_{\delta\eta} | < \eta$. The above inequalities, together with the boundedness of $f(x,y)$ and the fact that ξ_j belongs to $G_{\delta\eta}$ readily gives the following theorem:

Let $f(x,y)$ satisfy the conditions stated above. Then for any finite sub-

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* Concerning the non-measurability of sets defined in a manner very similar to that of $G_{\delta\eta}$, see Hobson, *loc. cit.*, third ed., Vol. 1, p. 727.

† W. H. Young, *Monatshefte für Mathematik und Physik*, Vol. 21 (1910), pp. 126-127.

division α of (a, b) with norm sufficiently small, we have, for a proper choice of ξ_i,

$$\left| \int_a^b f(x, y)\, dx - \sum_i f(\xi_i, y)\alpha_i \right| < \epsilon.$$

If $f(x, y)$ is unbounded but summable in x for each y, and such that for e any measurable part of (a, b) with me sufficiently small, $\left| \int_e f(x, y)\, dx \right|$ is arbitrarily small independent of y, we shall say that $F(y)$ is equi-convergent. The continuity of $F(y)$ would then follow from Vitali's theorem,* and this gives inequality (2) and (3) above, while (1) would follow from the equi-convergence of $F(y)$. We could then proceed to (4), thus obtaining the following theorem:

Let $f(x, y)$ $(a \leqq x \leqq b,\ c \leqq y \leqq d)$ be continuous in y for each x, summable in x for each y, and such that $F(y)$ is equi-convergent. Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to select from α with norm sufficiently small a sub-set β, and on each interval of β a point ξ_j for which

$$\left| \int_a^b f(x, y)\, dx - \sum_j f(\xi_j, y)\beta_j \right| < \epsilon,$$

and at the same time $b - a - m\beta < \epsilon'$.

If it is known that $f(x, y)$ is continuous in y at y_0 only, the other conditions in either the first or second case above remaining unchanged, it does not follow that the set of x-points for which $| f(x, y_0) - f(x, y) | < \eta$ when $| y_0 - y | < \delta$ is measurable.† But if we assume further that $f(x, y)$ is such that this set is measurable for each pair δ and η, it is then possible to establish the first theorem above for some interval about y_0 if $f(x, y)$ is bounded, and the second theorem in case $f(x, y)$ is not bounded but is such that $F(y)$ is equi-convergent.

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. 23 (1907), p. 137.

† Hobson, *loc. cit.*, third ed., Vol. 1, p. 727.

NON-INVOLUTORIAL BIRATIONAL TRANSFORMATIONS BELONGING TO A SPECIAL LINEAR LINE COMPLEX.

By H. A. Davis.

Introduction. The most general non-involutorial Cremona space transformation which belongs to a special linear line complex has been studied synthetically by M. Pieri.* He finds the general transformation to be of order $n+2n'-3$. In the present paper a transformation of the same order is discussed which has properties quite different from the one studied by Pieri.

1. *Synthetic Discussion.* Denote by T and Γ respectively the non-involutorial transformation and the special linear complex to which it belongs. Consider two superimposed ordinary spaces Σ and Σ' such that $\Sigma \sim \Sigma'$ under T. To each point P of Σ (or P' of Σ') corresponds the unique Γ-ray PP'. These two representations of the lines of Γ upon the points of Σ and of Σ' shall be designated by M and M' respectively. It is clear that $T=M^{-1}M'$, and $T^{-1}=M'^{-1}M$.†

A Γ-pencil (A, α) with vertex A on the directrix d of Γ and plane α not containing d corresponds in M to a curve $\Delta_{n-1}: A^{n-2}$, and in M' to $\Delta'_{n'-1}: A^{n'-1}$, hence, under T, $\Delta_{n-1}: A^{n-2} \sim \Delta'_{n'-1}: A^{n'-2}$. A Γ-pencil (B, β) with vertex B not on d and plane $\beta: d$ corresponds in M to a conic $\Delta_2: B$, and in M' to $\Delta_2': B$, hence, under T, $\Delta_2: B \sim \Delta_2': B$. Two pencils, one of each type, with a line in common form a composite Γ-regulus, hence a Γ-regulus R corresponds in M to a curve Δ_{n+1} of genus 0, and in M' to a $\Delta'_{n'+1}$ of genus 0, and, under T, $\Delta_{n+1} \sim \Delta'_{n'+1}$.

The surface F, image in M of a linear Γ-congruence Q_1 with directrix g is cut by a plane through d in d^{n-2} and a Δ_2; and by a plane through g in g and a Δ_{n-1}. Hence, Q_1 corresponds in M to a surface $F_n: d^{n-2}g$, and in M' to $F'_{n'}: d^{n'-2}g$, hence, under T, $F_n: d^{n-2}g \sim F'_{n'}: d^{n'-2}g$.

2. *The Equations of T.* There are in all $\infty^4 \mid F_n \mid$ and $\infty^4 \mid F'_{n'} \mid$, associated with the ∞^4 linear Γ-congruences. But a single pencil $\mid F_n \mid$, together with the corresponding pencil $\mid F'_{n'} \mid$, is sufficient to determine T.

The base of a pencil $\mid Q_1 \mid$ of linear Γ-congruences is a Γ-regulus R on

* "Sulle trasformazioni cirazionali dello spazio inerenti a un complesso lineare speciale, *Circolo Matematico di Palermo*, Vol. 6 (1892), pp. 234-244.

† *loc. cit.*

a quadric H. The directrices of the congruences $|Q_1|$ form the regulus R' associated with R on H. The directrix d of Γ belongs to R'. Associated with $|Q_1|$ are the pencils $|F_n|$ and $|F'_{n'}|$.

Through a generic point P of space passes a single surface $F_n: d^{n-2}g$, where g is the directrix of the Q_1 associated with F_n. The corresponding surface is $F'_{n'}: d^{n'-2}g$. The unique transversal of d and g through P meets $F'_{n'}$ in one residual point P', image of P in T.

Denote by $x_1 = x_2 = 0$, $x_1x_3 - x_2x_4 = 0$, $x_1/x_2 + x_4/x_3 = k$, and $x_1/x_4 = x_2/x_3 = m$, the directrix d, the quadric H, the regulus R, and the regulus R' respectively.

If we select the pencils $|F_n|$ and $|F'_{n'}|$ so that neither contains d as a tact locus, we obtain the T discussed by Pieri. We shall consider the case in which d is a tact locus for both pencils. They may then be written

$$(1) \qquad |F_n| \equiv U(x_1 - \rho x_4) + V(x_2 - \rho x_3) = 0,$$

$$(2) \qquad |F'_{n'}| \equiv U'(x_1 - \rho x_4) + V'(x_2 - \rho x_3) = 0,$$

where

$$U = \sum_{i=2}^{n} u_i x_1^{n-i} x_2^{i-2}, \quad V = \sum_{i=2}^{n} v_i x_1^{n-i} x_2^{i-2}, \quad U' = \sum_{i=2}^{n'} u_i' x_1^{n'-i} x_2^{i-2},$$

$$V' = \sum_{i=2}^{n'} v_i' x_1^{n'-i} x_2^{i-2}, \quad u_i = \sum_{k=1}^{4} a_{ik} x_k, \quad v_i = \sum_{k=1}^{4} b_{ik} x_k,$$

$$u_i' = \sum_{k=1}^{4} a'_{ik} x_k, \quad v_i' = \sum_{k=1}^{4} b'_{ik} x_k.$$

Each surface of the pencils $|F_n|$ and $|F'_{n'}|$ contains a line $g \equiv x_1 - \rho x_4 = 0$, $x_2 - \rho x_3 = 0$, of R'. Through a point $P(y)$ of space passes a single surface of $|F_n|$ for which

$$(3) \qquad \rho = [y_1 U(y) + y_2 V(y)]/[y_4 U(y) + y_3 V(y)].$$

The transversal t of d and g through P meets d and g in points whose coördinates are $(0, 0, U, -V)$ and $(\rho y_1, \rho y_2, y_2, y_1)$ respectively. Any point of t has coördinates

$$(4) \qquad x_1 = \mu\rho y_1, \quad x_2 = \mu\rho y_2, \quad x_3 = \mu y_2 + \lambda U, \quad x_4 = \mu y_1 - \lambda V.$$

When (4) is substituted in (2) factors $\lambda\rho$, $(\mu\rho)^{n'-2}$, and $H = y_1y_3 - y_2y_4$ cancel out leaving

$$(5) \qquad \bar{U}'V - \bar{V}'U = 0,$$

where now

$$U = U(y), \quad V = V(y), \quad \bar{U}' = \sum_{i=2}^{n'} \bar{u}_i' y_1^{n'-i} y_2^{i-2}, \quad \bar{V}' = \sum_{i=2}^{n'} \bar{v}_i' y_1^{n'-i} y_2^{i-2},$$

$$\bar{u}_i' = [(a'_{i1}\rho + a'_{i4})y_1 + (a'_{i2}\rho + a'_{i3})y_2]\,\mu + (a'_{i3}U - a'_{i4}V)\lambda,$$

$$\bar{v}_i' = [(b'_{i1}\rho + b'_{i4})y_1 + (b'_{i2}\rho + b'_{i3})y_2]\,\mu + (b'_{i3}U - b'_{i4}V)\lambda,$$

ρ having the value given by (3).

The values of λ and μ obtained from (5) may be written

(6) $$\lambda = J_1 H + (Uy_1 + Vy_2)K, \quad \mu = J_1(Uy_4 + Vy_3), \qquad \text{where}$$

$$J_1 = U\left(V\frac{\partial U'}{\partial y_3} - U\frac{\partial V'}{\partial y_3}\right) - V\left(V\frac{\partial U'}{\partial y_4} - U\frac{\partial V'}{\partial y_4}\right), \quad K = UV' - U'V.$$

When (6) is substituted in (4) a factor $Uy_1 + Vy_2$ appears, leaving for the equations of T^{-1},

(7) $$x_1 = y_1 J_1, \quad x_2 = y_2 J_1, \quad x_3 = y_3 J_1 + UK, \quad x_4 = y_4 J_1 - VK.$$

The equations of T are

(8) $$y_1 = x_1 J_1', \ y_2 = x_2 J_1', \ y_3 = x_3 J_1' - U(x)K(x), \ y_4 = x_4 J_1' + V(x)K(x),$$

where

$$J_1' = U'(x)[V'(x)\partial U(x)/\partial x_3 - U'(x)\partial V(x)/\partial x_3] - V'(x)[V'(x)\partial U(x)/\partial x_4 - U'(x)\partial V(x)/\partial x_4].$$

It is evident that T aind T^{-1} are of orders $n + 2n' - 3$ and $2n + n' - 3$ respectively.

3. *The System* $\infty^4 \,|\, F_n \,|$. The Plücker equation of Γ is $p_{12} = 0$. The Plücker coördinates of any Γ-line PP' are

$$p_{12} = 0, \ p_{13} = -Uy_1, \ p_{14} = Vy_1, \ p_{23} = -Uy_2, \ p_{42} = -Vy_2, \ p_{34} = Uy_4 + Vy_3.$$

The ∞^4 linear Γ-congruences $|\, Q_1 \,|$ are the intersections of $p_{12} = 0$ with

$$\alpha_{12}p_{12} + \alpha_{13}p_{13} + \alpha_{14}p_{14} + \alpha_{23}p_{23} + \alpha_{42}p_{42} + \alpha_{34}p_{34} = 0.$$

These $\infty^4 \,|\, Q_1 \,|$ correspond in M to the $\infty^4 \,|\, F_n \,|$,

(9) $$(\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}y_4)U - (\alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3)V = 0.$$

The directrix g of a Q_1 of this system has coördinates

(10) $$p_{12} = \alpha_{34}, \quad p_{13} = \alpha_{42}, \quad p_{14} = \alpha_{23}, \quad p_{23} = \alpha_{14}, \quad p_{42} = \alpha_{13}, \quad p_{34} = \alpha_{12},$$

$$\alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23} = 0.$$

The equations of g may be written

$$\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}y_4 = 0, \quad \alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3 = 0.$$

Each surface F_n of (9) evidently contains d^{n-2} and g, the directrix of its associated congruence.

Consider two surfaces of (9),

$$F_1 \equiv (\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}y_4)U - (\alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3)V = 0,$$
$$F_2 \equiv (\beta_{13}y_1 + \beta_{23}y_2 - \beta_{34}y_4)U - (\beta_{14}y_1 - \beta_{42}y_2 + \beta_{34}y_3)V = 0.$$

The elimination of U and V from F_1 and F_2 gives the quadric

$$H \equiv (\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}y_4)(\beta_{14}y_1 - \beta_{42}y_2 + \beta_{34}y_3)$$
$$-(\alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3)(\beta_{13}y_1 + \beta_{23}y_2 - \beta_{34}y_4) = 0.$$

This quadric H contains d, g_1 and g_2, the g_1 and g_2 being the directrices of the linear Γ-congruences Q_1 and Q_2, images in M^{-1} of F_1 and F_2. These lines d, g_1 and g_2 determine a regulus R' on H. The Γ-regulus R associated with R' on H is the base of the pencil of linear Γ-congruence defined by Q_1 and Q_2. $[F_1H] = d^{n-2}g\Delta_{n+1}$, where Δ_{n+1} is the image in M of R.

A generic plane $\delta \equiv y_4 = \sum_{i=1}^{3} k_iy_i$ meets F_n in a curve $C_n : D^{n-2}$, where $[\delta, d] = D$. In δ, the equation of C_n is

$$(\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}\sum_{1}^{3} k_iy_i)U - (\alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3)V = 0,$$

where y_4 in U and V is replaced by $\sum_{1}^{3} k_iy_i$. The tangents to C_n at D as given by the coefficient of $y_3{}^2$ in C_n are

$$k_3\left(\frac{\partial U}{\partial y_3} + k_3\frac{\partial U}{\partial y_4}\right) + \left(\frac{\partial V}{\partial y_3} + k_3\frac{\partial V}{\partial y_4}\right) = 0.$$

Since this expression is independent of α_{ik}, it follows that the $\infty^4 \mid F_n \mid$ are mutually tangent in d. Hence, $[F_nF'_n] = d^{(n-2)^2+(n-2)}\Delta_{n+1}C_{2n-3}$, where C_{2n-3} and d form the base of the system $\infty^4 \mid F_n \mid$.

4. *The Pencil* $\mid F_{n-1} \mid$. Suppose the directrix g of a linear Γ-congruence Q_1 meets d. Then from (10), $\alpha_{34} = 0$, and $\alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23} = 0$. The associated surface F_n is then composite, being

$$(\alpha_{13}y_1 + \alpha_{23}y_2)(\alpha_{23}U + \alpha_{42}V) = 0. \tag{11}$$

The $\alpha_{13}y_1 + \alpha_{23}y_2 = 0$ is the plane (d, g). The surface $F_{n-1} \equiv \alpha_{23}U + \alpha_{42}V$

$= 0$ is the image in M of the Γ-bundle on $G(0, 0, \alpha_{42}, \alpha_{23})$, the point of intersection of d and g.

The section of F_{n-1} by a generic plane $\delta \equiv y_4 = \sum_1^3 k_i y_i$ is a curve $C_{n-1}: D^{n-2}$, where $[d, \delta] = D$. In δ, the equation of C_{n-1} is $\alpha_{23}U + \alpha_{42}V = 0$, where y_4 in U and V is replaced by $\sum_1^3 k_i y_i$. The tangents to C_{n-1} at D as given by the coefficient of y_3 in C_{n-1} are

$$\alpha_{23}\left(\frac{\partial U}{\partial y_3} + k_3 \frac{\partial U}{\partial y_4}\right) + \alpha_{42}\left(\frac{\partial V}{\partial y_3} + k_3 \frac{\partial V}{\partial y_4}\right) = 0.$$

Since this expression depends upon α_{ik}, it follows that the $\infty^1 \mid F_{n-1} \mid$ are not mutually tangent in d. $[R_{n-1}, F_{n-1}] = d^{(n-2)^2} C_{2n-3}$. This pencil of surfaces $\mid F_{n-1} \mid$, together with the associated pencil $\mid F'_{n'-1} \mid$, image in M' of the ∞^1 Γ-bundles on d, furnishes a simple way of setting up the equations of the T. Let

$$\mid F_{n-1} \mid \equiv \alpha_{23}U + \alpha_{42}V = 0, \quad \mid F'_{n'-1} \mid \equiv \alpha_{23}U' + \alpha_{42}V' = 0.$$

where U, V, U', and V' have the values given in section 2.

Through a generic point $P(y)$ of space passes a single surface F_{n-1} of the pencil $\mid F_{n-1} \mid$, for which $\alpha_{23}/\alpha_{42} = V(y)/-U(y)$. The line t through P and $G(0, 0, \alpha_{42}, \alpha_{23})$ meets $F'_{n'-1}$, image in M' of the bundle G, in a unique point P', image in T of P. Any point of t has coördinates.

$$x_1 = \lambda y_1, \quad x_2 = \lambda y_2, \quad x_3 = \lambda y_3 + \mu U, \quad x_4 = \lambda y_4 - \mu V.$$

When the ratio λ/μ is determined so that this point lies on $F'_{n'-1}$, the result is (7).

5. *The $T_{3,3}$ in a Plane Through d.* A plane $\gamma \equiv x_1 = \sigma x_2$ through d cuts $\mid F_{n-1} \mid \equiv \alpha_{23}U(x) + \alpha_{42}V(x) = 0$ and $\mid F'_{n'-1} \mid \equiv \alpha_{23}U'(x) + \alpha_{42}V'(x) = 0$ in residual pencils of lines

$$\mid l \mid \equiv \alpha_{23}(ax) + \alpha_{42}(bx) = 0 \quad \text{and} \quad \mid l' \mid \equiv \alpha_{23}(a'x) + \alpha_{42}(b'x) = 0$$

respectively, where $(ax) = a_2x_2 + a_3x_3 + a_4x_4$,

$$a_2 = \sum_{i=2}^{n} \sigma a_{i1} + a_{i2})\sigma^{n-i}, \quad a_3 = \sum_{i=2}^{n} a_{i3}\sigma^{n-i}, \quad a_4 = \sum_{i=2}^{n} a_{i4}\sigma^{n-i}, \quad \text{etc.}$$

The vertices of $\mid l \mid$ and $\mid l' \mid$ are respectively $L(\lambda)$ and $L'(\lambda')$, where $\lambda_1 = \sigma\lambda_2$, $\lambda_2 = \mid a_3b_4 \mid$, $\lambda_3 = -\mid a_2b_4 \mid$, $\lambda_4 = \mid a_2b_3 \mid$, $\lambda_1' = \sigma\lambda_2'$, $\lambda_2' = \mid a_3'b_4' \mid$, $\lambda_3' = -\mid a_2'b_4' \mid$, $\lambda_4' = \mid a_2'b_3' \mid$.

Through a generic point $P(y)$ of γ passes one line of $\mid l \mid$, for which

$\alpha_{23}/\alpha_{42}=(by)/-(ay)$. The corresponding line $l'\equiv(by)(a'x)-(ay)(b'x)=0$ is met by the line through $P(y)$ and $G[0,(ay),-(by)]$ in a point P', image of P in T. The T_3^{-1} in γ is thus found to be

(12) $$x_2=y_2j_1,\quad x_3=y_3j_1+(ay)k,\quad x_4=y_4j_1-(by)k,$$

where $j_1=(ay)[a_3'(by)-b_3'(ay)]-(by)[a_4'(by)-b_4'(ay)]$,

and $k=(ay)(b'y)-(by)(a'y)$.

The T_3 is

(13) $$y_2=x_2j_1',\quad y_3=x_3j_1'-(a'x)k(x),\quad y_4=x_4j_1'+(b'x)k(x),$$

where $j_1'=(a'x)[a_3(b'x)-b_3(a'x)]-(b'x)[a_4(b'x)-b_4(a'x)]$.

The conic $k:LL'$ is pointwise invariant under $T_{3,3}$.

The points L and L' are evidently fundamental under T_3 and T_3^{-1} respectively.

The points P_1' and P_2', intersection residual to L' of $k(x)=0$ with the pair of lines $j_1'(x)=0$, are fundamental for T_3^{-1}.

Since (13) may be written

$$y_2=x_2j_1'(x),$$
$$y_3=[x_4(a'x)+x_3(b'x)][b_4(a'x)-a_4(b'x)]+[b_2(a'x)-a_2(b'x)]x_2(a'x),$$
$$y_4=-[x_4(a'x)+x_3(b'x)][b_3(a'x)-a_3(b'x)]+[b_2(a'x)-a_2(b'x)]x_2(b'x),$$

it follows that the points P_3', P_4', intersection of the conic $x_4(a'x)+x_3(b'x)=0$ with the line $x_2=0$, are fundamental for T_3^{-1}.

The homaloidal nets of T_3 and T_3^{-1} are respectively

$$\infty^2\,|f_3'|:L'^2P_1'P_2'P_3'P_4',\quad\text{and}\quad\infty^2\,|f_3|:L^2P_1P_2P_3P_4.$$

The images in T_3^{-1} of $(a'x)=0$, $k(x)=0$, and $f_3'(x)=0$ are respectively $(ay)j_2=0$, $kj_2j_3=0$, and $(\alpha y)j_1j_2^2j_3=0$. The factors $(ay)=0$, $k=0$, $(\alpha y)=0$, $j_1=0$, $j_2=0$, and $j_3=0$ are the images respectively of the proper points of $(a'x)=0$, the proper points of $k(x)=0$, the proper points of $f_3'(x)=0$, the pair of points $P_3'P_4'$, the point L', and the pair of points $P_1'P_2'$.

$$j_2=(ay)[a_3'(b'y)-b_3'(a'y)]-(by)[a_4'(b'y)-b_4'(a'y)],$$
$$j_3=(ay)[a_3(by)-b_3(ay)]-(by)[a_4(by)-b_4(ay)].$$

The images in T_3^{-1} of $j_1'(x)=0$, $j_2'(x)=0$, and $j_3'(x)=0$ are respectively $j_2^2j_3=0$, $j_1j_2j_3=0$, and $j_1j_2^2=0$. The jacobian of T_3^{-1} is now seen to be made up of $j_1:L^2P_1P_2$, $j_2:LL'P_1P_2P_3P_4$, and $j_3:L^2P_3P_4P_1'P_2'$. Similarly the jacobian of T_3 is composed of $j_1':L'^2P_1'P_2'$, $j_2':LL'P_1'P_2'P_3'P_4'$, and $j_3':L'^2P_3'P_4'P_1P_2$.

As the plane γ generates the pencil on d, the $T_{3,3}$ generates the space $T_{n+2n'-3,2n+n'-3}$. The equations of the latter may readily be obtained from (12) and (13) by replacing (ay), (by), $(a'x)$, $(b'x)$, and σ by U, V, $U'(x)$, $V'(x)$, *and* x_1/x_2 respectively.

Since the point L is the section by γ of C_{2n-3}, the latter may be represented by $x_1=\sigma\lambda_2$, $x_2=\lambda_2$, $x_3=\lambda_3$, $x_4=\lambda_4$. Similarly, $C'_{2n'-3}$ may be written $x_1=\sigma\lambda_2'$, $x_2=\lambda_2'$, $x_3=\lambda_3'$, $x_4=\lambda_4'$.

6. *The Homaloidal Webs in the* $T_{n+2n'-3,2n+n'-3}$. The image in T^{-1} of a generic plane $\sum_{i=1}^{4} c_i x_i = 0$ is a surface $F_{2n+n'-3} \equiv J_1 \sum_{i=1}^{4} c_i y_i + (c_3 U - c_4 V) K = 0$. A plane $\delta \equiv y_4 = \sum_{i=1}^{3} k_i y_i$ meets $F_{2n+n'-3}$ in a curve $C: D^{2n+n'-6}$, where $[\delta, d] = D$. The tangents to C at D as given by the coefficient of $y_3{}^3$ in C are

$$(k_3\bar{U}+\bar{V})\left[c_3\left(\bar{U}\frac{\partial V'}{\partial y_4}-\bar{V}\frac{\partial U'}{\partial y_4}\right)-c_4\left(\bar{U}\frac{\partial V'}{\partial y_3}-\bar{V}\frac{\partial U'}{\partial y_3}\right)\right]=0,$$

where

$$\bar{U}=\partial U/\partial y_3+k_3\partial U/\partial y_4, \quad V=\partial V/\partial y_3+k_3\partial V/\partial y_4.$$

Since the factor $k_3\bar{U}+\bar{V}$ is independent of c_i, it follows that, of the $2n+n'-6$ sheets of the surfaces of the homaloidal web through d, $n-2$ are mutually tangent there. It follows that the homaloidal web of T^{-1} is

$$\infty^3 \left| F'_{n+2n'-3} \right| : d^{n+2n'-6} C'^2_{2n'-3} \gamma'_{4n+2n'-10}.$$

Similarly, that of T is

$$\infty^3 \left| F_{2n+n'-3} \right| : d^{2n+n'-6} C^2_{2n-3} \gamma_{2n+4n'-10}.$$

7. *The F- and P-Systems of* $T_{2n+n'-3,n+2n'-3'}$. As the plane γ of section 5 describes the pencil on d, the points L, L', P_1P_2, $P_1'P_2'$, P_3P_4, and $P_3'P_4'$ describe respectively the curves C_{2n-3}, $C'_{2n'-3}$, $\gamma_{2n+4n'-10}$, $\gamma'_{4n+2n'-10}$, d and d. It is evident that C_{2n-3} (or $C'_{2n'-3}$) meets d in $2n-4$ (or $2n'-4$) points, and is rational. Also, $\gamma_{2n+4n'-10}$ (or $\gamma'_{4n+2n'-10}$) meets d in $2n+4n'-12$ (or $4n+2n'-12$) points. It is of genus $n'-3$ (or $n-3$).

The images in T^{-1} of d, $U'(x)=0$, and $K(x)=0$ are respectively $J_1=0$, $UJ_1{}^{n'-2}J_2=0$, and $KJ_1{}^{n+n'-4}J_2J_3=0$, where

$$J_1=U\left(V\frac{\partial U'}{\partial y_3}-U\frac{\partial V'}{\partial y_3}\right)-V\left(V\frac{\partial U'}{\partial y_4}-U\frac{\partial V'}{\partial y_4}\right),$$

$$J_2=U\left(V'\frac{\partial U'}{\partial y_3}-U'\frac{\partial V'}{\partial y_3}\right)-V\left(V'\frac{\partial U'}{\partial y_4}-U'\frac{\partial V'}{\partial y_4}\right),$$

$$J_3=U\left(V\frac{\partial U}{\partial y_3}-U\frac{\partial V}{\partial y_3}\right)-V\left(V\frac{\partial U}{\partial y_4}-U\frac{\partial V}{\partial y_4}\right).$$

The factors $U=0$, $K=0$, $J_1=0$, $J_2=0$, and $J_3=0$ are the images respectively of the proper points on $U'(x)=0$, the proper points on $K(x)=0$, the line d, the curve $C'_{2n'-3}$, and the curve $\gamma'_{4n+2n'-10}$.

These surfaces $J_i=0$ can also be obtained from the curves $j_i=0$ of section 5.

The jacobian of T^{-1} is composed of

$$\begin{aligned} &J_1{}^2: d^{2n+n'-6}C^2{}_{2n-3}\gamma_{2n+4n'-10},\\ &J_2: d^{n+2n'-6}C_{2n-3}C'_{2n'-3}\gamma_{2n+4n'-10},\\ &J_3: d^{3n-6}C^2{}_{2n-3}\gamma'_{4n+2n'-10}. \end{aligned}$$

The J_1, J_2, and J_3 are of orders $2n+n'-4$, $n+2n'-4$, and $3n-4$ respectively.

The $J_1=0$ and $J_3=0$ are ruled Γ-surfaces generated by the pairs of lines $j_1=0$ and $j_3=0$ of section 5. The $J_2=0$ is not ruled but is generated by the conic $j_2=0$. The image in T^{-1} of a line g is a curve of order $n+2n'-3$. If g meets d, its image in the $T_3{}^{-1}$ in the plane (g,d) is a cubic curve. Hence, a point of d corresponds in T^{-1} to a curve of order $n+2n'-6$. The $J_1=0$ contains a single infinity of such curves.

The pointwise invariant surface is

$$K_{n+n'-2}: d^{n+n'-4}C_{2n-3}C'_{2n'-3}\gamma_{2n+4n'-10}\gamma'_{4n+2n'-10}.$$

A plane $\delta \equiv y_4 = \sum_{i=1}^{3} k_i y_i$ meets $J_2=0$ in a curve $C_2: D^{n+2n'-6}$, where $[\delta, d]=D$. The tangents to C_2 at D as given by the coefficient of $y_3{}^2$ in C_2 are

$$\left(\frac{\partial U}{\partial y_3}\frac{\partial V}{\partial y_4}-\frac{\partial U}{\partial y_4}\frac{\partial V}{\partial y_3}\right)\left[k_3\left(\frac{\partial U}{\partial y_3}+k_3\frac{\partial U}{\partial y_4}\right)+\left(\frac{\partial V}{\partial y_3}+k_3\frac{\partial V}{\partial y_4}\right)\right]=0.$$

Similarly, the tangents to $C_3=[J_3,\delta]$ at D are

$$\left(\frac{\partial U'}{\partial y_3}\frac{\partial V'}{\partial y_4}-\frac{\partial U'}{\partial y_4}\frac{\partial V'}{\partial y_3}\right)\left[k_3\left(\frac{\partial U}{\partial y_3}+k_3\frac{\partial U}{\partial y_4}\right)+\left(\frac{\partial V}{\partial y_3}+k_3\frac{\partial V}{\partial y_4}\right)\right]=0.$$

It follows that $J_2=0$ and $J_3=0$ are tangent $n-2$ times in d.

$$[J_1,J_2]=d^{(2n+n'-6)(n+2n'-6)}C^2{}_{2n-3}\gamma_{2n+4n'-10}(2n'-4)l_i,$$

where the $(2n'-4)l_i$ are the images in T^{-1} of the $(2n'-4)$ points of intersection of $C'_{2n'-3}$ and d.

$$[J_1,J_3]=d^{(2n+n'-6)(3n-6)}C^4{}_{2n-3}(2n+2n'-8)l_i,$$

where the l_i are the images in T^{-1} of $(2n+2n'-8)$ of the points of inter-

section of $\gamma'_{4n+2n'-10}$ and d. Each of the remaining $2n-4$ points of intersection corresponds to d itself and is already counted.

$$[J_2, J_3] = d^{(n+2n'-6)(3n-6)+n-2} C^2_{2n-3} (3n+4n'-12) l_i,$$

where the l_i are the images of the points of intersection of $C'_{2n'-3}$ and $\gamma'_{4n+2n'-10}$.

$$\begin{aligned}
[J_1, K] &= d^{(2n+n'-6)(n+n'-4)} C^2_{2n-3} \gamma_{2n+4n'-10}, \\
[J_2, K] &= d^{(n+2n'-6)(n+n'-4)} C_{2n-3} C'_{2n'-3} \gamma_{2n+4n'-10}, \\
[J_3, K] &= d^{(3n-6)(n+n'-4)} C^2_{2n-3} \gamma'_{4n+2n'-10}.
\end{aligned}$$

The jacobian of T is composed of

$$\begin{aligned}
&J_1'^2 : d^{n+2n'-6} C'^2_{2n'-3} \gamma'_{4n+2n'-10}, \\
&J_2' : d^{2n+n'-6} C_{2n-3} C'_{2n'-3} \gamma'_{4n+2n'-10}, \\
&J_3' : d^{3n'-6} C'^2_{2n'-3} \gamma_{2n+4n'-10}, \quad \text{where}
\end{aligned}$$

$$\begin{aligned}
J_1' &= U'\left(V' \frac{\partial U}{\partial x_3} - U' \frac{\partial V}{\partial x_3}\right) - V'\left(V' \frac{\partial U}{\partial x_4} - U' \frac{\partial V}{\partial x_4}\right), \\
J_2' &= U'\left(V \frac{\partial U}{\partial x_3} - U \frac{\partial V}{\partial x_3}\right) - V'\left(V \frac{\partial U}{\partial x_4} - U \frac{\partial V}{\partial x_4}\right), \\
J_3' &= U'\left(V' \frac{\partial U'}{\partial x_3} - U' \frac{\partial V'}{\partial x_3}\right) - V'\left(V' \frac{\partial U'}{\partial x_4} - U' \frac{\partial V'}{\partial x_4}\right),
\end{aligned}$$

where $U = U(x)$, etc.

West Virginia University,
Morgantown, W. Va.

ON THE POSSIBLE FORMS OF DISCRIMINANTS OF ALGEBRAIC FIELDS I.

By William R. Thompson.

Let $K_{(\theta)}$ be an algebraic field of n-th degree, and let d be the discriminant of the field. Furthermore, let

$$(1) \qquad p = \prod_{i=1}^{r} P_i^{e_i}, \quad \text{where} \quad N_{(P_i)} = p^{f_i},$$

be the prime-ideal decomposition of an arbitrary rational prime, $p > 1$, in the field. Then Dedekind * has shown that the rational integer, $\varepsilon \geqq 0$, such that p^{ε} exactly divides d, is dependent upon (1) and that if none of the exponents (e_i) are divisible by p, then

$$(2) \qquad \varepsilon = \sum_{i=1}^{r} f_i(e_i - 1).$$

Ore † has treated the general case, where all or any of the exponents (e_i) may be divisible by p, and given the possible values of ε for any given prime-ideal decomposition. Furthermore, he has given the maximal value of ε for fields of n-th degree.

Let $N_{(n,p)}$ denote this maximal value. Then if we have the representation of n in a p-adic system

$$(3) \qquad n = \sum_{\alpha=0}^{q} b_\alpha p^\alpha, \quad \text{where} \quad 0 \leq b_\alpha < p$$

and b_α is a rational integer; and J is the aggregate number of these coefficients (b_α) which are different from zero; then Ore has shown that

$$(4) \qquad N_{(n,p)} = \sum_{\alpha=0}^{q} [b_\alpha(\alpha + 1)p^\alpha] - J.$$

The equivalent of the above is given in the paper ‡ previously mentioned wherein Ore has suggested the interest of ascertaining what other values are possible for ε for algebraic fields of the same degree. It is the purpose of

* R. Dedekind, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 29 (1882), pp. 1-56.

† Ö. Ore, *Mathematische Annalen*, Vol. 96 (1927), pp. 313-352.

‡ Ore, *loc. cit.*

the present communication to present the solution of this problem, which may be stated briefly as follows:

A. If $p > 2$, ε can assume all values from zero to $N_{(n,p)}$ inclusive except only as follows:

$\varepsilon \neq \alpha p^a - 1$ (where α is a positive rational integer) if $n = p^a$ or if $\alpha > 1$ and $n = p^a + 1$.

B. If $p = 2$ the result is formally the same as given for $p > 2$ except that $\varepsilon \neq 1$ always.

During the course of the proof of these relations which follow, it will become evident that certain other relations are developed which make possible a partial reversal of proof in order to establish criteria whereby, in certain cases, a knowledge of the power of p exactly dividing the discriminant of a field of n-th degree suffices to determine uniquely the prime-ideal decomposition of p in *any* such field. The statement and proof of these relations, however, will be given in another communication.

1. Let $K_{(\theta)}$, n, d, p, ε and $N_{(n,p)}$ be defined as above; and let p have the prime-ideal decomposition (1). Then let the representation of n in the p-adic system be given as in (3) and accordingly $N_{(n,p)}$ will be given by (4). Then it is well known that

$$n = \sum_{i=1}^{r} e_i f_i, \quad \text{and} \quad e_i > 0 < f_i, \tag{5}$$

As indicated above, the foundation of the proof to follow is found in certain theorems given by Ore.* These may be stated in the following form:

For each prime-ideal, P_i, there exists a rational integer $\rho_i \geqq 0$, (called the *supplemental* number) such that if $S_i \geqq 0$ be a rational integer such that e_i is exactly divisible by p^{S_i} then ρ_i is determined as follows:

(6) If $S_i = 0$, then $\rho_i = 0$; and if $S_i \neq 0$, then $1 \leqq \rho_i \leqq e_i S_i$, and in this latter alternative ρ_i is restricted by the condition that if there exists a positive rational integer, ν_i, such that ρ_i is exactly divisible by p^{ν_i}, then ν_i shall not exceed ρ_i/e_i, i. e., $\nu_i \leqq [\rho_i/e_i]$, and in any case,

$$\varepsilon = \sum_{i=1}^{r} f_i(e_i - 1 + \rho_i). \tag{7}$$

This theorem will be designated as Ore's First Theorem. The following existance theorem given in the same article will be called Ore's Second Theorem:

* *Loc. cit.*

For any set of rational integers, designated by

$$(8) \qquad \left\{ \begin{matrix} e_1, \cdots, e_r \\ f_1, \cdots, f_r \\ \rho_1, \cdots, \rho_r \end{matrix} \right\}$$

and satisfying the conditions of (5) *and* (6) *there exists an algebraic field of n-th degree such that its discriminant is exactly divisible by* p^ε *where* ε *has the value given in* (7). *Furthermore, in such a field the prime-ideal decomposition of* p *is given by* (1).

2. We may proceed from these two theorems and that relative to $N_{(n,p)}$ given in (3) and (4) to find what other values of ε are attainable for fields of n-th degree. Accordingly, let $\mathcal{E}_p^{(n)}$ be defined as the set of rational integers which are attainable values of ε for algebraic fields of n-th degree. Then, obviously,

(9) $N_{(n,p)}$ is the greatest component of $\mathcal{E}_p^{(n)}$.

Now in (8), obviously, by taking $e_i = 1$ for every i, then $S_i = 0$ whence by (6) $\rho_i = 0$ for every i whence (7) gives $\varepsilon = 0$, whence

(10) 0 is a component of $\mathcal{E}_p^{(n)}$ for every n and p.

(9) and (10) by Ore's Theorems then give the obvious but useful

THEOREM 1. $\mathcal{E}_p^{(n)}$, *the set of attainable values of* ε *for fields of n-th degree, is a set of a finite number of rational integers including* 0 *and* $N_{(n,p)}$ *as least and greatest component respectively.*

Obviously, there are instances [where $n = 1$ in (3)] where these two extremes are equal, in which case 0 is the only component of $\mathcal{E}_p^{(n)}$ but the set is never *void*.

Now consider the case, $p > n$. Then by (4) we have

$$(11) \qquad N_{(n,p)} = n - 1;$$

and in (8) we may choose $f_i = 1$ for every i; and as $p > n$, (5) gives $p > e_i$ whence by Ore's First Theorem or that of Dedekind we have in this case

$$(12) \qquad \varepsilon = n - r$$

where r can be any positive rational integer not exceeding n whence by Theorem 1 and (11) we have proved the

THEOREM 2. *For* $p > n$, $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$.

3. Now, before we attempt the proof of the general theorem stated in the introduction, it is expedient to prove certain incidental theorems some of

which have a wider application than that to be utilized at present, as has been mentioned above.

Consider the set of numbers in (8). It consists of three arrays of rational integers, each containing r numbers; the three arrays (the orders, degrees and supplemental numbers, respectively, of the prime-ideal divisors of p) being arranged in the *form* of a matrix. Obviously any permutation of the columns of this matrix gives the same value to ε. Such a matrix satisfying the conditions given in (5) and (6) as indicated under (8) will be called a *critical matrix*. Furthermore, it is evident that the set of possible values for each ρ_i depends upon p and e_i only.

Accordingly, if $r > 1$, we may take any positive rational integer, $r' < r$, and form the two critical matrices for fields of degree less than n

$$(13) \qquad \left\{ \begin{matrix} e_1, \cdots, e_{r'} \\ f_1, \cdots, r_{r'} \\ \rho_1, \cdots, \rho_{r'} \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} e_{r'+1}, \cdots, e_r \\ f_{r'+1}, \cdots, f_r \\ \rho_{r'+1}, \cdots, \rho_r \end{matrix} \right\}.$$

Now, let n', n'', ε' and ε'' be defined by

$$(14) \qquad n' = \sum_{i=1}^{r'} e_i f_i, \quad n'' = \sum_{i=r'+1}^{r} e_i f_i,$$

$$\varepsilon' = \sum_{i=1}^{r'} f_i(e_i - 1 + \rho_i), \text{ and } \varepsilon'' = \sum_{i=r'+1}^{r} f_i(e_i - 1 + \rho_i).$$

Then by (5) and (7) we have

$$(15) \qquad n = n' + n'' \text{ and } \varepsilon = \varepsilon' + \varepsilon''.$$

Now, in the present discourse, if A and B are sets of rational integers, let $A + B$ be assigned the following meaning:

(16) If η is a component of $A + B$; then there exists a component of A, say a, and a component of B, say b, such that $\eta = a + b$;
and if a is a component of A, and b is a component of B, then $(a + b)$ is a component of $(A + B)$.

Now, if we have any two critical matrices whatever, we may write them in the form of (13) and employ the definitions under (14); but with no prior assumption as to the matrix in (8). However, let n' and n'' be restricted by the equation

$$n' + n'' = n$$

then, obviously, by a reversal of proof we may construct the critical matrix (8) by the fusion of those of (13) and thus establish the existence of a field of n-th degree wherein the discriminant is exactly divisible by $p^{\varepsilon'+\varepsilon''}$, i. e.,

$$\varepsilon = \varepsilon' + \varepsilon''.$$

Accordingly, we have proved the

THEOREM 3. *If n' and n'' are two positive rational integers such that their sum is equal to n then $\mathcal{E}_p^{(n)}$ includes $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')}$.*

Also, inasmuch as by Theorem 1 zero is always a component of $\mathcal{E}_p^{(n)}$ for any n, we have to Theorem 3 the

COROLLARY 1. $\mathcal{E}_p^{(n)}$ includes $\mathcal{E}_p^{(n')}$ provided $n \geq n'$.

4. In order to facilitate calculation let us divide the set, $\mathcal{E}_p^{(n)}$ dichotomously as follows:

Let $H_p^{(n)}$ be a set such that if and only if η is a component of $H_p^{(n)}$ then there exist two positive rational integers, n' and n'', such that $n = n' + n''$ and η is a component of $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')}$.

Obviously, by Theorem 3, therefore

(17) $\mathcal{E}_p^{(n)}$ includes $H_p^{(n)}$.

Accordingly, let $A_p^{(n)}$ be defined as the set of rational integers such that if and only if η is a component of $\mathcal{E}_p^{(n)}$ but not of $H_p^{(n)}$, then η is a component of $A_p^{(n)}$.

Then $\mathcal{E}_p^{(n)}$ is the union of the two mutually exclusive sets, $H_p^{(n)}$ and $A_p^{(n)}$, which we may call the *heritage* and the *acquisition* respectively of $\mathcal{E}_p^{(n)}$.

Now, by (13), (14) and (15) and the above definitions any value of ε obtained from a critical matrix (8) wherein $r > 1$ lies in the heritage, $H_p^{(n)}$, and not in $A_p^{(n)}$. Otherwise stated we have proved

(18) if ε is a component of $A_p^{(n)}$, then $r = 1$.

Now in (8) let us consider the case $r = 1$ and let the superfluous subscript, i, then be dropped. Then the critical matrix becomes simply

$$(19) \qquad \left\{ \begin{matrix} e \\ f \\ \rho \end{matrix} \right\} \quad \text{and} \quad \varepsilon = f(e - 1 + \rho).$$

Now, if in (19) $f > 1$ then there exists another critical matrix under (8) where $r = 2$, namely

$$(20) \qquad \left\{ \begin{matrix} e, & e \\ 1, & f-1 \\ \rho, & \rho \end{matrix} \right\} \quad \text{and} \quad \varepsilon = f(e - 1 + \rho),$$

which value of ε is the same as in (19) but by (18) this is not a component of $A_p^{(n)}$. Accordingly, by (18) we have in (8)

(21) if ε is a component of $A_p^{(n)}$, then $r = f_i = 1$, and $e_i = n$..

Therefore, by Ore's First Theorem we have the

THEOREM 4. *If ε is a component of $A_p^{(n)}$ it corresponds to a critical matrix of the form*

$$\left\{ \begin{matrix} n \\ 1 \\ \rho \end{matrix} \right\} \quad \textit{and} \quad \varepsilon = n - 1 + \rho,$$

where ρ is a rational integer defined by the relations:

If $S \geqq 0$ is a rational integer such that n is exactly divisible by p^S, then

$$\begin{aligned} &\textit{if } S = 0, \quad \textit{then} \quad \rho = 0, \\ &\textit{and if } S \neq 0, \quad \textit{then} \quad 1 \leq \rho \leq nS \end{aligned}$$

and in this latter alternative ρ is restricted by the condition that if there exists a positive rational integer, ν, such that ρ is exactly divisible by p^ν, then ν shall not exceed ρ/n.

Accordingly, by the definition of $A_p^{(n)}$ and Ore's Theorems we have

THEOREM 5. *That ε be a component of $A_p^{(n)}$ it is necessary and sufficient that the conditions of Theorem 4 be satisfied and that ε be not a component of $H_p^{(n)}$.*

5. In Section 2 we have obtained a solution for the case $p > n$ and in the succeeding sections have prepared for certain phases of the handling of the other cases $(p \leq n)$. However, before attempting the general solution there are a few additional contingencies for which provision should be made.

In order to illustrate this need as well as to extend the domain of the solution, let us consider the case, $p = n$. Then by Theorem 2 and the definition of $H_p^{(n)}$ we have

(22) $$H_p^{(p)} = \mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} \quad \text{where} \quad n' + n'' = p$$

whence, by the definition in (16), we have

(23) $$H_p^{(p)} = 0, \cdots, p - 2.$$

Now in Theorem 4 for $n = p$ we have

(24) $$S = 1 \quad \text{and} \quad \rho = 1, \cdots, p \quad \text{and}$$

(25) $$A_p^{(p)} = p, \cdots, 2p-1$$

whence (as $N_{(p,p)} = 2p-1$) we have by the definitions of Section 4 the

THEOREM 6. *$\mathcal{E}_p^{(p)} = 0, \cdots, N_{(p,p)}$ except $p-1$.*

Here we note the first instance of a number, η, a rational integer such that

(26) $0 \leq \eta \leq N_{(n,p)}$ and yet η is not a component of $\mathcal{E}_p^{(n)}$.

Such a number will be called an *exceptional number* relative to $\mathcal{E}_p^{(n)}$. Furthermore, in (26) by Theorem 1 we have $0 \neq \eta \neq N_{(n,p)}$.

Now, if η in (26) is also an exceptional number relative to $\mathcal{E}_p^{(\mu)}$ for every $\mu > n$ (where μ is a positive rational integer) then we shall call η a *universal exception* relative to p. Obviously, by Cor. 1 of Theorem 3, then η is not a component of $\mathcal{E}_p^{(\mu)}$ for any positive rational integer, μ.

On the other hand, if η is an exceptional number relative to $\mathcal{E}_p^{(n)}$ but $\eta-2, \eta-1, \eta+1$ and $\eta+2$ are components of $\mathcal{E}_p^{(n)}$; then η will be called a *regular exception* relative to $\mathcal{E}_p^{(n)}$.

Obviously, by Theorem 6 we have

(27) $\mathcal{E}_p^{(p)}$ has the single exceptional number, $p-1$,

which is regular for $p > 2$; and for $p = 2$ the exception $(\eta = 1)$ is not regular.

Now, suppose that 1 is not a component of $\mathcal{E}_2^{(\mu)}$ for $\mu < n$. Then by definition 1 is not in $H_2^{(n)}$; and by Theorem 4

1 is not in $A_2^{(n)}$; whence

1 is not in $\mathcal{E}_2^{(n)}$; whence, obviously, by complete induction we have the

THEOREM 7. *The number 1 is a universal exception relative to the rational prime, 2; that is, the number 1 is not a component of $\mathcal{E}_2^{(n)}$ for any n.*

Obviously, from the definitions we have to this theorem the

COROLLARY 1. $d \not\equiv 2 \pmod 4$.

This is merely a restatement of Theorem 7 and is essentially the same as part of a result obtained by Stickelsberger* concerning the *discriminant of the equation, D.* The equation here as usual being by implication the equation of θ, there exists the well known relation

(28) $$D = k^2 d$$

* L. Stickelsberger, *Proceedings of the International Congress*, Zürich (1897), pp. 182-193.

where k is a rational integer called the index of θ. Obviously, then by the corollary above we have

$$(29) \qquad D \not\equiv 2 \pmod 4$$

which is a part of the result of Stickelsberger mentioned above. Another proof of the Theorem of Stickelsberger has recently been given by I. Schur.*

Now, let us assume that for any two positive rational integers, n' and n'', such that $n' + n'' = n$, the sets $\mathcal{E}_p^{(n')}$ and $\mathcal{E}_p^{(n'')}$ have at most one regular exception each and no other exceptions unless $p = 2$ and that in this case the only other exception is the universal exception, 1. Then, obviously, by Theorems 2 and 6 we have

$$(30) \quad \text{If } n' > 1 < n'', \text{ then } \mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} = 0, \cdots, (N_{(n',p)} + N_{(n'',p)}) \\ \text{except 1 if } p = 2.$$

and if either n' or $n'' = 1$, then $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} = \mathcal{E}_p^{(n-1)}$.

6. We are now prepared to prove by the method of complete induction the following general theorem.

THEOREM 8. *If α is a positive rational integer and $p > 2$; then, if $n = p^\alpha$, then $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$ except $\alpha p^\alpha - 1$, if $\alpha > 1$* and $n = p^\alpha + 1$, *then $\mathcal{E}_p^{(n)} = \mathcal{E}_p^{(n-1)}$, and in every other case $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$; and if $p = 2$ then $\mathcal{E}_p^{(n)}$ is formally the same as given for $p > 2$ except that* 1 *is a universal exception.*

Now, by Theorem 2 and 6 we have verified Theorem 8 for the case, $n \leq p$.

Let k be a positive rational integer such that Theorem 8 is verified for the case, $n \leqq p^k$. By the statement preceding there is at least one possible value for k; namely, the number 1. It remains, accordingly, but to establish that given a value for k above then the Theorem 8 can be verified for the case, $n \leq p^{k+1}$; and, obviously, it suffices to make the demonstration for the case, $p^k + 1 \leq n \leq p^{k+1}$; and in so doing we may refer to Theorem 8 for the enunciation of the components of any set, $\mathcal{E}_p^{(\mu)}$, provided that μ is a positive rational integer not exceeding p^k. Furthermore, $\mathcal{E}_2^{(n)}$ does not contain 1.

Accordingly, let us consider the case, $n = p^k + 1$. Then $N_{(n,p)} = N_{(p^k,p)}$ and $\mathcal{E}_p^{(p^k)} = 0, \cdots, N_{(p^k,p)}$ except $kp^k - 1$ and except 1 if $p = 2$.

Now, if $k = 1$ and $p = 2$, then $kp^k - 1 = 1$ whence

$$(31) \qquad \mathcal{E}_2^{(3)} = 0, \cdots, N_{(3,2)} \text{ except } 1,$$

* I. Schur, *Mathematische Zeitschrift*, Vol. 29 (1929), pp. 464-465.

and if $k=1$ and $p>2$ we have

(32) $\mathcal{E}_p^{(n)}$ contains $\mathcal{E}_p^{(p-1)}+\mathcal{E}_p^{(2)}$ which contains $p-1$, whence we have, for $n=p^k+1$ and $k=1$,

(33) $$\mathcal{E}_p^{(n)}=0,\cdots,N_{(n,p)} \text{ (except 1 if } p=2)$$

This verifies Theorem 8 for the case, $n=p+1$.

Now, if $k>1$ in this same case $(n=p^k+1)$ it may readily be verified that there exist no two rational integers greater than *one*, n' and n'', such that

$$n=n'+n'' \text{ and } N_{(n',p)}+N_{(n'',p)}\geqq kp^k-1$$

whence by (30) as $\mathcal{E}_p^{(p^k)}$ does not contain kp^k-1 we have in this case

$$H_p^{(n)} \text{ does not contain } kp^k-1$$

and by Theorem 4 for $n=p^k+1$ we have $A_p^{(n)}$ does not contain kp^k-1 where $k>1$. Therefore, by Theorem 8, restricted, and 1 of Theorem 3 we have, as $N_{(p^k,p)}=N_{(p^k+1,p)}$

(34) $$\text{If } k>1, \text{ then } \mathcal{E}_p^{(n)}=\mathcal{E}_p^{(p^k)} \text{ if } n=p^k+1.$$

Now, consider the case, $n=bp^k$ where b is a rational integer, and $1<b<p$. Obviously, $p=2$ is excluded from this case. Then by the definition of k and (30) if we set $n'=(b-1)p^k$ and $n''=p^k$ then if $\mathcal{E}_p^{(n')}$ is as given in Theorem 8 we have

(35) $$\text{as } n=n'+n'', \ H_p^{(n)} \text{ includes } 0,\cdots,(N_{(n',p)}+N_{(n'',p)})$$

and by (3) and (4) we have in this case $N_{(n',p)}+N_{(n'',p)}=N_{(n,p)}$ whence (35) and Theorem 1 give (if $\mathcal{E}_p^{(n')}$ is correctly given by Theorem 8 where $n'=(b-1)p^k$) then

(36) $$\mathcal{E}_p^{(n)}=0,\cdots,N_{(n,p)};$$

(where $n=bp^k$ as provided above) which is as given by *Theorem 8*. But for $b=2$, $n'=p^k$; whence by definition of k, $\mathcal{E}_p^{(n)}$ is in this case correctly given by Theorem 8; whence by complete induction we have by (36) for any rational integer, $b>1$ and $<p$

(37) $$\text{if } n=bp^k \text{ then } \mathcal{E}_p^{(n)}=0,\cdots,N_{(n,p)},$$

which is as given in Theorem 8.

Now consider every other case for $n<p^{k+1}$; i. e., $n>p^k+1$ and such that there exists no positive rational integer, b, such that $n=bp^k$. Then n is given in p-adic form by (3) where $q=k$; then

$$(38) \qquad n = \sum_{a=0}^{k} b_a p^a, \quad \text{where} \quad 0 \leq b_a < p$$

and b_a is a rational integer. Obviously, as $n > p^k + 1$ we have $b_k \neq 0$. Now, let $b = b_k$ and $n' = bp^k$. Then in (30) by substitution we have $n'' = n - bp^k$ whence (38) gives

$$(39) \qquad n'' = \sum_{a=0}^{k-1} b_a p^a < p^k \quad \text{and, if } b = 1, \text{ then } n'' > 1;$$

whence by (30), (37) and the definition of k we have in this case

$$(40) \qquad \mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)} \qquad (\text{except } 1 \text{ if } p = 2).$$

Accordingly, we have shown that if Theorem 8 is verified for $n \leq p^k$, then it can be verified for $n < p^{k+1}$. It remains but to show that then it can be verified for $n = p^{k+1}$.

Consider this case, $n = p^{k+1}$. Then for any two positive rational integers, n' and n'', such that $n = n' + n''$, it can be deduced from the definition of $N_{(n,p)}$ and the relations (3) and (4), by replacement of n by n' and n'' in the argument successively, that in any instance we have

$$(41) \qquad \text{setting } k' = k + 1, \quad \text{then} \quad k'p^{k'} - 2 \geq N_{(n',p)} + N_{(n'',p)};$$

and, indeed, that equality exists only when $n' \equiv 0 \bmod p^k$).

Now, let $n' = (p-1)p^k$. Then $n'' = p^k$, and in (41) we have

$$(42) \qquad k'p^{k'} - 2 = N_{(n';p)} + N_{(n'',p)};$$

whence by (41), (30) and the definition of $H_p^{(n)}$ we have by Theorem 8 (restricted to the domain verified)

$$(43) \quad \text{if } n = p^{k'}, \text{ then } H_p^{(n)} = 0, \cdots, (k'p^{k'} - 2), \quad (\text{except } 1 \text{ if } p = 2).$$

We now turn to a consideration of the components of $A_p^{(n)}$ for $n = p^{k'}$. These are given by Theorems 4 and 5 by reference to (43) by

$$(44) \qquad \text{if } n = p^{k'}, \text{ then } A_p^{(n)} = k'p^{k'}, \cdots, N_{(n,p)};$$

whence by (43) we have verified Theorem 8 for the case, $n = p^{k+1}$, which was all that remained to be done in order to establish this theorem completely.

Theorem 8 is stated in another form in the introduction. It may be verified readily that the two statements are equivalent and that every possible case is covered.

YALE UNIVERSITY.

TWO-DIMENSIONAL CHAINS.

By A. Arwin.

In the present paper I have set myself the problem of finding a generalization in two or more dimensions of the usual chains of fractions, which we may briefly speak of as one-dimensional chains, and their periodicity. Of earlier papers on this theme I mention those of Berwick * and Daus,† and above all those of Perron,‡ who has done close investigation of the chains which he calls " Jacobi-chains," and of their convergence, periodicity and order of approximation. The two-dimensional Jacobi-chains do not, however, enjoy the property of periodicity in conjunction with that of the " best possible order of approximation," § as the one-dimensional chains for a quadratic irrationality do. It is therefore of interest to know that it is possible also in the case of cubic irrationalities to form chains of periodicity together with the highest order of approximation by following up a generalization of the one-dimensional chains along lines, which will be explained below. The best order of approximation of two cubic irrationalities μ_1, μ_2 is the one given by

$$(1) \qquad |\mu_1 - x_r/z_r| \leqq k_1/z_r^{3/2}, \quad |\mu_2 - y_r/z_r| \leq k_2/z_r^{3/2},$$

where x, y, z are rational integers, k_1 and k_2 constants $\nrightarrow 0$, greater than a readily assigned numerical quantity.¶

The principle to be applied may briefly be characterized as follows. Let us have the one-dimensional chain, homogeneously written

$$(2) \qquad \omega_0^{(1)} = \omega_{r+1}^{(1)}x_{r+1} + \omega_{r+1}^{(2)}x_r, \quad \omega_0^{(2)} = \omega_{r+1}^{(1)}z_{r+1} + \omega_{r+1}^{(2)}z_r,$$

and, as known, the order of approximation

$$\omega_0^{(1)}/\omega_0^{(2)} - x_r/z_r = \delta_r/z_r^2$$

where $\omega_0^{(i)}$, $\omega_{r+1}^{(i)}$ are quadratic, algebraic integers and the constant δ_r limited, possibly $\rightarrow 0$. On account of this equation the relations (2) will give

* W. E. H. Berwick, *Proceedings of the London Mathematical Society* (2), Vol. 12 (1912).

† P. H. Daus, *American Journal of Mathematics*, Vol. 44 (1922).

‡ O. Perron, *Mathematische Annalen*, Bd. 64 (1907); *Sitzungsberichte der Königlischen Bayerschen Akademie der Wissenschaften zu München*, Bd. 37 and 38.

§ O. Perron, *Irrationalzahlen*, 1921, s. 135.

¶ O. Perron, *Mathematische Annalen*, Bd. 83 (1921).

$$(-1)^{r+1}\omega_{r+1}^{(1)} = \omega_0^{(2)}\delta_r/z_r$$

and for $\overline{\omega}_{r+1}^{(1)}$, the conjugate of $\omega_{r+1}^{(1)}$

$$(-1)^{r+1}\overline{\omega}_{r+1}^{(1)} = \overline{\omega}_0^{(2)}z_r(\overline{\omega}_0^{(1)}/\overline{\omega}_0^{(2)} - \omega_0^{(1)}/\omega_0^{(2)} + \delta_r/z_r^2).$$

With increasing r we have the norm

$$N(\omega_{r+1}^{(1)}) \sim \omega_0^{(2)}\overline{\omega}_0^{(2)}(\overline{\omega}_0^{(1)}/\overline{\omega}_0^{(2)} - \omega_0^{(1)}/\omega_0^{(2)})\delta_r.$$

As however $\omega_{r+1}^{(1)}$ is an algebraic integer, its norm is a rational integer and therefore $\nrightarrow 0$. Hence, since the chain cannot break off, $\delta_r \nrightarrow 0$ and $N(\omega_{r+1}^{(1)})$ as well as $N(\omega_{r+1}^{(2)})$ are limited for all r, from which we easily infer that in the one-dimensional chain from a quadratic irrationality the formation is periodic. From this point of view, we are going to investigate the two-dimensional chains formed by cubic irrationalities. Let us therefore take μ_1 and μ_2 from a cubic field and let us assume, as is usually done (a proof of this statement however we omit here) that

$$x - \mu_1 z = 0, \qquad y - \mu_2 z = 0 \tag{4}$$

form a vector in space. Then we have to determine a set of points $A_r(x_r, y_r, z_r)$ which shall approximate the vector (4). Let us have, for example, constructed $A_r(x_r, y_r, z_r)$ in the shortest distance ρ_r from (4) and h_r from a plane at right angles to (4). Proceeding from A_r with the circular disc $\pi(\rho_r - \epsilon_1)^2$, $\epsilon_1 > 0$ but otherwise arbitrarily small, we move it along the vector (4) until the next point A_{r+1} with $\rho_{r+1} < \rho_r$ falls on it. In this way we construct a set of points A_r, A_{r+1} etc. with decreasing $\rho_r > \rho_{r+1} > \cdots$. The existence of such a sequence follows from the theorem of Minkowski,* that says: A convex body with a "lattice point" as centre and volume 8 must always without this centre have at least one further lattice point. But we may infer more than only the existence of the set A_r. Then, since no point is given on or within the cylinder $2\pi\rho_r^2 h_{r+1}$, A_r and A_{r+1} excepted, we may contract ρ_r and h_{r+1} arbitrarily little ϵ_1 and ϵ_2, and have no lattice point at all on or within the cylinder $2\pi(\rho_r - \epsilon_1)^2(h_{r+1} - \epsilon_2)$. By reason of this theorem of Minkowski we therefore infer the following important inequality

$$2\pi(\rho_r - \epsilon_1)^2(h_{r+1} - \epsilon_2) < 8.$$

$$2\pi\rho_r^2 h_{r+1} < 8 + 4\pi\rho_r h_{r+1}\epsilon_1 + 2\pi\rho_r^2\epsilon_2 + 2\pi\epsilon_1^2\epsilon_2 - 2\pi\epsilon_1^2 h_{r+1} - 4\pi\epsilon_1\epsilon_2\rho_r,$$

or, since ϵ_1 and ϵ_2 are arbitrarily small, for example, also

$$2\pi\rho_r^2 h_{r+1} < 16, \qquad \rho_r^2 h_{r+1} < 8/\pi. \tag{5}$$

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* H. Minkowski, *Diophantische Approximationen*, s. 60.

The plane

(6) $$x\mu_1 + y\mu_2 + z = 0$$

is perpendicular to the vector (4), whence h_{r+1} is computed as

(7) $$h_{r+1} = (x_{r+1}\mu_1 + y_{r+1}\mu_2 + z_{r+1})/(\mu_1^2 + \mu_2^2 + 1)^{1/2}$$

and also

(7′) $$\rho_r^2 = R_r^2 \sin^2 \delta_r = \frac{(x_r - \mu_1 z_r)^2 + (y_r - \mu_2 z_r)^2 + (\mu_1 y_r - \mu_2 z_r)^2}{\mu_1^2 + \mu_2^2 + 1}.$$

From our construction of the set A_r it immediately follows that

(8) $$\mu_1 - x_r/z_r = \epsilon_r^{(1)}, \quad \mu_2 - y_r/z_r = \epsilon_r^{(2)}, \quad \epsilon_r^{(1)} \text{ and } \epsilon_r^{(2)} \to 0.$$

Hence we have

$$z_{r+1}\left[1 - \frac{\mu_1\epsilon_{r+1}^{(1)} + \mu_2\epsilon_{r+1}^{(2)}}{\mu_1^2 + \mu_2^2 + 1}\right]\left[(x_r - \mu_1 z_r)^2 + (y_r - \mu_2 z_r)^2 + (\mu_1 y_r - \mu_2 z_r)^2\right]$$
$$< (8/\pi)(\mu_1^2 + \mu_2^2 + 1)^{1/2}$$

and therefore, for example,

$$(x_r - \mu_1 z_r)^2 < \text{constant}/z_{r+1}, \text{ const.} = (9/\pi)(\mu_1^2 + \mu_2^2 + 1)^{1/2},$$

that is,

(9) $$|\mu_1 - x_r/z_r| < \text{constant}/z_{r+1}.$$

Proceeding as above the set of points A_r will give the following approximations

(9′) $$\mu_1 - x_r/z_r = k_r^{(1)}/z_r^{3/2}, \qquad \mu_2 - y_r/z_r = k_r^{(2)}/z_r^{3/2},$$

where $k_r^{(i)}$ might eventually tend to zero; but, as already said, if μ_1 and μ_2 are cubic, independent irrationalities, Mr. Perron* has proved that actually $k_r^{(i)}$ does not tend to zero but has the order of a numerical constant. From this fact and (9′) we also conclude, and this is important, that in

(10) $$z_{r+1} = \tau_r z_r,$$

τ_r for all r is limited since

$$z_{r+1} < (9/\pi)(\mu_1^2 + \mu_2^2 + 1)z_r/\delta^2,$$
$$\delta \geq 1/2\sigma, \quad \sigma = \prod \rho_i, \quad \rho_i = \sum_{\nu=1}^{2} |\mu_\nu^{(i)} - \mu_\nu| \qquad (i = 1, 2).$$

Let us now put

(11) $$A_{r+1,r} = y_{r+1}z_r - y_r z_{r+1}, \quad B_{r+1,r} = z_{r+1}x_r - z_r x_{r+1},$$
$$C_{r+1,r} = x_{r+1}y_r - x_r y_{r+1},$$

and by means of (9′) compute for example

* O. Perron, *Mathematische Annalen*, Bd. 83 (1921).

$$B_{r+1,r}=(z_{r+1}z_r/z_r^{3/2})[-k_r^{(1)}+k_{r+1}^{(1)}(z_r/z_{r+1})^{3/2}],$$

from which we further conclude, that $B_{r+1,r}$ has an order of infinity of at most $z_r^{1/2}$ and in the same way all $A_{m,n}$, $B_{m,n}$, $C_{m,n}$ $(m, n=r, r+1$ and $r+2)$. Hence we write

$$(12)\quad A_{r+1,r}=x_{r+1,r}^{(1)}z_r^{1/2},\quad B_{r+1,r}=x_{r+1,r}^{(2)}z_r^{1/2},\quad C_{r+1,r}=x_{r+1,r}^{(3)}z_r^{1/2},$$

where now all $x_{m,n}^{(i)}$ are limited, eventually $\to 0$. Let us further set

$$(13)\quad \mu_1=(\omega_0^{(1)}/\omega_0^{(3)}),\quad \mu_2=(\omega_0^{(2)}/\omega_0^{(3)}),$$

where $\omega_0^{(i)}$ are algebraic integers, form with x_r, y_r, z_r the expressions

$$(14)\quad \begin{aligned}\omega_0^{(1)}&=x_{r+2}\omega_r^{(1)}+x_{r+1}\omega_r^{(2)}+x_r\omega_r^{(3)},\\ \omega_0^{(2)}&=y_{r+2}\omega_r^{(1)}+y_{r+1}\omega_r^{(2)}+y_r\omega_r^{(3)},\\ \omega_0^{(3)}&=z_{r+2}\omega_r^{(1)}+z_{r+1}\omega_r^{(2)}+z_r\omega_r^{(3)},\end{aligned}$$

and solve with regard to $\omega_r^{(i)}$

$$(14')\quad \begin{aligned}\Delta_{r+2}\omega_r^{(1)}&=A_{r+1,r}\omega_0^{(1)}+B_{r+1,r}\omega_0^{(2)}+C_{r+1,r}\omega_0^{(3)},\\ -\Delta_{r+2}\omega_r^{(2)}&=A_{r+2,r}\omega_0^{(1)}+B_{r+2,r}\omega_0^{(2)}+C_{r+2,r}\omega_0^{(3)},\\ \Delta_{r+2}\omega_r^{(3)}&=A_{r+1,r+1}\omega_0^{(1)}+B_{r+1,r+1}\omega_0^{(2)}+C_{r+1,r+1}\omega_0^{(3)}.\end{aligned}$$

Let us set

$$(14'')\quad \omega_0^{(i)}=a_0^{(i)}\omega_1+a_1^{(i)}\omega_2+a_2^{(i)}\qquad (i=1,2,3)$$

1, ω_1 and ω_2 forming a base in the cubic field; $\Delta_{r+2}\omega_r^{(i)}$ are from (14′) also algebraic integers. By (9′) and (12) the first relation (14′) yields

$$(15)\quad \begin{aligned}\Delta_{r+2}\omega_r^{(1)}&=(\omega_0^{(3)}/z_r)[x_r(y_{r+1}z_r-y_rz_{r+1})+y_r(z_{r+1}x_r-x_{r+1}z_r)\\ &\quad+z_r(x_{r+1}y_r-x_ry_{r+1})+k_r^{(1)}x_{r+1,r}^{(1)}+k_r^{(2)}x_{r+1,r}^{(2)}+x_{r+1,r}^{(3)}]\\ &=(\omega_0^{(3)}/z_r)[k_r^{(1)}x_{r+1,r}^{(1)}+k_r^{(2)}x_{r+1,r}^{(2)}+x_{r+1,r}^{(3)}].\end{aligned}$$

From this relation we infer that $\Delta_{r+2}\omega_r^{(1)}$ is for increasing r at least of the order $1/z_r$, or possibly of a less order $1/z_r^{1+\epsilon}$, $\epsilon>0$. Let us further with (12) and (14″) form the conjugate integers $\Delta_{r+2}\overline{\omega}_r^{(1)}$ and $\Delta_{r+2}\overline{\overline{\omega}}_r^{(1)}$. We have for example

$$(15)\quad \begin{aligned}\Delta_{r+2}\overline{\omega}_r^{(1)}=z_r^{1/2}[&(a_0^{(1)}x_{r+1,r}^{(1)}+a_0^{(2)}x_{r+1,r}^{(2)}+a_0^{(3)}x_{r+1,r}^{(3)})\overline{\omega}_1\\ &+\overline{\omega}_2(a_1^{(1)}x_{r+1,r}^{(1)}+a_1^{(2)}x_{r+1,r}^{(2)}+a_1^{(3)}x_{r+1,r}^{(3)})\\ &+(a_2^{(1)}x_{r+1,r}^{(1)}+a_2^{(2)}x_{r+1,r}^{(2)}+a_2^{(3)}x_{r+1,r}^{(3)})],\end{aligned}$$

and a similar expression in $\Delta_{r+2}\overline{\overline{\omega}}_r^{(1)}$, from which we infer that these conjugate values are of an order of infinity of at most $z_r^{1/2}$ each. Hence the norm $N(\Delta_{r+2}\omega_r^{(1)})$ must either tend to zero or rest limited, different from zero. In the same way we have $N(\Delta_{r+2}\omega_r^{(2)})$ zero or limited and finally

$N(\Delta_{r+2}\omega_r^{(3)})$. We shall in the following deduce an upper limit for Δ_{r+2}. Since however $\Delta_{r+2}\omega_r^{(i)}$ are algebraic integers the formation of A_r would in the former case go to an end, and this must occur for quadratic irrationalities $\omega_0^{(i)}$ with the order of approximation (9′), but it presumes that a linear relation between the initial elements $\omega_0^{(i)}$ is existing. In general the latter must therefore occur, whence the two expressions (15″) must have the order of infinity of precisely $z_r^{1/2}$ as (15′) of precisely $1/z_r$. Hence the above relation must lead to a set of algebraic integers of limited norm just as formerly in the one-dimensional case and quadratic irrationalities. In two dimensions, however, the periodicity is not yet proved. But, writing

$$(16) \qquad \Omega_0^{(i)} = \omega_0^{(i)}/\omega_0^{(3)}, \quad \Omega_r^{(i)} = \Delta_{r+2}\omega_r^{(i)}/\Delta_{r+2}\omega_r^{(3)} \qquad (i = 1, 2).$$

whence

$$(17) \quad \Omega_0^{(1)} = \frac{x_r + x_{r+1}\Omega_r^{(2)} + x_{r+2}\Omega_r^{(1)}}{z_r + z_{r+1}\Omega_r^{(2)} + z_{r+2}\Omega_r^{(1)}}, \quad \Omega_0^{(2)} = \frac{y_r + y_{r+1}\Omega_r^{(2)} + y_{r+2}\Omega_r^{(1)}}{z_r + z_{r+1}\Omega_r^{(2)} + z_{r+2}\Omega_r^{(1)}}$$

we see, since $\Delta_{r+2}\omega_r^{(i)}$ $(i = 1, 2, \text{ and } 3)$ are just proved to have the precise order $1/z_r$, that in

$$(16') \qquad \Omega_r^{(i)} = \frac{\alpha_r^{(i)}\omega_1 + s_r^{(i)}\omega_2 + j_r^{(i)}}{R_r},$$

not only $R_r < M$ and $N(\alpha_r^{(i)}\omega_1 + s_r^{(i)}\omega_2 + j_r^{(i)}) < M$, but $\Omega_r^{(i)}$ itself must be limited and hence also the expression $\alpha_r^{(i)}\omega_1 + \beta_r^{(i)}\omega_2 + j_r^{(i)}$. From this fact we are able to conclude that the formation must become periodic. We write namely for

$$\omega_1 = \frac{a_0^{(1)}\omega^2 + b_0^{(1)}\omega + c_0}{R_0^{(1)}}, \qquad \omega_2 = \frac{a_0^{(2)}\omega^2 + b_0^{(2)}\omega + c_0^{(2)}}{R_0^{(2)}}$$

the numerators of $\Omega_r^{(i)}$ after multiplication with $R_0^{(1)}R_0^{(2)}$

$$\begin{aligned} a_r\omega^2 + b_r\omega + c_r &= \eta_r^{(1)} \\ a_r\overline{\omega}^2 + b_r\overline{\omega} + c_r &= \eta^{(2)} \\ a_r\overline{\overline{\omega}}^2 + b_r\overline{\overline{\omega}} + c_r &= \eta_r^{(3)} \end{aligned}$$

where a_r, b_r, c_r are rational integers, $\eta_r^{(i)}$ limited for all r, and compute, since $\omega + \overline{\omega} + \overline{\overline{\omega}} = -A_1$, $\omega^3 + A_1\omega^2 + A_2\omega + A_3 = 0$, three relations

$$-A_1 - \omega + \frac{b_r}{a_r} = \frac{\eta_r^{(2)} - \eta_r^{(3)}}{a_r(\overline{\omega} - \overline{\overline{\omega}})}.$$

But these three relations are in contradiction if $|a_r| \to \infty$. Hence a_r, b_r, c_r are limited, and the formation periodic. But it is remarkable that the same elements need not follow in the same order in the set of periods; that is,

the periods need not hold the same elements in the same order. The periodicity can be mixed. If $\omega_0^{(i)}$ belong to a field higher than the cubic we get questions and problems almost as in the case of cubic and higher irrationalities in the one-dimensional chains, questions as these, if R_r in (16′) be limited, if $\Omega_r^{(i)}$ themselves are limited, etc. We shall now prove that Δ_r for all r is limited. From

$$\Delta_{r+2} = \begin{vmatrix} x_{r+2} & x_{r+1} & x_r \\ y_{r+2} & y_{r+1} & y_r \\ z_{r+2} & z_{r+1} & z_r \end{vmatrix}, \; \mu_1 - x_r/z_r = k_r^{(1)}/z_r^{3/2}, \; \mu_2 - y_r/z_r = k_r^{(1)}/z_r^{3/2}$$

where $k_r^{(i)}$ and τ_r are limited $\nrightarrow 0$ for increasing r, we infer

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 z_r^3 \begin{vmatrix} \mu_1 - k_{r+2}^{(1)}/z_{r+2}^{3/2}, & \mu_1 - k_{r+1}^{(1)}/z_{r+1}^{3/2}, & \mu_1 - k_r^{(1)}/z_r^{3/2} \\ \mu_2 - k_{r+2}^{(2)}/z_{r+2}^{3/2}, & \mu_2 - k_{r+1}^{(2)}/z_{r+1}^{3/2}, & \mu_2 - k_r^{(2)}/z_r^{3/2} \\ 1, & 1, & 1 \end{vmatrix},$$

and further

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 \begin{vmatrix} \phi_{r+1}^{(1)}, & \phi_r^{(1)}, & (\mu_1 - k_r^{(1)}/z_r^{3/2})z_r^{3/2} \\ \phi_{r+1}^{(2)}, & \phi_r^{(2)}, & (\mu_2 - k_r^{(2)}/z_r^{3/2})z_r^{3/2} \\ 0, & 0, & 1 \end{vmatrix},$$

that is in

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 \begin{vmatrix} \phi_{r+1}^{(1)}, & \phi_r^{(1)} \\ \phi_{r+1}^{(2)}, & \phi_r^{(2)} \end{vmatrix}$$

all elements limited. Hence an upper limit of Δ_{r+2} is readily deduced. By means of the above geometrical considerations it is yet in general possible to limit Δ_{r+2} to the values of some few of the smallest integers.

Let us then propose any relation

$$x_{r+3} = \alpha x_{r+2} + \beta x_{r+1} + \gamma x_r$$

given in rational fractions α, β and γ with

$$(19') \qquad \begin{aligned} \begin{vmatrix} x_{r+3} & y_{r+2} & z_{r+1} \end{vmatrix} &= \Delta_{r+3}, \\ \begin{vmatrix} x_{r+2} & y_{r+1} & z_r \end{vmatrix} &= \Delta_{r+2}. \end{aligned}$$

Then we compute $\gamma = \Delta_{r+3}/\Delta_{r+2}$ and hence the recursion formula

$$(19'') \qquad \Delta_{r+2}x_{r+3} = m_{r+3}x_{r+2} + n_{r+3}x_{r+1} + \Delta_{r+3}x_r,$$

for if x'_{r+3}, y'_{r+3}, z'_{r+3} be any solutions of (19′) we can always determine m_{r+3} and n_{r+3} as integers from (19″), whence this formula must afford all solutions of (19′). Let us then substitute $r+1$ for r in (14′), whence the relations

$$
(20)\qquad
\begin{aligned}
&\omega_r^{(1)}x_{r+2} + \omega_r^{(2)}x_{r+1} + \omega_r^{(3)}x_r = \omega_{r+1}^{(1)}x_{r+3} + \omega_{r+1}^{(2)}x_{r+2} + \omega_{r+1}^{(3)}x_r,\\
&\omega_r^{(1)}y_{r+2} + \omega_r^{(2)}y_{r+1} + \omega_r^{(3)}y_r = \omega_{r+1}^{(1)}y_{r+3} + \omega_{r+1}^{(2)}y_{r+2} + \omega_{r+1}^{(3)}y_r,\\
&\omega_r^{(1)}z_{r+2} + \omega_r^{(2)}z_{r+1} + \omega_r^{(3)}z_r \ + \omega_{r+1}^{(1)}z_{r+3} + \omega_{r+1}^{(2)}z_{r+2} + \omega_{r+1}^{(3)}z_r.
\end{aligned}
$$

Solving this system with respect to $\omega_{r+1}^{(i)}$ we have proved the following relations

$$
(21)\qquad
\begin{aligned}
\Delta_{r+3}\omega_{r+1}^{(1)} &= \Delta_{r+2}\omega_r^{(3)},\\
\Delta_{r+3}\Delta_{r+2}\omega_{r+1}^{(2)} &= -\Delta_{r+2}m_{r+3}\omega_r^{(3)} + \Delta_{r+3}\Delta_{r+2}\omega_r^{(1)},\\
\Delta_{r+3}\Delta_{r+2}\omega_{r+1}^{(3)} &= -\Delta_{r+2}n_{r+3}\omega_r^{(3)} + \Delta_{r+3}\Delta_{r+2}\omega_r^{(2)},
\end{aligned}
$$

also written in form

$$
(21')\qquad
\begin{aligned}
\Delta_{r+3}\Omega_r^{(1)} &= m_{r+3} + \Delta_{r+2}(\Omega_{r+1}^{(2)}/\Omega_{r+1}^{(1)}),\\
\Delta_{r+3}\Omega_r^{(2)} &= n_{r+3} + \Delta_{r+2}(1/\Omega_{r+1}^{(1)}),
\end{aligned}
$$

which suggests that we search for m_{r+3} and n_{r+3} as characteristic integers in $\Omega_r^{(i)}$. Consider the plane

$$
xA_{r+2,r+1} + yB_{r+2,r+1} + zC_{r+2,r+1} = \Delta_{r+3}
$$

carrying all points x_{r+3}, y_{r+3}, z_{r+3}, and determine the intersection of this plane with the vector (4). We find the coordinates

$$
(22)\qquad
\begin{aligned}
x^0{}_{r+3} &= \frac{\mu_1\Delta_{r+3}}{A_{r+2,r+1}\mu_1 + B_{r+2,r+1}\mu_2 + C_{r+2,r+1}},\\
y^0{}_{r+3} &= \frac{\mu_2\Delta_{r+3}}{A_{r+2,r+1}\mu_1 + B_{r+2,r+1}\mu_2 + C_{r+2,r+1}},\\
z^0{}_{r+3} &= \frac{1}{A_{r+2,r+1}\mu_1 + B_{r+2,r+1}\mu_2 + C_{r+2,r+1}},
\end{aligned}
$$

and by varying m_{r+3}, n_{r+3} so as to make (22) satisfy (19″) we compute further

$$
\begin{aligned}
m^0{}_{r+3} &= -\Delta_{r+3}\,\frac{A_{r+1,r}\mu_1 + B_{r+1,r}\mu_2 + C_{r+1,r}}{A_{r+2,r+1}\mu_1 + B_{r+2,r+1}\mu_2 + C_{r+2,r+1}} = \Delta_{r+3}\,\frac{\omega_r^{(1)}}{\omega_r^{(3)}} = \Delta_{r+3}\Omega_r^{(1)},\\
n^0{}_{r+3} &= -\Delta_{r+3}\,\frac{A_{r+2,r}\mu_1 + B_{r+2,r}\mu_2 + C_{r+2,r}}{A_{r+2,r+1}\mu_1 + B_{r+2,r+1}\mu_2 + C_{r+2,r+1}} = \Delta_{r+3}\,\frac{\omega_r^{(2)}}{\omega_r^{(3)}} = \Delta_{r+3}\Omega_r^{(2)}.
\end{aligned}
$$

$m^0{}_{r+3}$ and $n^0{}_{r+3}$ are easy to characterize as the way that we have to proceed along the edges of a tetrahedron in order to reach just the point $\Delta_{r+2}\, x^0{}_{r+3}$, $y^0{}_{r+3}$, $z^0{}_{r+3}$ and consequently m_{r+3} and n_{r+3} are rational integers in $m^0{}_{r+3}$, $n^0{}_{r+3}$, which lead to the "best" approximating point x_{r+3}, y_{r+3}, z_{r+3} close by $x^0{}_{r+3}$, $y^0{}_{r+3}$, $z^0{}_{r+3}$. To determine the integers m_{r+3} and n_{r+3} as well as Δ_r we have to proceed from $\Delta_{r+3} = 1$, decide, as above explained, on the "best" integers in $\Omega_r^{(1)}$ and $\Omega_r^{(2)}$, compute x_{r+3}, y_{r+3}, z_{r+3} and test, if $\rho_{r+2} > \rho_{r+3}$ is true.

If so, the point A_{r+3} is found; if not, we have to repeat this process with $\Delta_{r+3} = 2$, 3 etc. until A_{r+3} will be determined; but we have no algorithm. which gives us m_r and n_r automatically, as in the case of the one-dimensional chains. As a special case of this mode of forming chains we may consider the "Jacobi-chains," in which without any exception Δ_r is equal to one, and m_{r+3}, n_{r+3} are determined as greatest integers in $\Omega_r^{(1)}$ and $\Omega_r^{(2)}$. But, what in this manner is won in simplicity, is a loss in generality, namely loss of periodicity in connection with the best order of approximation, and of this simple proof of convergence. Hence, in order to attain the generalization of the one-dimensional chains to two dimensions we have to release the conditions $\Delta_r = 1$ and permit mixed periodicity. As already indicated, a formation with any quadratic irrationalities must break off, and this is readily proved directly also with $k_r^{(i)} \to 0$. For if $\omega_0^{(i)}$ are quadratic irrationalities there exists in integers a_0, b_0, c_0 a relation

$$a_0(\omega_0^{(1)}/\omega_0^{(3)}) + b_0(\omega_0^{(2)}/\omega_0^{(3)}) + c_0 = 0$$

from which by the approximation (9) it follows that

$$a_0x_r + b_0y_r + c_0z_r \to 0\,;$$

that is

$$a_0x_r + b_0y_r + c_0z_r = 0$$

for $n > N$, which is impossible. But as the Jacobi-chains in two dimensions also possess the same property,* and as they are easy to form, we shall use them for the purpose of solving in general a problem from the theory of numbers. It is however of some interest to show that also with quadratic elements a periodic chain is possible to construct in the following simple way.

$$7 + 3(7)^{1/2} = 14 + \cfrac{1}{\cfrac{3(7)^{1/2}+7}{14}}; \qquad 5 + 2(7)^{1/2} = 10 + \cfrac{\cfrac{7-(7)^{1/2}}{14}}{\cfrac{3(7)^{1/2}+7}{14}}$$

$$\frac{3(7)^{1/2}+7}{14} = 1 + \cfrac{1}{\cfrac{3(7)^{1/2}+7}{1}}; \qquad \frac{7-(7)^{1/2}}{14} = 0 + \cfrac{\cfrac{(7)^{1/2}+2}{1}}{\cfrac{3(7)^{1/2}+7}{1}},$$

$$\frac{3(7)^{1/2}+7}{1} = 14 + \cfrac{1}{\cfrac{3(7)^{1/2}+7}{14}}; \qquad \frac{(7)^{1/2}+2}{1} = 4 + \cfrac{\cfrac{7+(7)^{1/2}}{14}}{\cfrac{3(7)^{1/2}+7}{14}}.$$

* O. Perron, *Sitzungsberichte der Königlichen Bayerschen Akademie der Wissenschaften zu München*, Bd. 38 (1908).

$$\frac{3(7)^{\frac{1}{2}}+7}{14} = 1 + \frac{1}{\dfrac{3(7)^{\frac{1}{2}}+7}{1}}; \qquad \frac{7+(7)^{\frac{1}{2}}}{14} = 0 + \frac{\dfrac{5+2(7)^{\frac{1}{2}}}{1}}{\dfrac{3(7)^{\frac{1}{2}}+7}{1}}.$$

We shall now treat the named problem of solving any ternary, quadratic form in a quadratic field. Let us therefore have the general ternary form

$$f(xyz) = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$$

$$= ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy,$$

and let us assume that by means of any substitution

$$(23) \qquad \begin{aligned} x &= \alpha_0 t + \alpha_1 u, \\ y &= \beta_0 t + \beta_1 u, \\ z &= \gamma_0 t + \gamma_1 u, \end{aligned}$$

the binary quadratic form $(p, q, r) = \psi(t, u)$ is brought into the quality

$$(24) \qquad f(x, y, z) = \psi(t, u).$$

This problem is treated already by Gauss. Putting the root of $\psi(t, u) = 0$ in (24) we have from (23) to each substitution a diophantine solution of

$$(25) \qquad f(x^0 y^0 z^0) = 0$$

in the quadratic field $K[(d)^{\frac{1}{2}}]$, $s^2 d$ or $d = p^2 - qr$, and we observe that the solutions of (25) in $K[(d)^{\frac{1}{2}}]$ can, when existing, be arranged after classes of forms in $K[(d)^{\frac{1}{2}}]$. Now it is interesting that conversely from each solution of (25) it is possible to derive a substitution (23). For, having found the solutions $\mu = x^0/z^0$, $\nu = y^0/z^0$, where x^0, y^0, z^0 are algebraic integers in $K[(d)^{\frac{1}{2}}]$, we form as above the Jacobi-chains, and from the stopping chain we shall see, that a substitution (23) is readily constructed. Hence: The necessary and sufficient condition that any ternary, quadratic form may have diophantine solutions in a quadratic field $K[(d)^{\frac{1}{2}}]$ is the existence of relations (24), which give all solutions. Let us have the ternary form

$$(26) \qquad x^2 + 2y^2 + 3z^2 + 4yz + 4xz + 3xy$$

and seek for example solutions in $K[(7)^{\frac{1}{2}}]$. For $\mu = x/z$, $\nu = y/z$ we find a pair of solutions from

$$\mu = -\frac{3\nu + 4}{2} \pm \frac{(\nu^2 + 8\nu + 4)^{\frac{1}{2}}}{2}, \qquad (\nu^2 + 8\nu + 4 = a^2)$$

$$\nu = 4 \pm (12 + a^2)^{\frac{1}{2}},$$

with, for example, $a=4$, yielding $-\mu=3(7)^{1/2}-6$, $\nu=2(7)^{1/2}-4$. From them we form the following chain

$$\nu=2(7)^{1/2}-4=1+\frac{1}{[2(7)^{1/2}+5]/3=\alpha_1^{(1)}};$$
$$-\mu=3(7)^{1/2}-6=1+\frac{[7+(7)^{1/2}]/3=\alpha_1^{(2)}}{[2(7)^{1/2}+5]/3};$$
$$\frac{7+(7)^{1/2}}{3}=3+\frac{1}{[(7)^{1/2}+2]/1=\alpha_2^{(1)}};$$
$$\frac{2(7)^{1/2}+5}{3}=3+\frac{2=\alpha_2^{(2)}}{[(7)^{1/2}+2]/1};$$

and the chain is breaking off, as it should, already with $\alpha_2^{(2)}=2$ falling in $K(1)$. Hence we compute the following equivalences

$$\nu=\frac{\alpha_2^{(2)}+4\alpha_2^{(1)}}{\alpha_2^{(2)}+3\alpha_2^{(1)}},\qquad -\mu=\frac{1+\alpha_2^{(2)}+6\alpha_2^{(1)}}{\alpha_2^{(2)}+3\alpha_2^{(1)}}.$$

Returning to homogeneous coordinates $\mu=x/z$, $\nu=y/z$, $\alpha_2^{(2)}=x'/z'$, $\alpha_2^{(1)}=y'/z'$ the unimodular substitution

$$\begin{aligned} -x&=x'+6y'+1z'\\ y&=x'+4y'+0z'\\ z&=x'+3y'+0z' \end{aligned}\tag{27}$$

transforms (26) into the equivalent form

$$3x'^2-y'^2+z'^2-12z'y'-5z'x'+8x'y',$$

and by means of the substitution $x'=2t$, $y'=u$, $z'=t$ this form is transformed in the binary form

$$-u^2+4tu+3t^2\tag{28}$$

having just the root $\alpha_2^{(1)}=2+(7)^{1/2}$ from the chain above. Hence the substitution

$$\begin{aligned} -x&=3t+6u\\ y&=2t+4u\\ z&=2t+3u \end{aligned}\tag{29}$$

formed by the chain lead directly from (26) to (28). The transformations of (28) into itself, and the transformations of it into equivalent forms gives rise to further solutions; and we see, how all solutions of (26) must be reached.

I will finally simply sketch the following generalization in the four

dimensional-space $(xyzt)$. The definition of an angle ω between two lines is, as in euclidean space, also here given by

$$\cos\omega = \sum \cos\delta_i \cos\phi_i,$$

where δ_i and ϕ_i represent the direction angles of the two lines with the axes x, y, z, t. Any "plane" has the equation

$$A_1x + A_2y + A_3z + A_4t + A_5 = 0,$$

and cuts a three-dimensional space out of $(xyzt)$. A three-dimensional cube can therefore, for example, exist in this "plane." Its distance from the point x_1, y_1, z_1, t_1 is

$$d = \pm \frac{A_1x_1 + A_2y_1 + A_3z_1 + A_4t_1 + A_5}{[\sum_1^4 A_i^2]^{1/2}}.$$

After these preliminaries we assume μ_1, μ_2 and μ_3 to belong to the same algebraic field and let

$$(30) \qquad x - \mu_1 t = 0, \quad y - \mu_2 t = 0, \quad z - \mu_3 t = 0$$

represent a vector in $(xyzt)$. A plane perpendicular to this vector at the vertex has the equation

$$(31) \qquad x\mu_1 + y\mu_2 + z\mu_3 + t = 0.$$

Let further $\alpha_n^{(i)}$ $(i = 1, 2, 3, 4)$ represent any lattice point; then we compute the cosine between (30) and the line from the vertex to $\alpha_n^{(i)}$ as

$$(32) \qquad \cos\omega_n = \frac{\sum \mu_i \alpha_n^{(i)}}{[\sum \mu_i^2 \cdot \sum \alpha_n^{(i)2}]^{1/2}}, \qquad \mu_4 = 1,$$

that is

$$\sin^2\omega_n = \frac{\sum \mu_i^2 \sum \alpha_n^{(i)2} - [\sum \mu_i \alpha_n^{(i)}]^2}{[\sum \mu_i^2 \sum \alpha_n^{(i)2}]} =$$

$$\frac{\sum_{i=1}^{3} (\alpha_n^{(i)} - \mu_i \alpha_n^{(4)})^2 + \sum_{\tau \neq i=1}^{3} (\mu_\tau \alpha_n^{(i)} - \mu_i \alpha_n^{(\tau)})^2}{\sum \mu_i^2 \sum \alpha_n^{(i)2}}$$

and the length d_{n+1} from $\alpha_{n+1}^{(i)}$ $(i = 1, 2, 3, 4)$ to (31)

$$(33) \qquad d_{n+1} = \frac{\sum_1^4 \mu_i \alpha_{n+1}^{(i)}}{[\sum \mu_i^2]^{1/2}}.$$

Any plane (31′) parallel to (31) has a point common with (30), and around this point in (31′) we lay, as we have already laid a circle in the plane o

the three-dimensional space, a sphere of radius r_n and of volume $4/3\pi r_n^3$. Also here we let this sphere glide along the vector and get a convex figure, by which the above theorem of Minkowski is to be proved with a constant 2^4 instead of the above 2^3. We then deduce the inequality

$$(4/3)\pi d_{n+1} r_n^3 < \text{constant}$$

where

$$r_n = R_n \sin \omega_n, \quad R_n = [\sum \alpha_n^{(i)2}]^{1/2}, \quad d_{n+1} = \frac{\sum \mu_i \alpha_{n+1}^{(i)}}{[\sum \mu_i^2]^{1/2}}.$$

From this our way of constructing new lattice points it follows that

$$\alpha_\tau^{(i)}/\alpha_\tau^4 - \mu_i = \epsilon_\tau^{(i)}, \quad \epsilon_\tau^{(i)} \to 0, \quad \tau \to \infty \qquad (i = 1, 2, 3).$$

Hence, as above

$$[\sum_{i=1}^{3} (\alpha_n^{(i)} - \mu_i \alpha_n^{(4)})^2 + \sum_{\tau \neq i=1}^{3} (\alpha_n^{(i)} \mu_\tau - \alpha_n^{(\tau)} \mu_i)^2]^{3/2} < k_1/\alpha_{n+1}^{(4)},$$

and, since all six terms are positive, so much the more

$$| \alpha_n^{(i)} - \mu_i \alpha_n^{(4)} | < k_2^{(i)}/\alpha_{n+1}^{(4)\,1/3} \qquad (i = 1, 2, 3)$$

and because of the theorem of Mr. Perron $\alpha_{n+1}^{(4)} = \tau_n^{(4)} \alpha_n^{(4)}$, that is

$$(34) \qquad \alpha_n^{(i)}/\alpha_n^{(4)} - \mu_i = k_n^{(i)}/\alpha_n^{(4)} \alpha_n^{(4)\,1/3} \qquad (i = 1, 2, 3)$$

$\tau_n^{(4)}$ and $k_n^{(i)}$ limited, and this is exactly our fundamental formula, from which other data follow. In the same way we may also give the generalization to any higher dimension than four.

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MALMÖ, SWEDEN.

TENSORS OF THE CALCULUS OF VARIATIONS.

By Marie M. Johnson.

In this paper tensors are discussed which are connected with the non-parametric problem of the calculus of variations. The integral

$$J=\int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx$$

is taken along arcs whose equations are $y_i=y_i(x)$ $(x_1 \leqq x \leqq x_2;\ i=1, \cdots, n)$ and which join two fixed points in the $(x, y_1, \cdots, y_n)$ space.

For the parametric problem with the integral

$$J=\int_{u_1}^{u_2} F(y_1, \cdots, y_n, y_1', \cdots, y_n')\, du$$

Murnaghan * has shown that the functions $F_{y_i'}$ form a covariant tensor of rank 1 and that there is a contravariant tensor associated with the equations of the geodesics. It is also possible for the parametric problem to prove that the expressions in Euler's differential equations form a covariant tensor. The Weierstrass E-function, the quadratic form used in the Legendre condition, and the expression in the transversality condition are all invariants.†

In the following pages it is shown that when the non-parametric case is considered the $n+1$ functions $(F-y_1'F_{y_1'}-\cdots-y_n'F_{y_n'})$, $F_{y_i'}$, instead of the n functions $F_{y_i'}$ of the parametric case, are the components of a covariant tensor. Likewise a function has to be added to the n expressions in Euler's differential equations to form a covariant tensor. An application is made of this latter tensor to deduce very simply the relations between canonical differential equations and their transforms by a canonical transformation of coördinates.‡ It is found further that the expression in the transversality condition is an invariant, while the quadratic form of the Legendre condition and the Weierstrass E-function transform so that a factor is introduced. In addition to these results it turns out that there is a covariant tensor of rank 1 which is connected with Jacobi's differential equations. Furthermore the laws of transformation of the two determinants which are used to find the conjugate points of Jacobi's condition are discussed.

* F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, pp. 86-90.

† G. A. Bliss, *Lecture Notes*, Autumn 1926.

‡ See for example, the chapter by Carathéodory in Riemann-Weber, *Die Differential- und Integralgleichungen der Mechanik und Physik*, Teil 1 (1925), pp. 201-205.

1. *Preliminary notions.* In this section we shall des of the non-parametric problem of the calculus of variatio formations of coördinates to which the necessary condition are to be subjected. The laws of transformation of tensors i space are also stated.

In the non-parametric problem of the calculus of varia to be minimized has the form

$$(1) \qquad J = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx \equiv \int_{x_1}^{x_2} F$$

The symbols y and y' stand for the sets $(y_1, \cdots, y_n)$ an respectively, and the primes indicate derivatives with respe tegral is taken along arcs E_{12} which join two fixed points 1 equations are

$$(2) \qquad y_i = y_i(x) \quad (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n).$$

Let $u, v_1, \cdots, v_n$ be new coördinates for which u is variable. The two systems of coördinates are related by transformation

$$(3) \qquad x = x(u, v_1, \cdots, v_n) \equiv x(u, v), \quad y_i = y_i(u, v_1, \cdots,$$
$$(i = 1, \cdots, n),$$

whose Jacobian

$$(4) \qquad \mathcal{J} \equiv \partial(x, y)/\partial(u, v)$$

is different from zero in the region of the $(n+1)$-dimensi discussion. The transformation sets up a one-to-one corres the points of a region of the (x, y) space and the points of t region of the (u, v) space. Also it possesses all the continui are needed in the following arguments. By a transformatio tives $y_i', y_i'', \cdots$ with respect to x along an arc in the $(x,$ expressed in terms of the derivatives with regard to u, v_i', along the corresponding arc in the (u, v) space. This gives

$$(5) \qquad y_i' = \frac{\dfrac{\partial y_i}{\partial u} + \dfrac{\partial y_i}{\partial v_a} v_a'}{\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v_a} v_a'} \qquad (i$$

and further similar formulas for y_i'', etc.

Whenever a Greek subscript or superscript occurs twice intended that a sum of terms shall be represented. The su

letting the index take on the values $1, \cdots, n$ and by summing the resulting expressions.

The inverse of transformation (3) and the solutions of equations (5) for v_i' are given by the following equations:

$$u = u(x, y_1, \cdots, y_n) \equiv u(x, y), \qquad v_i = v_i(x, y_1, \cdots, y_n) \equiv v_i(x, y),$$

$$(6) \qquad v_i' = \frac{\dfrac{\partial v_i}{\partial x} + \dfrac{\partial v_i}{\partial y_a} y_a'}{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y_a} y_a'} \qquad (i = 1, \cdots, n).$$

The Jacobian of this transformation from (x, y) to (u, v) coördinates is the reciprocal of the determinant in (4).

Every arc (2) in the (x, y) space has in the (u, v) space a corresponding arc whose equations are found as follows. By substituting equations (2) in the first equation of (6) we obtain

$$(7) \qquad u = u(x, y(x)).$$

In order to be able to solve for x we make the assumption that $du/dx \neq 0$ along the arcs considered. This is fundamental throughout the succeeding discussions. Let the solution for x of equation (7) be given by $x = x(u)$. By means of this result and equations (2) the first group of n equations in (6) becomes $v_i = v_i(u)$ $(u_1 \leqq u \leqq u_2;\ i = 1, \cdots, n)$ which are the equations of the corresponding arc in the (u, v) space. On this arc the images of the points defined by x_1 and x_2 are determined by values u_1 and u_2 which are found by putting x_1 and x_2 in equation (7).

We will now define tensors whose components in the (x, y) space are functions of x, y_i and the successive derivatives $y_i', y_i'', \cdots$ of y_i with regard to x. Suppose that a set of $n + 1$ functions $A_0(x, y, y', \cdots)$, $A_i(x, y, y', \cdots)$ is transformed by every transformation (3) and associated equations (5) into a new set $a_0(u, v, v', \cdots)$, $a_i(u, v, v', \cdots)$ $(i = 1, \cdots, n)$ in such a way that

$$a_0 = A_0 \partial x/\partial u + A_a \partial y_a/\partial u, \qquad a_i = A_0 \partial x/\partial v_i + A_a \partial y_a/\partial v_i \qquad (i = 1, \cdots, n).$$

Then the functions A_0, A_i are the components in the (x, y) space of a covariant tensor of rank 1 and the functions a_0, a_i are the components of the same tensor in the (u, v) space. If $n + 1$ functions $A^0(x, y, y', \cdots)$, $A^i(x, y, y', \cdots)$ are transformed so that

$$a^0 = A^0 \partial u/\partial x + A^a \partial u/\partial y_a, \qquad a^i = A^0 \partial v_i/\partial x + A^a \partial v_i/\partial y_a \qquad (i = 1, \cdots, n),$$

then A^0, A^i are the components of a contravariant tensor of rank 1. These definitions can be extended readily to those of tensors of higher rank.*

2. *Tensors associated with Euler's differential equations.* The non-parametric problem of the calculus of variations in the (x, y) space is transformed to the (u, v) space by equations (3). First we state the relationship between the integrands of the integrals in the two spaces. By differentiating this relation n components of a covariant tensor are obtained. With these results it is easy to find the covariant tensor of rank 1 which is associated with Euler's equations.

In the (u, v) space let the integral to be minimized be

$$I = \int_{u_1}^{u_2} f(u, v, v')\,du. \tag{8}$$

The integrals in (1) and (8) will have the same value provided they are taken over corresponding arcs in the two spaces and provided

$$f(u, v, v') = F(x, y, y')\;dx/du, \tag{9}$$

in which the right side is a function of (u, v, v') by means of equations (3) and (5).

Differentiation of relation (9) with respect to v_i' shows that

$$f_{v_i'} = (F - y_a' F_{y_a'})\partial x/\partial v_i + F_{y_a'}\,\partial y_a/\partial v_i \qquad (i = 1, \cdots, n), \tag{10}$$

and then it can be verified that

$$(f - v_a' f_{v_a'}) = (F - y_a' F_{y_a'})\partial x/\partial u + F_{y_a'}\,\partial y_a/\partial u. \tag{11}$$

Equations (10) and (11) and the definition of a covariant tensor of rank 1 establish the following theorem.

THEOREM 1. *The functions* $(F - y_a' F_{y_a'})$, $F_{y_i'}$ $(i = 1, \cdots, n)$ *are the components of a covariant tensor of rank* 1.

COROLLARY. *The function*

$$(F - y_a' F_{y_a'})\,dx + F_{y_a'}\,dy_a,$$

in terms of which the transversality condition is stated, is an invariant.

For the differentials dx, dy_i form a contravariant tensor of rank 1. The inner product † of this tensor and the tensor in Theorem 1 gives the expression in the transversality condition which is, therefore, an invariant.

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* L. P. Eisenhart, *Riemannian Geometry*, p. 10.

† L. P. Eisenhart, *loc. cit.*, p. 13.

The tensor associated with the Euler differential equations * may now be obtained. The Euler equations for the integral I are $df_{v_i'}/du - f_{v_i} = 0$. The expressions in these equations can be computed with the help of formulas (9) and (10) and hence we find for $i = 1, \cdots, n$, after some computation,

$$(12) \qquad \frac{d}{du} f_{v_i'} - f_{v_i} = \left[-y_a' \left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) \frac{\partial x}{\partial v_i} + \left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) \frac{\partial y_a}{\partial v_i} \right] \frac{dx}{du} .$$

These equations together with the equation formed by multiplying the above equations by $(-v_i')$ $i = 1, \cdots, n$ respectively and adding give the following theorem.

THEOREM 2. *The* $n+1$ *functions*

$$-\left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) dy_a, \quad \left(\frac{d}{dx} F_{y_i'} - F_{y_i} \right) dx \qquad (i = 1, \cdots, n)$$

are the components of a covariant tensor of rank 1.

3. *Canonical variables, equations, and transformations.* In this section we consider canonical variables and the Hamilton function. The Euler differential equations are replaced by canonical equations. Lastly, a discussion of canonical transformations is given with an example in which the variables are canonical.

The canonical variables † (x, y, z) which are introduced by the equations

$$(13) \qquad z_i = F_{y_i'}(x, y, y') \qquad (i = 1, \cdots, n)$$

are used in place of the element (x, y, y'). If the functional determinant of these equations with respect to y_i' is different from zero, we expect to find solutions $y_i' = P_i(x, y, z)$. Now the definition of the Hamilton function is

$$(14) \qquad H(x, y, z) \equiv y_a' F_{y_a'} - F,$$

in which $P_i(x, y, z)$ is substituted for y_i'. We may prove readily that $P_i = H_{z_i}$ so that the solutions of equations (13) for y_i' in terms of (x, y, z) are also expressible in the form $y_i' = H_{z_i}$. In changing from the element (x, y, y') to the canonical variables (x, y, z) the following property is important. Every solution $y_i(x)$ of class ‡ C'' of the Euler differential equa-

* O. Bolza, *Vorlesungen über Variationsrechnung*, p. 51.

† Riemann-Weber, *loc. cit.*, p. 186.

‡ O. Bolza, *loc. cit.*, p. 13.

tions defines a set of functions $y_i(x)$, $z_i(x)$ by means of equations (13) which satisfy the canonical equations *

$$dy_i/dx - H_{z_i} = 0, \quad dz_i/dx + H_{y_i} = 0 \qquad (i = 1, \cdots, n), \tag{15}$$

and conversely.

Let u, v_i, w_i and $h(u, v, w)$ represent the canonical variables and the Hamilton function in the transformed space. After inserting in equations (10) and (11) the notations of canonical variables we can solve for H and z_i. This gives the equations †

$$\begin{aligned} -H &= -h\partial u/\partial x + w_a \partial v_a/\partial x, \\ z_i &= -h\partial u/\partial y_i + w_a \partial v_a/\partial y_i \qquad (i = 1, \cdots, n). \end{aligned} \tag{16}$$

We may state now Theorem 1 in terms of canonical variables. It says that the set of $n + 1$ functions, $-H$, z_i $(i = 1, \cdots, n)$, are the components of a covariant tensor of rank 1.

If canonical variables are used, the invariance of the expression in the transversality condition which is given in the corollary to Theorem 1 may be stated as follows:

$$-hdu + w_a dv_a = -Hdx + z_a dy_a. \tag{17}$$

It will be shown that the transformation between the two $(2n + 1)$-dimensional spaces, (x, y, z) and (u, v, w), which is given by equations (3) and the last n equations in (16), has the property that the solutions of the canonical equations (15) of the original problem go into the solutions of the canonical equations of the transformed problem, a result which will be seen to depend upon formula (17). The transformation is then a somewhat special case of transformations which are called canonical.‡

The following is the definition of a canonical transformation.§ A transformation from (u, v, w) to (x, y, z) coördinates

$$x = x(u, v, w), \quad y_i = y_i(u, v, w), \quad z_i = z_i(u, v, w) \quad (i = 1, \cdots, n), \tag{18}$$

is said to be canonical if it is a one-to-one transformation with its functional determinant $\partial(x, y, z)/\partial(u, v, w)$ not zero, and if three functions $H(x, y, z)$, $h(u, v, w)$, $\Psi(x, y, z)$, exist for which the relation

* Riemann-Weber, *loc. cit.*, p. 191.

† Riemann-Weber, *loc. cit.*, p. 200.

‡ *Encyklopädie der Mathematischen Wissenschaften*, Band 2, Teil 1, Heft 1 (1915), pp. 343-346.

§ See, for example, Carathéodory's definition in Riemann-Weber, *loc. cit.*, pp. 201-204.

$$-hdu + w_a dv_a = -Hdx + z_a dy_a + d\Psi, \tag{19}$$

is identically satisfied.

We shall consider the problem of the calculus of variations which is connected with the integral

$$J = \int_{x_1}^{x_2} (-H + z_a y_a' + d\Psi/dx)\, dx.$$

Since the Euler equations of a function which is the derivative of another function are identically zero,* the function $d\Psi/dx$ may be omitted in calculating these equations for this problem. The Euler equations are found to be the canonical equations

$$dz_i/dx + H_{y_i} = 0, \quad -(dy_i/dx - H_{z_i}) = 0 \qquad (i = 1, \cdots, n).$$

Hence by Theorem 2 when the dependent variables are $y_1, \cdots, y_n, z_1, \cdots, z_n$ we have the theorem:

THEOREM 3. *The expressions*

$$-(H_{y_a} dy_a + H_{z_a} dz_a), \quad (dz_i/dx + H_{y_i})\, dx,$$
$$-(dy_i/dx - H_{z_i})\, dx \qquad (i = 1, \cdots, n),$$

are the $2n + 1$ *components of a tensor of rank* 1 *with respect to transformations from the* (x, y, z)*- to the* (u, v, w)*-space.*

The transformation equations for this tensor are deduced by Carathéodory in the reference cited in a quite different and much less simple manner.

A fundamental property of tensors is that if the components of a tensor are zero in one coördinate system, then they are zero in every other coördinate system. Thus we have the following corollary:

COROLLARY. *By means of a canonical transformation the solutions of the canonical equations* (15) *of the original problem are transformed into the solutions of the canonical equations of the new problem.*

Since the relation (17) is a special case of (19), the transformation defined by equations (3) and the last n equations in (16) comes under the definition of canonical transformations, provided that the functional determinant is not zero. Since the equations $h_{w_i} = v_i'$ are always satisfied, as indicated in the second paragraph of Section 3, and with the help of the value of the determinant

$$| \partial v_i/\partial y_j - v_i' \partial u/\partial y_j | = \mathfrak{J}^{-1} dx/du \quad (i, j = 1, \cdots, n), \tag{20}$$

* O. Bolza, *loc. cit.*, p. 153.

where J is the Jacobian (4), it can be shown from equations (3) and the last n equations (16) that the functional determinant of x, y_i, z_i with respect to u, v_i, w_i has the value

$$\partial(x, y, z)/\partial(u, v, w) = \mathfrak{J}\mathfrak{J}^{-1} dx/du = dx/du,$$

which is different from zero along the arcs considered.

4. *Tensor properties of the conditions of Weierstrass and Legendre.* The Weierstrass E-function, in terms of which the necessary condition of Weierstrass* is expressed, may be written in the form

$$\begin{aligned} E(x, y, y', Y') = F(x, y, Y') - Y_\alpha' F_{y_\alpha'}(x, y, y') \\ + [y_\alpha' F_{y_\alpha'}(x, y, y') - F(x, y, y')], \end{aligned}$$

in which Y' is the symbol for the set $(Y_1', \cdots, Y_n')$. In the (u, v) space we define

$$(21) \qquad \begin{aligned} e(u, v, v', V') = f(u, v, V') - V_\alpha' f_{v_\alpha'}(u, v, v') \\ + [y_\alpha' f_{v_\alpha'}(u, v, v') - f(u, v, v')], \end{aligned}$$

where V' is the symbol for the set $(V_1', \cdots, V_n')$.

Transform the element (u, v, v') by the transformation (6), while the equations of transformation for the sets V' and Y' are those for the sets v' and y' with v' and y' replaced by V' and Y' respectively. From relation (9) and the equations of transformation for the sets V' and Y' it is seen that

$$f(u, v, V') = \{F(x, y, Y')\}\{1/[\partial u/\partial x + (\partial u/\partial y_\alpha) Y_\alpha']\}.$$

If this result is substituted in formula (21) together with those given in formulas (10) and (11) and the equations of transformation for the set V', then we obtain an expression for e which can be reduced to the following form:

$$(22) \qquad e(u, v, v', V') = \{E(x, y, y', Y')\}\{\partial x/\partial u + (\partial x/\partial v_\alpha) V_\alpha'\}.$$

This proves the theorem:

THEOREM 4. *The Weierstrass E-function is transformed by every transformation* (3) *and associated equations so that equation* (22) *is identically satisfied.*

The statement of the necessary condition of Legendre† involves the quadratic form $F_{y_\alpha' y_\beta'} \eta_\alpha \eta_\beta$. Concerning this form we find

* Riemann-Weber, *loc. cit.*, p. 182.

† Riemann-Weber, *loc. cit.*, p. 183.

THEOREM 5. *The function*

$$F_{y_\alpha' y_\beta'}\eta_\alpha\eta_\beta(1/dx)$$

is an invariant if the variations η_i $(i=1,\cdots,n)$ *are transformed by the relations* (31) *given below.*

The proof is as follows. Consider the one-parameter family of arcs through points 1 and 2

$$y_i = y_i(x,a) \quad (x_1 \leqq x \leqq x_2;\ i=1,\cdots,n), \tag{23}$$

which contains the extremal E_{12} for $a=0$. Then the set of variations η_i along E_{12} are defined by the equations

$$\eta_i(x) = y_{ia}(x,0) \qquad (i=1,\cdots,n). \tag{24}$$

In the (u,v) space the equations of the family of curves (23) have to be found. Substitute equations (23) into the first equation of transformation (6) and obtain

$$u = u(x, y(x,a)). \tag{25}$$

If $du/dx \neq 0$, the solution for x will be given by $x = x(u,a)$. When this result and equations (23) are put in the first group of n equations in (6), we secure the equations

$$v_i = v_i[x(u,a),\, y(x(u,a),\, a)] \equiv v_i(u,a) \quad (i=1,\cdots,n), \tag{26}$$

which represent the curves (23) in the (u,v) space. And as before we define

$$\zeta_i(u) = v_{ia}(u,0) \qquad (i=1,\cdots,n). \tag{27}$$

In order to find the law of transformation between η_i and ζ_i differentiate equations (26) with regard to the parameter and then set $a=0$ so that

$$\zeta_i = [(dv_i/dx)x_a + (\partial v_i/\partial y_\alpha)y_{\alpha a}]_{a=0}. \tag{28}$$

If the solution for x of equation (25) is substituted in (25), an identity is procured which is differentiated with respect to the parameter. This gives the relation

$$0 = [(du/dx)x_a + (\partial u/\partial y_\alpha)y_{\alpha a}]_{a=0}. \tag{29}$$

When the value of x_a from (29) and the definitions of η_i are put in (28), we find

$$\zeta_i = (\partial v_i/\partial y_\alpha - v_i'\partial u/\partial y_\alpha)\eta_\alpha \qquad (i=1,\cdots,n). \tag{30}$$

The determinant of the coefficients of η_i is the determinant in (20). By solving for η_i we have the desired law of transformation between η_i and ζ_i:

$$\eta_i = (\partial y_i/\partial v_a - y_i' \partial x/\partial v_a)\zeta_a \qquad (i = 1, \cdots, n). \tag{31}$$

By differentiating equations (10) with respect to v_j' it is found that

$$f_{v_i' v_j'} = F_{y_a' y_\beta'}(\partial y_a/\partial v_i - y_a' \partial x/\partial v_i)(\partial y_\beta/\partial v_j - y_\beta' \partial x/\partial v_j)\, du/dx.$$

Multiply by $\zeta_i\zeta_j$ and sum for $i, j = 1, \cdots, n$. On account of relations (31) this gives

$$f_{v_a' v_\beta'}\zeta_a\zeta_\beta(1/du) = F_{y_a' y_\beta'}\eta_a\eta_\beta(1/dx),$$

so that the theorem is proved.

5. *A tensor associated with Jacobi's differential equations.* A tensor associated with Jacobi's differential equations is deduced by a direct method although it can be found from the law of transformation of the integrand of the second variation of the integral J.

It is necessary to make a few preliminary remarks in regard to notations in the two spaces for Jacobi's equations and to derive some properties of the variations. If the family of arcs (23) is substituted in the integral J, the function $J(a)$ has the second derivative

$$J''(0) = \int_{x_1}^{x_2} (F_{y_a} y_{aaa} + F_{y_a'} y'_{aaa} + 2\Omega(x, \eta, \eta'))\, dx,$$

where the function 2Ω is defined as follows:

$$2\Omega = F_{y_a y_\beta}\eta_a\eta_\beta + 2F_{y_a y_\beta'}\eta_a\eta_\beta' + F_{y_a' y_\beta'}\eta_a'\eta_\beta'. \tag{32}$$

Since the family of arcs (23) contains the extremal E_{12} for $a = 0$, the Euler equations $dF_{y_i'}/dx - F_{y_i} = 0$ are satisfied for $a = 0$. This result and the vanishing of the values of $y_{iaa}(x, 0)$ at the limits x_1 and x_2 (see equations (37)) give us finally, by integrating the second term of the integrand in $J''(0)$ by parts,

$$J''(0) = \int_{x_1}^{x_2} 2\Omega(x, \eta, \eta')\, dx. \tag{33}$$

The Jacobi differential equations* are by definition the equations

$$d\Omega_{\eta_i'}/dx - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n). \tag{34}$$

If in the (u, v) space the corresponding family of arcs (26) is substituted in the integral I, the function $I(a)$ is obtained. By a reduction

* G. A. Bliss, *Calculus of Variations*, First Carus Monograph, p. 163.

which is similar to that for the second variation in the (x, y) space we find that

$$I''(0) = \int_{u_1}^{u_2} 2\omega(u, \zeta, \zeta')\,du.$$

where 2ω has the form

$$(35) \qquad 2\omega = f_{v_\alpha v_\beta}\zeta_\alpha\zeta_\beta + 2f_{v_\alpha v_\beta'}\zeta_\alpha\zeta_\beta' + f_{v_\alpha' v_\beta'}\zeta_\alpha'\zeta_\beta'.$$

The Jacobi differential equations in this space are

$$(36) \qquad d\omega_{\zeta_i'}/du - \omega_{\zeta_i} = 0 \qquad (i = 1, \cdots, n).$$

Let the sets y_{i1}, y_{i2} denote the values of y_i $(i = 1, \cdots, n)$ for x_1 and x_2 respectively. Since the curves whose equations are given in (23) go through the points 1 and 2, we have $y_{i1} = y_i(x_1, a)$, $y_{i2} = y_i(x_2, a)$. By differentiation with respect to a we find

$$(37) \qquad \begin{aligned} 0 &= y_{ia}(x_1, 0), & 0 &= y_{ia}(x_2, 0), \\ 0 &= y_{iaa}(x_1, 0), & 0 &= y_{iaa}(x_2, 0) \end{aligned} \qquad (i = 1, \cdots, n).$$

The set of variations (24) is now seen to satisfy the conditions

$$(38) \qquad \eta_i(x_1) = \eta_i(x_2) = 0 \qquad (i = 1, \cdots, n).$$

Since the values u_1 and u_2 of the independent variable u in the (u, v) space locate points in that space which correspond respectively to the points determined by x_1 and x_2 in the (x, y) space, then the conditions (38) should imply $\zeta_i(u_1) = \zeta_i(u_2) = 0$, and conversely. This result is proved if it is observed that the determinant of the coefficients of η_i in (30) has the value $\mathcal{J}^{-1}dx/du$ which is not zero. The converse is proved similarly by means of equations (31).

In deriving the tensor associated with Jacobi's equations (34) equations (12) are used. In these equations substitute for y_i and v_i the equations of the corresponding families of arcs (23) and (26) respectively. After this result is differentiated with regard to a, let a be set equal to zero. Since the arc E_{12} obtained for $a = 0$ is an extremal, we note that the Euler equations $dF_{y_i'}/dx - F_{y_i} = 0$ are satisfied for $a = 0$. By means of this fact and the definitions of the functions Ω and ω it is easily shown that

$$(39) \qquad \begin{aligned} \frac{d}{du}\,\omega_{\zeta_i'} - \omega_{\zeta_i} = &- y_a'\left(\frac{d}{dx}\,\Omega_{\eta_a'} - \Omega_{\eta_a}\right)\frac{dx}{du}\,\frac{\partial x}{\partial v_i} \\ &+ \left(\frac{d}{dx}\,\Omega_{\eta_a'} - \Omega_{\eta_a}\right)\frac{dx}{du}\,\frac{\partial y_a}{\partial v_i} \qquad (i = 1, \cdots, n). \end{aligned}$$

These equations together with the tensor equation which is secured by multi-

plying equations (39) by $-v_i'$ $(i=1,\cdots,n)$, respectively, and adding establish the following theorem:

THEOREM 6. *The* $n+1$ *functions*

$$-y_a'(d\Omega_{\eta_a'}/dx-\Omega_{\eta_a})dx, \quad (d\Omega_{\eta_i'}/dx-\Omega_{\eta_i})dx \qquad (i=1,\cdots,n),$$

are the components of a covariant tensor of rank 1.

6. *Tensor properties of the determinants which determine conjugate points.* Conjugate points in Jacobi's condition can be determined easily in either one of two ways, involving in the one case the complete $2n$-parameter family of extremals and in the other case an n-parameter family of extremals which pass through the fixed point 1 and contain the extremal E_{12}. In each case the conjugate points are defined by the zeros of a determinant.* We will find the law of transformation of each determinant when the variables are subjected to a transformation (3).

In the first place we will consider the complete $2n$-parameter family of extremals

$$(40) \qquad y_i=y_i(x,a_1,\cdots,a_n,b_1,\cdots,b_n)\equiv y_i(x,a,b) \quad (i=1,\cdots,n),$$

which contains the extremal E_{12} for the set of parameter values $(a_0,b_0)\equiv(a_{10},\cdots,a_{n0},b_{10},\cdots,b_{n0})$. The points 3 conjugate to point 1 on the extremal E_{12} are determined by the zeros $x\neq x_1$ of the determinant

$$(41) \qquad D(x,x_1,a_0,b_0)=\begin{vmatrix} y_{ia_j} & y_{ib_j} \\ y_{ia_j}(1) & y_{ib_j}(1)\end{vmatrix} \quad (i,j=1,\cdots,n).$$

In this determinant and following expressions where the parameters are not indicated, they are assumed to have the values given by the set (a_0,b_0). Also when the value of a function is taken at point 1 we write $y_{ia_j}(1)$ for example.

In changing to (u,v) coördinates it is necessary to find the equations of the family of extremals in the (u,v) space which correspond to the family (40). Using the method by which the equations (26) were obtained the equations of the family of curves (40) in the (u,v) space are found to be

$$(42) \qquad v_i=v_i[x(u,a,b),y(x,a,b)]\equiv v_i(u,a,b) \quad (i=1,\cdots,n).$$

The conjugate points on the corresponding extremal e_{12} are determined by the zeros $u\neq u_1$ of the determinant

* G. A. Bliss, *loc. cit.*, pp. 148-151.

$$(43) \qquad d(u, u_1, a_0, b_0) = \begin{vmatrix} v_{ia_j} & v_{ib_j} \\ v_{ia_j}(1) & v_{ib_j}(1) \end{vmatrix} \qquad (i, j = 1, \cdots, n).$$

In order to transform determinant (41) the following results will be needed. As equations (29) and (28) were found in Section 4, we find now the equations (44) when the set of parameters (a, b) is given the value (a_0, b_0) of the extremal E_{12}:

$$(44) \qquad \begin{aligned} 0 &= (du/dx)x_{a_j} + (\partial u/\partial y_a)y_{aa_j}, \\ 0 &= (du/dx)x_{b_j} + (\partial u/\partial y_a)y_{ab_j}, \qquad (j = 1, \cdots, n), \\ v_{ia_j} &= (dv_i/dx)x_{a_j} + (\partial v_i/\partial y_a)y_{aa_j}, \\ v_{ib_j} &= (dv_i/dx)x_{b_j} + (\partial v_i/\partial y_a)y_{ab_j}, \qquad (i, j = 1, \cdots, n). \end{aligned}$$

Since each of the expressions for $v_{ia_j}(u, a_0, b_0)$ and $v_{ib_j}(u, a_0, b_0)$ contains $n + 1$ terms, it is advisable to increase the order of the determinant (43) from $2n$ to $2n + 2$. With the help of equations (44) this determinant may be written as follows:

$$d(u, u_1, a_0, b_0) =$$

$$\begin{vmatrix} 1 & \dfrac{du}{dx}x_{a_j} + \dfrac{\partial u}{\partial y_a}y_{aa_j} & 0 & \dfrac{du}{dx}x_{b_j} + \dfrac{\partial u}{\partial y_a}y_{ab_j} \\ \dfrac{dv_i}{du} & \dfrac{dv_i}{dx}x_{a_j} + \dfrac{\partial v_i}{\partial y_a}y_{aa_j} & 0 & \dfrac{dv_i}{dx}x_{b_j} + \dfrac{\partial v_i}{\partial y_a}y_{ab_j} \\ 0 & \left(\dfrac{du}{dx}\right)_1 x_{a_j}(1) + \left(\dfrac{\partial u}{\partial y_a}\right)_1 y_{aa_j}(1) & 1 & \left(\dfrac{du}{dx}\right)_1 x_{b_j}(1) + \left(\dfrac{\partial u}{\partial y_a}\right)_1 y_{ab_j}(1) \\ 0 & \left(\dfrac{dv_i}{dx}\right)_1 x_{a_j}(1) + \left(\dfrac{\partial v_i}{\partial y_a}\right)_1 y_{aa_j}(1) & \left(\dfrac{dv_i}{du}\right)_1 & \left(\dfrac{dv_i}{dx}\right)_1 x_{b_j}(1) + \left(\dfrac{\partial v_i}{\partial y_a}\right)_1 y_{ab_j}(1) \end{vmatrix}$$

where the subscript 1 indicates that the value is taken at point 1. The determinant may now be factored easily so that the following relation is found:

$$(45) \qquad d(u, u_1, a_0, b_0) = \mathfrak{J}^{-1}(\mathfrak{J}^{-1})_1 D(x, x_1, a_0, b_0) \frac{dx}{du}\left(\frac{dx}{du}\right)_1,$$

where $\mathfrak{J}^{-1}$ is the Jacobian of the transformation (6) and the subscript 1 denotes that the value is taken at point 1.

We can state now the following theorem:

THEOREM 7. *When there is given a $2n$-parameter family of extremals which includes the extremal E_{12} for the set of parameters (a_0, b_0), the determinant $D(x, x_1, a_0, b_0)$ for the determination of points conjugate to point* 1 *on the extremal E_{12} is transformed as in formula* (45).

Secondly, we consider an n-parameter family of extremals $y_i = y_i(x, a_1, \cdots, a_n) \equiv y_i(x, u)$, which includes the extremal E_{12} for the particular para-

metric values $(a_0) \equiv (a_{10}, \cdots, a_{n0})$ and all the extremals pass through the point 1. The points 3 conjugate to point 1 on the extremal arc E_{12} are determined by the zeros $x \neq x_1$ of the determinant

$$\Delta(x, a_0) \equiv | y_{ia_j}(x, a_0) | \qquad (i, j = 1, \cdots, n).$$

In the (u, v) space the conjugate points on the extremal e_{12}, corresponding to the extremal E_{12}, are determined by the zeros $u \neq u_1$ of the determinant

$$\delta(u, a_0) \equiv | v_{ia_j}(u, a_0) | \qquad (i, j = 1, \cdots, n),$$

where $v_i = v_i(u, a)$ are the equations of the corresponding family of extremals. The method of the first case now shows the law of transformation of the determinant $\Delta(x, a_0)$ to be

$$\delta(u, a_0) = \mathcal{J}^{-1} \Delta(x, a_0) \, dx/du. \tag{46}$$

Hence we have

THEOREM 8. *When there is given an n-parameter family of extremals through point* 1 *which includes the extremal E_{12} for the set of parameters* (a_0), *the determinant* $\Delta(x, a_0)$ *for the determination of points conjugate to point* 1 *on the extremal E_{12} is transformed as in formula* (46).

CAYLEY'S DEFINITION OF NON-EUCLIDEAN GEOMETRY.

By JAMES PIERPONT.

1. Cayley in his Sixth Memoir upon Quantics (1859) laid the foundations of non-euclidean geometry. Although this paper is often referred to, its true significance seems to have been entirely overlooked. Led by Klein, whose autographed lectures on Nicht Euklidische Geometrie (1892) have enjoyed the widest popularity, another point of view has been adopted. This however depends on finding a *projective* definition of coördinates and cross ratios; and this can be done rigorously only by an intricate piece of reasoning.

The purpose of the present paper is to develop the implications of Cayley's paper and to show how naturally they lend themselves to a systematic and rigorous treatment of this subject.

2. Cayley takes four numbers x_1, x_2, x_3, x_4 and regards their ratios $x_1 : x_2 : x_3 : x_4$ as defining a point whose coördinates are the x's. In the notes to volume 2 which Cayley prepared for his "Collected Papers" he says, p. 605;—"As to my memoir, the point of view was that I regarded "coördinates" not as distances or ratios of distances but as an assumed fundamental notion, not requiring or admitting of explanation." It is well to remember in this connection that the notion of an abstract group is also due to Cayley (1854).

A straight is defined as the points $x_i = la_i + mb_i$, $(i = 1, 2, 3, 4)$, where a_i, b_i are the coördinates of two points a, b, and l, m are parameters. A plane is defined by $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$. It is easy to show that if a, b are two points of a plane, then all points on their join also lie on the plane. Also all points common to two planes lie on a straight.

To define distance and angle Cayley introduces the quadratic forms

$$(1) \qquad (x, x) = F(x, x) = \sum a_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \qquad (i, j = 1, 2, 3, 4)$$

of non-vanishing determinant $a = \det | a_{ij} |$ and

$$(2) \qquad G(u, u) = \sum a^{ij}u_iu_j, \qquad (a^{ij} = \text{Minor of } a_{ij}/a).$$

The distance δ between two points x, y we define by

$$(3) \qquad \cos(\delta/c) = F(x, y)/[F(x, x)F(y, y)]^{1/2},$$

and the angle ϕ between two planes u, v we define by

$$(4) \qquad \cos(\phi/c') = G(u, v)/[G(u, u)G(v, v)]^{1/2},$$

Cayley takes $c = c' = 1$. We, following Klein, choose them so that δ, ϕ may be real when possible. From (3) we have

$$(5) \qquad \sin(\delta/c) = \Delta^{1/2}/[F(x,x)F(y,y)]^{1/2}, \qquad (\Delta = F(x,x)F(y,y) - F(x,y)^2$$

3. Let a, b, c be three points on a straight, let $\gamma = \text{dist}(ab)$, $\eta = \text{dist}(bc)$ $\delta = \text{dist}(ac)$. We wish to show that $\delta = \gamma + \eta$. A similar proof holds for the addition of angles. Consider

$$\Delta = \cos[(\gamma + \eta)/c] = \cos(\gamma/c)\cos(\eta/c) - \sin(\gamma/c)\sin(\eta/c)$$

and let us write for the sake of brevity (a, b) as ab, etc. Then

$$\Delta = [ab/(aa \cdot bb)^{1/2}][bc/(bb \cdot cc)^{1/2}] \\ -(aa \cdot bb - ab^2)^{1/2}(bb \cdot cc - bc^2)^{1/2}/(aa \cdot bb)^{1/2}(bb \cdot cc)^{1/2}.$$

Let us denote the numerator of the last term on the right by $A^{1/2}$. Then

$$A = (aa \cdot bb - ab^2)(bb \cdot cc - bc^2).$$

Now

$$\begin{array}{ll} b = \alpha a + \beta c, & (bb) = \alpha^2(aa) + 2\alpha\beta(ac) + \beta^2(cc) \\ (ab) = \alpha(aa) + \beta(ac), & (bc) = \alpha(ac) + \beta(cc) \end{array}$$

$$aa \cdot bb - ab^2 = \beta^2(aa \cdot cc - ac^2)$$
$$bb \cdot cc - bc^2 = \alpha^2(aa \cdot cc - ac^2)$$
$$A = \alpha^2\beta^2(aa \cdot cc - ac^2)^2.$$

Hence

$$\Delta = [ab \cdot bc - \alpha\beta(aa \cdot cc - ac^2)]/bb(aa \cdot cc)^{1/2} = B/C,$$
$$ab \cdot bc = \alpha^2 \cdot aa \cdot ac + \alpha\beta(ac^2 + aa \cdot cc) + \beta^2\, ac \cdot cc.$$

Hence

$$B = \alpha^2 \cdot aa \cdot ac + 2\alpha\beta\, ac^2 + \beta^2\, ac \cdot cc = (ac)(bb)$$

Thus

$$\Delta = (ac)(bb)/(bb)[(aa)(cc)]^{1/2} = (ac)/[(aa)(cc)]^{1/2} = \cos(\delta/c).$$

Hence

$$\cos(\delta/c) = \cos[(\gamma + \eta)/c] \quad \text{or} \quad \delta = \gamma + \eta.$$

We can now express the parameters in $x_i = \alpha a_i + \beta b_i$. We find at once

$$(6) \qquad \begin{aligned} \alpha &= [(xx)/(aa)]^{1/2} \sin[(x,b)/c]/\sin[(a,b)/c], \\ \beta &= [(xx)/(bb)]^{1/2} \sin[(a,x)/c]/\sin[(a,b)/c], \end{aligned}$$

where we have set $(x, b) = \text{dist}(x, b)$, etc.

In fact

$$(xx) = \alpha^2(aa) + 2\alpha\beta(ab) + \beta^2(bb).$$

Let $$\gamma = \operatorname{dist}(a, x) \qquad \delta = \operatorname{dist}(a, b),$$

then by (3)

$$(7) \qquad (xx) = \alpha^2 (aa) + \beta^2 (bb) + 2\alpha\beta[(aa)(bb)]^{1/2} \cos(\delta/c).$$

Also by (3)

$$\begin{aligned}[(aa)(xx)]^{1/2} \cos(\gamma/c) = (ax) &= \alpha(aa) + \beta(ab) \\ &= \alpha(aa) + \beta[(aa)(bb)]^{1/2} \cos(\delta/c).\end{aligned}$$

Hence squaring

$$(xx)\cos^2(\gamma/c) = \alpha^2(aa) + 2\alpha\beta[(aa)(bb)]^{1/2}\cos(\delta/c) + \beta^2(bb)\cos^2(\delta/c),$$

or using (7)

$$(xx)\cos^2(\gamma/c) = (xx) - \beta^2(bb) + \beta^2(bb)\cos^2(\delta/c)$$

or $$(xx)\sin^2(\gamma/c) = \beta^2(bb)\sin^2(\delta/c).$$

This is the second equation in (6); we get α similarly.

4. The plane

$$(8) \qquad (g, x) = \sum a_{ij} g_i x_j = \sum_j x_j \sum_i a_{ij} g_i = \sum x_j u_j = 0$$

we call the absolute polar of g, or simply the polar of g. The distance between g and any point x on this plane is such that $\cos \delta/c = 0$. Hence $\delta = \pi c/2$. We call g the pole of (8). We ask how many poles g, does a plane

$$(9) \qquad c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0,$$

have i. e. how many points g are there such that the polar of g, $(gx) = 0$, is identical with (9). For (8) to be identical with (9) it is necessary and sufficient that

$$\rho c_j = a_{j1} g_1 + a_{j2} g_2 + a_{j3} g_3 + a_{j4} g_4.$$

Since the determinant a of $F(xx)$ is $\neq 0$, we have

$$(10) \qquad g_i = \rho(a^{i1} c_1 + a^{i2} c_2 + a^{i3} c_3 + a^{i4} c_4).$$

Thus $g_1 : g_2 : g_3 : g_4$ is uniquely determined. The plane (9) has thus a unique pole whose coördinates are given by (10).

The equations of the straight joining the points a, b are $x_i = \alpha a_i + \beta b_i$. Suppose b lies on the polar of a. Set distance $(a, x) = s$, then distance $(x, b) = \pi c/2 - s$. Then (6), (7) give

$$\alpha = [(xx)/(aa)]^{1/2} \cos(s/c), \quad \beta = [(xx)/(bb)]^{1/2} \sin(s/c),$$
$$\beta/\alpha = [(aa)/(bb)]^{1/2} \tan(s/c).$$

The point x on the straight lies on the plane $L(x) = u_1 x_1 + \cdots u_4 x_4 = 0$,

when α, β are such that $\alpha L(a) + \beta L(\beta) = 0$, which determines β/α unless $L(b) = 0$, in which case $x = b$. Thus a straight cuts a plane but once.

When b lies on the polar of a we say b is at a quadrant's distance from a.

5. The distance δ from a point x to the plane (9) we define in this way: Let g be the pole of (9). The join of g, x cuts (9) in a point h and by definition distance $(g, h) = \pi c/2$. We define $\delta = \mathrm{dist}(x, h)$. Let $\eta = \mathrm{dist}(g, x)$, then $\eta + \delta = \pi c/2$. Now

$$\cos(\eta/c) = (g, x)/[(g, g)(x, x)]^{1/2} = \cos[(\pi c/2 - \delta)/c] = \sin(\delta/c).$$

Thus

$$(11) \qquad \sin(\delta/c) = (g, x)/[(g, g)(x, x)]^{1/2}.$$

This expression involves $g_1 \cdots g_4$ which are not given directly. Now

$$(12) \qquad \begin{aligned}(g, x) &= \sum a_{ij} g_i x_j = \sum_j x_j \sum_i a_{ij} g_j = \sum_j x_j \sum_i a_{ij} \rho \sum_k a^{ik} c_k \\ &= \rho \sum_j x_j \sum_k c_k \sum_i a_{ij} a^{ik} = \rho \sum_j c_j x_j.\end{aligned}$$

$$(13) \qquad \begin{aligned}(g, g) &= \sum a_{ij} g_i g_j = \rho^2 \sum_{ij} a_{ij} \sum_\alpha a^{i\alpha} c_\alpha \sum_\beta a^{j\beta} c_\beta \\ &= \rho^2 \sum_{\alpha\beta} c_\alpha c_\beta \sum_i a^{i\alpha} \sum_j a_{ij} a^{j\beta} = \rho^2 \sum_{\alpha\beta} a^{\alpha\beta} c_\alpha c_\beta \\ &= \rho^2 G(c, c).\end{aligned}$$

These in (11) give

$$(14) \qquad \begin{aligned}\sin(\delta/c) &= \rho(c_1 x_1 + \cdots + c_4 x_4)/[F(g, g) F(x, x)]^{1/2} \\ &= (c_1 x_1 + \cdots + c_4 x_4)/[F(x, x) G(c, c)]^{1/2}.\end{aligned}$$

The four planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ define a tetrahedron which we call τ. The distance δ_i of a point x to the plane x_i is given by

$$(15) \qquad \sin(\delta_i/c) = x_i/(a^{ii})^{1/2}[(x, x)]^{1/2}.$$

Thus

$$x_1 : x_2 : x_3 : x_4 = (a^{11})^{1/2} \sin(\delta_1/c) : (a^{22})^{1/2} \sin(\delta_2/c) : (a^{33})^{1/2} \sin(\delta_3/c) : (a^{44})^{1/2} \sin(\delta_4/c).$$

This gives a geometric interpretation of the coördinates of a point in terms of distance.

Let V_i be the vertex opposite the face $x_i = 0$ of τ; i. e. the point whose coördinates v are all $= 0$ except v_i. Its polar plane is $(v, x) = 0$

$$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 = 0.$$

The distance η_i of V_i to the plane

$$(16) \qquad u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0$$

is given by

$$\sin(\eta_i/c) = u_i/(a_{ii})^{1/2}[G(u,u)]^{1/2}.$$

Hence

$$u_i = (a_{ii})^{1/2}[G(uu)]^{1/2}\sin(\eta_i/c).$$

Hence

$$(17) \quad u_1 : u_2 : u_3 : u_4 \\ = (a_{11})^{1/2}\sin(\eta_1/c) : (a_{22})^{1/2}\sin(\eta_2/c) : (a_{33})^{1/2}\sin(\eta_3/c) : (a_{44})^{1/2}\sin(\eta_4/c).$$

6. The angle ϕ between the two planes

$$(18) \quad u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \quad v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0,$$

is given by (4). Let us call g, h the poles of these planes; Then

$$g_i = \rho\sum_a a^{ia}u_a, \qquad h_j = \rho\sum_\beta a^{j\beta}v_\beta.$$

The distance δ between g, h is given by

$$\cos(\delta/c) = \sum_{ij} a_{ij}g_ih_j/[F(g,g)F(h,h)]^{1/2},$$

$$\sum_{ij} a_{ij}g_ih_j = \rho^2\sum_{ij} a_{ij}\sum_a a^{ia}u_a\sum_\beta a^{j\beta}v_\beta = \rho^2\sum_{a\beta}u_av_\beta\sum_i a^{ia}\sum_j a_{ij}a^{j\beta}$$
$$= \rho^2\sum_{a\beta}a^{a\beta}u_av_\beta = \rho^2G(u,v).$$

Hence using (13)

$$\cos(\delta/c) = G(u,v)/[G(u,u)G(v,v)]^{1/2}.$$

Comparing this with (4) gives

$$(19) \qquad \cos(\phi/c') = \cos(\delta/c) \quad \text{or} \quad \phi = c'\delta/c.$$

When $\phi = \pi/2$ we say the planes (18) are orthogonal; the distance between their poles is

$$(20) \qquad \delta = (\pi/2)c/c'.$$

The angle ϕ_{ij} between the faces $x_i = 0$, $x_j = 0$ of the tetrahedron τ is given by

$$(21) \qquad \cos(\phi_{ij}/c') = a^{ij}/(a^{ii}a^{jj})^{1/2}.$$

We define the angle θ between two straights l, m meeting at a point a in this way. Let b, c be points on l, m at a quadrants distance from a. If β, γ are the polar planes of b, c we define θ as angle between β, γ. Now we saw the distance δ between b, c is related to θ by $\phi/c' = \delta/c$. Hence we can define θ as c/c' times the distance between b, c.

7. Let us introduce new variables $\xi_1, \xi_2, \xi_3, \xi_4$ setting

$$x_i = \sum_j c_{ij}\xi_j, \tag{22}$$

where $c = \det | c_{ij} | \neq 0$. Setting $c^{ij} =$ minor c_{ij}/c, we have

$$\xi_i = \sum_j c^{ji} x_j. \tag{23}$$

Then

$$\begin{aligned} F(x, x) = \sum a_{ij} x_i x_j &= \sum_{ij} a_{ij} \sum_\lambda c_{i\lambda}\xi_\lambda \sum_\mu c_{j\mu}\xi_\mu \\ &= \sum_{\lambda\mu} \xi_\lambda \xi_\mu \sum_{ij} a_{ij} c_{i\lambda} c_{j\mu}. \end{aligned}$$

Hence setting

$$\alpha_{\lambda\mu} = \sum_{ij} a_{ij} c_{i\lambda} c_{j\mu}, \tag{24}$$

$$F(x, x) = \sum_{\lambda\mu} \alpha_{\lambda\mu} \xi_\lambda \xi_\mu = \Phi(\xi, \xi). \tag{25}$$

The distance δ between two points g, h is given by

$$\cos(\delta/c) = \sum a_{ij} g_i h_j / [F(g, g) F(h, h)]^{1/2}. \tag{26}$$

Suppose (23) makes γ, η correspond to g, h, then conversely (22) gives

$$g_i = \sum_\lambda c_{i\lambda}\gamma_\lambda, \qquad h_j = \sum_\mu c_{j\mu}\eta_\mu.$$

These in (26) give

$$\cos(\delta/c) = [\sum_{ij} a_{ij} \sum_\lambda c_{i\lambda}\gamma_\lambda \sum_\mu c_{j\mu}\eta_\mu] / [\Phi(\gamma, \gamma)\Phi(\eta, \eta)]^{1/2} = N/D.$$

Now by (20)

$$N = \sum_{\lambda\mu} \gamma_\lambda \eta_\mu \sum_{ij} a_{ij} c_{i\lambda} c_{j\mu} = \sum_{\lambda\mu} \gamma_\lambda \eta_\mu \alpha_{\lambda\mu}.$$

Thus

$$\cos(\delta/c) = \Phi(\gamma, \eta) / [\Phi(\gamma, \gamma)\Phi(\eta, \eta)]^{1/2}, \tag{27}$$

i. e. $\cos \delta/c$ is an invariant of the quadratic form (1) relative to the linear transformations (22). The plane

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0, \tag{28}$$

becomes in the new variables

$$\sum v_i \xi_i = 0 \quad \text{where} \quad v_i = \sum_\lambda c_{\lambda i} u_\lambda. \tag{29}$$

The distance δ of point x to the plane (28) is given by

$$\sin(\delta/c) = \rho(u_1 x_1 + \cdots + u_4 x_4) / [F(g, g) F(x, x)]^{1/2},$$

where g is the pole of (28).

Passing to the ξ variables this gives

(30) $$\sin(\delta/c) = \rho(v_1\xi_1 + \cdots + v_4\xi_4)/[\Phi(\gamma,\gamma)\Phi(\xi,\xi)]^{1/2},$$

where γ corresponds to g. Thus $\sin(\delta/c)$ is an invariant of the form (1). Now

$$\Phi(\gamma,\gamma) = \sum_{ij} \alpha_{ij}\gamma_i\gamma_j = \rho^2 \sum_{ij} \alpha_{ij} \sum_{\lambda} \alpha^{i\lambda}v_\lambda \sum_{\mu} \alpha^{j\mu}v_\mu = \rho^2 \sum_{\lambda\mu} \alpha^{\lambda\mu}v_\lambda v_\mu.$$

Hence

(31) $$\sin(\delta/c) = (v_1\xi_1 + \cdots + v_4\xi_4)/[\Psi(v,v)\Phi(\xi,\xi)]^{1/2},$$

where

$$\Psi(v,v) = \sum_{\lambda\mu} \alpha^{\lambda\mu}v_\lambda v_\mu.$$

If δ_i is the distance from Point P $(\xi_1, \xi_2, \xi_3, \xi_4)$ to the plane $\xi_i = 0$, we have

(32) $$\sin(\delta_i/c) = \xi_i/[\alpha^{ii}\Phi(\xi,\xi)]^{1/2},$$

or

(33) $$\xi_1 : \xi_2 : \xi_3 : \xi_4 = (\alpha^{11})^{1/2}\sin(\delta_1/c) : (\alpha^{22})^{1/2}\sin(\delta_2/c) : (\alpha^{33})^{1/2}\sin(\delta_3/c) : (\alpha^{44})^{1/2}\sin(\delta_4/c).$$

Geometrically the transformation (22) or (23) represents replacing the tetrahedron of reference τ by a tetrahedron τ' whose faces are $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$.

Let us refer the straight

(34) $$x_i = la_i + mb_i, \qquad (i = 1, 2, 3, 4),$$

to the τ' tetrahedron. By (23), $\xi_i = \sum_j c^{ji}x_j$.

As x lies on (34),

$$\begin{aligned}\xi_i &= \sum_j c^{ji}(la_i + mb_i)\\ &= l\alpha_i + m\beta_i,\end{aligned}$$

where by (23), α_i, β_i are the coördinates a, b referred to τ'. From algebra we know that a real transformation (23) will reduce the form (1) to one of three types

(35) $$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 0,$$

(36) $$\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 = 0,$$

(37) $$\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 = 0.$$

To (35) corresponds the geometry first discovered by Riemann. To (36)

corresponds the geometry discovered by Lobachevsky and Bolyai. Finally to (37) corresponds a geometry which I have called L-geometry.*

8. The distance $\delta = ds$ between x and $x + dx$ is given by

$$\cos(ds/c) = (x, x + dx)/[(x, x)(x + dx, x + dx)]^{1/2} = A/B = 1 - ds^2/2c^2 + \cdots.$$

Now

$$(x, x + dx) = (x, x) + (x, dx),$$
$$(x + dx, x + dx) = (x, x) + 2(x, dx) + (dx, dx),$$
$$B = (x, x)[1 + 2(x, dx)/(x, x) + (dx, dx)/(x, x)]^{1/2}.$$
$$\begin{aligned} A/B &= [1 + (x, dx)/(x, x)]\,[1 + 2(x, dx)/(x, x) + (dx, dx)/(x, x)]^{-1/2} \\ &= [1 + (x, dx)/(x, x)] \cdot [1 - (1/2)\{2(x, dx)/(x, x) + (dx, dx)/(x, x)\} \\ &\qquad + (3/8)\{2(x, dx)/(x, x) + (dx, dx)^2/(x, x)\} + \cdots], \\ &= 1 - (x, dx)/(x, x) - (dx, dx)/2(x, x) + (3/2)(x, dx)^2/(x, x)^2 + \cdots, \\ &\quad + (x, dx)/(x, x) - (x, dx)^2/(x, x)^2 - \cdots, \\ &= 1 - (dx, dx)/2(x, x) + (1/2)(x, dx)^2/(x, x)^2 + \cdots \\ &= 1 - ds^2/2c^2 + \cdots. \end{aligned}$$

Hence

(38) $$ds^2/c^2 = (dx, dx)/(x, x) - (x, dx)^2/(x, x)^2.$$

Since only the ratios $x_1 : x_2 : x_3 : x_4$ have been used, if we like, we may set

(39) $$(x, x) = \sum a_{ij} x_i x_j = h^2, \quad \text{a constant.}$$

Then

(40) $$(x, dx) = 0 \quad \text{and} \quad ds^2 = c^2/h^2 (dx, dx).$$

If we choose c, h so that $c = h$, then

(41) $$ds^2 = \sum a_{ij} dx_i dx_j.$$

Let us define the angle θ between two curves meeting at a point a, as the angle between their tangents, whose equations are, say

$$x_i = a_i \cos s/c + b_i \sin s/c, \quad x_i' = a_i \cos s'/c + \beta_i \sin s'/c.$$

We have at the point a for which $s = s' = 0$,

$$dx_i/ds = b_i/c, \qquad dx_i'/ds' = \beta_i/c.$$

If $\delta = \operatorname{dist}(b, \beta)$ we set $\theta = \delta/c$ and have

$$\cos(\delta/c) = \sum a_{ij} b_i \beta_j / [(b, b)(\beta, \beta)]^{1/2}$$

or

$$\cos\theta = \sum a_{ij} (dx_i/ds)(dx_j'/ds')\,[F(b, b)F(\beta, \beta)]^{1/2}.$$

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* *Monatshefte für Mathematik und Physik*, Vol. 35 (1928).

If we suppose $(b, b) = (\beta, \beta) = c^2$ we have

$$\cos\theta = \sum a_{ij}(dx_i/ds)(dx_j'/ds'). \tag{42}$$

Let us find the curves for which

$$\delta \int ds = 0,$$

where ds is given by (41).

Performing the variation we get

$$\int ds \sum_j \delta x_j \sum_i a_{ij}\, d^2x_i/ds^2 = 0.$$

From the relation (39) we get $\sum a_{ij}x_i\delta x_j = 0$, hence introducing a Lagrangean multiplier H we get

$$\int ds \sum_j \delta x_j \sum_i a_{ij}\{d^2x_i/ds^2 + Hx_i\} = 0,$$

or

$$\sum_i a_{ij}(d^2x_i/ds^2 + Hx_i) = 0, \qquad (j = 1, 2, 3, 4). \tag{43}$$

Multiplying these four equations respectively by x_1, x_2, x_3, x_4, and adding, gives

$$\sum_{ij} a_{ji}x_j\, d^2x_i/ds^2 + H\sum_{ij} a_{ij}x_ix_j = 0,$$

or

$$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + Hc^2 = 0. \tag{44}$$

From (39) we have differentiating twice

$$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + \sum_{ij} a_{ij}(dx_i/ds)(dx_j/ds) = 0,$$

or

$$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + 1 = 0.$$

This in (44) gives $H = 1/c^2$ which put in (43) gives the four linear equations

$$\sum a_{ji}(d^2x_i/ds^2 + x_i/c^2) = 0, \qquad (j = 1, 2, 3, 4).$$

The determinant of the system of equations is $a \neq 0$. Hence this system admits only the solution

$$d^2x_i/ds^2 + x_i/c^2 = 0, \qquad (i = 1, 2, 3, 4).$$

Hence integrating

$$\begin{aligned} x_i &= a_i \cos s/c + b_i \sin(s/c) \\ &= \alpha a_i + \beta b_i, \end{aligned}$$

which are the equations of a straight as we saw § 4.

9. We have now gone far enough to show how the geometry belonging to the quadratic form (1) with non-vanishing determinant may be developed. It remains only to put this abstract geometry in relation to the geometry of our physical space.

In two papers,* I have considered the optics of space of constant curvature taking as fundamental assumption that the path of a ray of light in such a space is given by Fermat's principle, $\delta \int nds = 0$. I showed that in such a space, light behaved in the main as in euclidean space. For constant index of refraction the path of a ray is a straight according to Cayley's definition. In elliptic and hyperbolic spaces it has long been known that rigid bodies exist which can be freely moved about without distortion, and I have shown the same holds in L-space in the paper referred to in § 7. Thus we have the means, partly mechanical and partly optical, to construct straight edges, planes etc., Also we are in position to construct measuring bars. This being so we can define a tetrahedron and define our coördinates by (15) where the a's are determined by one of the forms (35), (36), or (37).

The indentification of our abstract space with our physical space is thus complete. Cayley appears to have had no interest in identifying his abstract geometry with our physical space, his "coördinates" have thus left a fundamental notion undefined.

In Klein's theory straight lines are the undefinabled. They may be identified with what we call straights in our physical world in a manner similar to that just outlined.

Disregarding such identification, but looking at these geometries as abstractions the only question would then be;—Which of these two abstract methods of procedure Cayley's or Klein's is the more direct and easy. For those who are familiar with projective geometry when developed *projectively* † there would be no preference.

But for those who have not gone thru such a strenuous course of training the advantage, so it seems to the present writer, lies with Cayley.

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* *Transactions of the American Mathematical Society*, Vol. 30 (1927), pp. 33-48; *American Journal of Mathematics*, Vol. 49 (1927), pp. 343-354.

† As for example, in the magistral treatise of Veblen and Young.

FAMILIES OF PLANE INVOLUTIONS OF GENUS 2 OR 3.

By Franklin G. Williams.

1. *Introduction.* A general method for finding families of plane involutions of order t for a pencil of curves of genus $p \leq (t-1)(t-2)/2$ has been developed recently * and applied to the case $t=3$ so that $p=0$ or 1, and to the cases $t=4$, $p=0$ or 1. In this paper, the case of pencils with $t=4$ has been completed by considering pencils of genus 2 or 3. The irreducible types of pencils of curves with $p=2$ are known.† For $p=3$, they are determined in this paper.

2. *General Method.* In a plane x, let $|C_1|$ denote a linear system of curves of order n_1 with an $i_1, i_1' \cdots$ fold point at $P_i, P_i', \cdots$, and let $|C_2|$ denote a similar linear system. Then $|C_1| + |C_2|$ is defined as a linear system of order $n_1 + n_2$, having an $i_1 + i_1'$-fold point at P_i, P_i'. Any curve $|C_1|$ together with any curve $|C_2|$ constitute a composite curve of the system $|C_1| + |C_2|$. Given a pencil of curves

$$(1) \qquad |C| = a_1C_1 + a_2C_2 = 0,$$

then $|C| + |\bar{C}|$ is a system containing $(a_1C_1 + a_2C_2)\bar{C} = 0$ as a composite pencil. Let

$$(2) \qquad |C| + |\bar{C}| = (a_1C_1 + a_2C_2)\bar{C} + a_3C_4 + \cdots + a_{r+1}C_{r+1} = 0.$$

The symbol $[C_1, C_2]$ is used to denote the number of variable intersections of a curve of $|C_1|$ with a curve of $|C_2|$. Then $[C_1, C_1 + C_2] = [C_1, C_1] + [C_1, C_2]$.

We wish to consider the cases in which any curve of $|C|$ meets any curve of $|C| + |\bar{C}|$ in a certain number, t, of variable points. We are going to consider the possible cases with $t=4$ and $p=2$ or 3.

The irreducible pencils of genus 2, mentioned in Section 1, are:

$$C_4: A^2 12B, \quad C_6: 8A^2 4B, \quad C_7: A^3 10B, \quad C_9: 8A^3 2B^2 C, \quad C_{13}: A^5 9B^4,$$

where the subscript denotes the order of the curve C, and a superscript denotes the multiplicity of a corresponding point.

As typical of the process employed, let (1) be the pencil, $C_4: A^2 12B$,

* F. R. Sharpe, *American Journal of Mathematics*, Vol. 50 (October, 1928).

† Michele di Franchis, *Rendiconti del Circolo Matematico di Palermo*, Vol. 13 (1899).

and $\bar{C}$ a line through A. Then $|C| + \bar{C}$ is the system, $C_5: A^3 10B$, for which $r = 4$.

Consider the pencil

$$z_2 C_1 - z_1 C_2 = 0, \tag{3}$$

and the algebraic system of $C_5: A^3 10B$

$$(b_1 z_3 + c_1) C_1 \bar{C} + (b_2 z_3 + c_2) C_2 \bar{C} + (b_3 z_3 + c_3) C_3 + (b_4 z_3 + c_4) C_4 + (b_5 z_3 + c_5) C_5 = 0 \tag{4}$$

where the b's are of arbitrary degrees $n - 1$, n respectively in z_1, z_2. Given a point in (x), then (3) and (4) determine a point in (z); and, conversely, given a point in (z), then (3) and (4) determine a group of 4 points in (x). Thus (3) and (4) determine an involution of order 4, I_4.

By solving (3) and (4) for the ratio $z_1 : z_2 : z_3$, we have

$$z_1 = C_1 u; \quad z_2 = C_2 u; \quad z_3 = v \tag{5}$$

where u and v are linear in $C_1\bar{C}$, $C_2\bar{C}$, $C_3\bar{C}$, $C_4\bar{C}$, $C_5\bar{C}$ with coefficients of orders $n - 1$, n respectively in C_1, C_2. Hence the lines of (z) correspond to the net of curves

$$a_1 C_1 u + a_2 C_2 u + C_3 v = 0. \tag{6}$$

The curves of the net (6) are therefore $C_{4n+5}: A^{2n+3} 10 B^{n+1} 2 C^n$, while $u = 0$ represents a curve $C_{4n+1}: A^{2n+1} 10 B^n 2 C^{n-1}$. Hence the curves $u = 0$ and $v = 0$ have

$$(4n + 1)(4n + 5) - (2n + 3)(2n + 1) - 10n(n + 1) - 2n(n - 1) = 8n + 2$$

intersections at points, D, other than at the basis points of $C_4: A^2 10B 2C$. We have therefore determined a family of involutions of the type

$$C_{4n+1}: A^{2n+1} 10 B^n 2 C^{n-1} (8n - 6) D.$$

Following the method outlined above and making use of the tables of linear systems of curves given by Jung,* the following families of involutions of genus 2 have been found:

* G. Jung, *Annali di Matematico*, Ser. 2, Vols. 14, 15.

Pencil $C_4: A^2 12B$

$\bar{C}$	Genus	Families of Involutions
$(C_4: A\,10B)$	—	$C_{4n}: A^{2n-1}10B^n 2C^{n-1}(8n-7)D$
$(C_4: A^2 8B)$	—	$C_{4n}: A^{2n}8B^n 4C^{n-1}(8n-8)D$
C_1	—	$C_{4n+1}: A^{2n}12B^n(8n-3)C$
$C_1: A$	—	$C_{4n+1}: A^{2n+1}10B^n 2C^{n-1}(8n-6)D$
$C_2: 4B$	—	$C_{4n+2}: A^{2n}4B^{n+1}8C^n(8n-4)D$
$C_3: A6B$	1	$C_{4n+3}: A^{2n+1}6B^{n+1}6C^n(8n-6)D$
$C_3: 8B$	1	$C_{4n+3}: A^{2n}8B^{n+1}4C^n(8n-3)D$
$(C_3: A8B)^2$	1	$C_{4n+2}: A^{2n}8B^{n+1}4C^{n-1}(8n-12)D$
$C_4: AB^2 8C$	2	$C_{4n}: A^{2n-1}B^{n+1}8C^n 3D^{n-1}(8n-9)E$
$C_6: A^2 7B^2 2C$	2	$C_{4n+2}: A^{2n}7B^{n+1}2C^n 3D^{n-1}(8n-10)E$
$C_4: 12B$	3	$C_{4n}: A^{2n-2}12B^n(8n-8)C$
$C_5: AB^3 11C$	3	$C_{4n+1}: A^{2n-1}B^{n+2}11C^n(8n-8)D$
$C_6: A^2 6B^2 4C$	3	$C_{4n+2}: A^{2n}6B^{n+1}4C^n 2D^{n-1}(8n-8)E$
$C_7: A^3 9B^2$	3	$C_{4n+3}: A^{2n+1}9B^{n+1}3C^{n-1}(8n-8)D$
$C_7: A^2 B^3 8C^2 D$	3	$C_{4n+3}: A^{2n}B^{n+2}8C^{n+1}D^n 2E^{n-1}(8n-9)F$
$C_5: A^2 B^2 10C$	4	$C_{4n+1}: A^{2n}B^{n+1}10C^n D^{n-1}(8n-5)E$
$C_5: A2B^2 10C$	4	$C_{4n+1}: A^{2n-1}2B^{n+1}10C(8n-6)D$
$C_6: A^4 12B$	4	$C_{4n+2}: A^{2n+2}12B^n(8n-4)C$
$C_6: A^2 5B^2 6C$	4	$C_{4n+2}: A^{2n}5B^{n+1}6C^n D^{n-1}(8n-6)E$
$C_6: A6B^2 6C$	4	$C_{4n+2}: A^{2n-1}6B^{n+1}6C^n(8n-7)D$
$C_7: A^2 10B^2$	4	$C_{4n+3}: A^{2n}10B^{n+1}2C^{n-1}(8n-7)D$
$C_7: A11B^2$	4	$C_{4n+3}: A^{2n-1}11B^{n+1}C^{n-1}(8n-4)D$
$C_7: A^2 B^3 7C^2 3D$	4	$C_{4n+3}: A^{2n}B^{n+2}7C^{n+1}3D^n E^{n-1}(8n-7)F$
$C_7: AB^3 8C^2 3D$	4	$C_{4n+3}: A^{2n-1}B^{n+2}8C^{n+1}3D^n(8n-8)E$
$C_8: A^2 B^4 10C^2$	4	$C_{4n}: A^{2n-2}B^{n+2}10C^n D^{n-2}(8n-16)E$
$C_{10}: A^4 8B^3 2C^2$	4	$C_{4n+2}: A^{2n}8B^{n+1}2C^n 2D^{n-2}(8n-16)E$
$C_7: A^3 7B^2 4C$	5	$C_{4n+3}: A^{2n+1}7B^{n+1}4C^n D^{n-1}(8n-4)E$
$C_7: A^2 9B^2 2C$	5	$C_{4n+3}: A^{2n}9B^{n+1}2C^n D^{n-1}(8n-5)E$
$C_7: A10B^2 2C$	5	$C_{4n+3}: A^{2n-1}10B^{n+1}2C^n(8n-6)D$
$C_8: A^4 10B^2$	5	$C_{4n}: A^{2n}10B^n 2C^{n-2}(8n-12)D$
$C_{10}: A^4 8B^3 C^2 2D$	5	$C_{4n+2}: A^{2n}8B^{n+1}C^n 2D^{n-1}E^{n-2}(8n-14)F$
$C_8: A^4 9B^2 2C$	6	$C_{4n}: A^{2n}9B^n 2C^{n-1}D^{n-2}(8n-10)E$

Pencil $C_6: 8A^2 4B$

$\bar{C}$	Genus	Families of Involutions
$(C_6: 7A^2 4C)$	—	$C_{6n}: 7A^{2n}B^{2n-2}4C^n(8n-8)D$
$(C_6: 7A^2 2C)$	—	$C_{6n}: 7A^{2n}B^{2n-1}2C^n 2D^{n-1}(8n-7)E$
$C_1: A$	—	$C_{6n+1}: A^{2n+1}7B^{2n}4C^n(8n-4)D$
$C_1: 2B$	—	$C_{6n+1}: 8A^{2n}2B^{n+1}2C^n(8n-5)D$
$C_3: 7A$	1	$C_{6n+3}: 7A^{2n+1}B^{2n}4C^n(8n-2)D$
$C_3: 8A$	1	$C_{6n+3}: 8A^{2n+1}2B^n 2C^{n-1}(8n-5)D$
$C_3: 6A2C$	1	$C_{6n+3}: 6A^{2n+1}2B^{2n}2C^{n+1}2D^n(8n-3)E$
$C_4: A^2 7B2C$	2	$C_{6n+4}: A^{2n+2}7B^{2n+1}2C^{n+1}2D^n(8n-1)E$
$C_4: 8AB^2 2C$	2	$C_{6n+4}: 8A^{2n+1}B^{n+2}2C^{n+1}D^n(8n-2)E$
$C_4: 8A2B2C$	3	$C_{6n+4}: 8A^{2n+1}2B^{n+1}2C^{n+1}8nD$
$C_7: A^3 7B^2 2C^2$	3	$C_{6n+1}: A^{2n+1}7B^{2n}2C^{n+1}2D^{n-1}(8n-8)E$
$C_9: 8A^3 B^2$	3	$C_{6n+3}: 8A^{2n+1}B^{n+1}C^{n-1}2D^{n-1}(8n-7)E$

$\bar{C}$	Genus	Families of Involutions
$C_9: A^2 7B^3 C^3 D$	3	$C_{6n+3}: A^{2n} 7B^{2n+1} C^{n+2} D^n 2E^{n-1} (8n-8) F$
$C_{12}: 7A^4 B^3 C^4 D^2$	3	$C_{6n}: 7A^{2n} B^{2n-1} C^{n+2} D^n 2E^{n-2} (8n-17) F$
$C_6: 6A^2 2B 4C$	4	$C_{6n}: 6A^{2n} 2B^{2n-1} 4C^n (8n-6) D$
$C_7: 8A^2 3B^2$	4	$C_{6n+1}: 8A^{2n} 3B^{n+1} C^{n-1} (8n-7) D$
$C_7: A^3 7B^2 C^2 2D$	4	$C_{6n+1}: A^{2n+1} 7B^{2n} C^{n+1} 2D^n E^{n-1} (8n-6) F$
$C_{10}: 8A^3 B^4 2C^2$	4	$C_{6n+4}: 8A^{2n+1} B^{n+3} 2C^{n+1} D^{n-1} (8n-8) E$
$C_{12}: 8A^4 B^3 C$	4	$C_{6n}: 8A^{2n} B^{n+1} C^{n-1} 2D^{n-2} (8n-14) E$
$C_{15}: 8A^5 B^4 C^2$	4	$C_{6n+3}: 8A^{2n+1} B^{n+2} C^n 2D^{n-2} (8n-15) E$
$C_{16}: A^6 7B^5 C^5 D^4 E$	4	$C_{6n+4}: A^{2n+2} 7B^{2n+1} C^{n+3} D^{n+2} E^{n-1} F^{n-2} (8n-17)$
$C_{10}: A^4 7B^3 C^3 D^2 E$	5	$C_{6n+4}: A^{2n+2} 7B^{2n+1} C^{n+2} D^{n+1} E^n F^{n-1} (8n-5) G$
$C_{15}: 8A^5 B^4 2C$	5	$C_{6n+3}: 8A^{2n+1} B^{n+2} 2C^{n-1} D^{n-2} (8n-13) E$
$C_{12}: 8A^4 B^2 2C$	5	$C_{6n}: 8A^{2n} B^n 2C^{n-1} D^{n-2} (8n-10) E$

Pencil $C_7: A^3 10B^2$

$\bar{C}$	Genus	Families of Involutions
$(C_7: A^3 9B^2)$	—	$C_{7n}: A^{3n} 9B^{2n} C^{2n-2} (8n-8) D$
$C_1: A$	—	$C_{7n+1}: A^{3n+1} 10B^{2n} (8n-4) C$
$C_3: A 7B$	1	$C_{7n+3}: A^{3n+1} 7B^{2n+1} 3C^{2n} (8n-3) D$
$C_6: A^2 7B^2 2C$	2	$C_{7n+6}: A^{3n+2} 7B^{2n+2} 2C^{2n+1} D^{2n} (8n-2) E$
$C_6: A^2 6B^2 4C$	3	$C_{7n+6}: A^{3n+2} 6B^{2n+2} 4C^{2n+1} 8nD$
$C_9: A^3 7B^3 C^2 2D$	3	$C_{7n+2}: A^{3n} 7B^{2n+1} C^{2n} 2D^{2n-1} (8n-9) E$
$C_{12}: A^4 7B^4 C^3 D^2 E$	3	$C_{7n+2}: A^{3n+1} 7B^{2n+2} C^{2n+1} D^{2n} E^{2n-1} (8n-10) F$
$C_7: A^3 8B^2 2C$	4	$C_{7n}: A^{3n} 8B^{2n} 2C^{2n-1} (8n-6) D$

Pencil $C_9: 8A^3 2B^2 C$

$\bar{C}$	Genus	Families of Involutions
$(C_9: 8A^3 2BC)$	—	$C_{9n}: 8A^{3n} 2B^{2n-1} C^n (8n-6) D$
$(C_9: 7A^3 B^2 2C^2)$	—	$C_{9n}: 7A^{3n} B^{3n-1} 2C^{2n} D^{n-1} (8n-6) E$
$C_3: 6A 2CD$	1	$C_{9n+3}: 6A^{3n+1} 2B^{3n} 2C^{2n+1} D^{n+1} (8n-4) E$
$C_4: A^2 7B 2CD$	2	$C_{9n+4}: A^{3n+2} 7B^{3n+1} 2C^{2n+1} D^{n+1} (8n-2) E$
$C_6: 8A^2 B$	2	$C_{9n+6}: 8A^{3n+2} B^{2n+1} C^{2n} D^n (8n-1) E$
$C_6: 6A^2 2B 2C^2$	2	$C_{9n+6}: 6A^{3n+2} 2B^{3n+1} 2C^{2n+2} D^n (8n-2) E$
$C_6: 7A^2 B 2CD$	3	$C_{9n+6}: 7A^{3n+2} B^{3n+1} 2C^{2n+1} D^{n+1} 8nE$
$C_7: A^3 7B^2 2C^2$	3	$C_{9n+7}: A^{3n+3} 7B^{3n+2} 2C^{2n+2} D^n 8nE$
$C_9: 8A^3 B^2 D$	3	$C_{9n}: 8A^{3n} B^{2n} C^{2n-2} D^n (8n-8) E$
$C_{12}: 7A^4 B^2 C^3 D^4$	3	$C_{9n+3}: 7A^{3n+1} B^{3n-1} C^{2n+1} D^{2n+2} E^{n-1} (8n-9) F$
$C_{12}: 8A^4 B^3 C^2$	3	$C_{9n+3}: 8A^{3n+1} B^{2n+1} C^{n+1} D^{2n-2} (8n-9) E$
$C_{12}: 8A^4 B^3 C$	4	$C_{9n+3}: 8A^{3n+1} B^{2n+1} C^{2n-1} D^{n-1} (8n-6) E$
$C_{10}: A^4 7B^3 C^3 D^2 E$	5	$C_{9n+1}: A^{3n+1} 7B^{3n} C^{2n+1} D^{2n} E^n (8n-5) F$
$C_{15}: 8A^5 B^4 CD$	5	$C_{9n+6}: 8A^{3n+2} B^{2n+2} C^{2n-1} D^n (8n-5) E$

Pencil $C_{13}: A^5 9B^4$

$\bar{C}$	Genus	Families of Involutions
$(C_{13}: A^5 8B^4 C^3)$	—	$C_{13}: A^{6n} 8B^{4n} C^{4n-1} (8n-5) D$
$C_6: A^2 7B^2 2C$	2	$C_{13n+6}: A^{5n+2} 7B^{4n+2} 2C^{4n+1} (8n-2) D$
$C_7: A^3 9B^2$	3	$C_{13n+7}: A^{5n+3} 9B^{4n+2} 8nC$
$C_{12}: A^4 7B^4 C^3 D^2$	3	$C_{13n+12}: A^{5n+4} 7B^{4n+4} C^{4n+3} D^{4n+2} (8n-1) E$

3. *Pencils of Curves of Genus 3.* In order to proceed further with the problem, it is necessary to have a complete list of pencils of curves of genus 3, irreducible in type.

Let r_i denote the multiplicity of the i-th basis point, s the number of basis points, and n the order of the curve, where $r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_s > 0$. Therefore

$$(1) \qquad \sum_{i=1}^{s} r_i^2 = n^2,$$

since there are no variable intersections and since the genus, p, is 3, therefore, $(n-1)(n-2)/2 = r_i(r_i-1)/2 + 3$. Hence

$$(2) \qquad \sum_{i=1}^{s} r_i = 3n + 4.$$

Multiplying (2) by r_3 and substracting (1) from the product, we get

$$(3) \qquad r_3 \sum_{i=4}^{s} r_i - \sum_{i=4}^{s} r_i^2 = r_3(3n+4) - r_3r_1 - r_3r_2 - n^2 + r_1^2 + r_2^2$$
$$= n(3r_3 - n) - r_3(r_1 + r_2 - 4) + r_1^2 + r_2^2 \geq 0,$$

since each term of the first summation is greater than or equal to the corresponding term of the second summation. And $n \geq r_1 + r_2 + r_3$, because we are seeking irreducible pencils. Either $n > 3r_3$, or $r_1 = r_2 = r_3$, and therefore $n(3r_3 - n) \leq (r_1 + r_2 + r_3)(2r_3 - r_1 - r_2)$; and, from (3), it follows

$$(4) \qquad \begin{gathered} (r_1 + r_2 + r_3)(2r_3 - r_1 - r_2) - r_3(r_1 + r_2 - 4) + r_1^2 + r_2^2 \geq 0, \\ r_3(r_3 + 2) - r_1r_2 \geq 0. \\ r_3 \leq r_2, \quad r_3 + 2 \geq r_1. \end{gathered}$$

Case I.

$$(5) \qquad r_3(r_3 + 2) - r_1r_2 = 0.$$

(6) Here $r_1 - 2 = r_2 = r_3 = R$, hence, from (3),

$$n(3R - n) - R(R + 2 + R - 4) + R^2 + 4R + 4 + R^2 \geq 0.$$

For the equality, $n^2 - 3Rn - 6R - 4 = 0$, $n = 3R + 2$. Denoting the number of additional points of multiplicity R by p_0, the number of multiplicity $R - 1$ by p_1, etc., (1) and (2) become

$$(R + 2)^2 + 2R^2 + p_0R^2 + p_1(R-1)^2 + p_2(R-2)^2 + \cdots + p_{r-1} = 9R^2 + 12,$$
$$R + 2 + 2R + p_0R + p_1(R-1) + p_2(R-2) + \cdots + p_{r-1} = 9R + 10.$$

Simplifying, we get

$$(\mathrm{A}) \qquad p_0R^2 + p_1(R-1)^2 + p_2(R-2)^2 + \cdots + p_{r-1} = 6R^2 + 8R,$$

(B) $\quad p_0R + p_1(R-1) + p_2(R-2) + \cdots + p_{r-1} = 6R + 8.$

Multiplying (B) by $(R-1)$ and subtracting from (A), we get

(C) $\quad p_0R - p_2(R-2) - \cdots = 6R + 8.$

From (B), $p_0 \leq (6+8)/R$, and from (C), $p_0 \geq (6+8)/R$. Therefore $p_0 = (6+8)/R$ where both R and p_0 are integers.

$$\begin{array}{llll} R=1 & p_0=14 & s=17 & n=5 \\ R=2 & p_0=10 & s=13 & n=8 \\ R=4 & p_0=8 & s=11 & n=14 \\ R=8 & p_0=7 & s=10 & n=26 \end{array}$$

Hence we have the following pencils:

$$C_5\colon A^3 16B, \quad C_8\colon A^4 12B^2, \quad C_{14}\colon A^6 10B^4, \quad C_{26}\colon A^{10} 9B^8.$$

Case II.

(7) $$r_3(r_3+2) - r_1r_2 > 0.$$

Here $r_3 = r_1 - 1$ or r_1 and hence $r_2 = r_1 - 1$ or r_1.

If $r_1 - 1 = r_2 = r_3 = \cdots = r$, $n \geq 3r+1$. Equation (3) gives

$$n(3r-n) - r(2r-3) + 2r^2 + 2r + 1 > 0,$$
$$n^2 - 3nr - 5r - 1 < 0.$$

In order to establish an upper limit for n, let $n = 3r + 1 + x$.

$$9r^2 + 1 + x^2 + 6r + 6rx + 2x - 9r^2 - 3r - 3rx - 5r - 1 < 0,$$
$$x^2 - 2r + 3rx + 2x < 0, \quad x^2 + (3r+2)x - 2r < 0.$$

It is seen that x cannot be a positive integer, so $n \leq 3r + 1$. Consider the possible cases when $n = 3r+1$. From (1), we have

(8) $$(r+1)^2 + p_0r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 + 6r + 1,$$

where the p's have the same significance as above.

(9) $$(r+1) + p_0r + p_1(r-1) + p_2(r-2) + \cdots + p_{r+1} = 9r + 7.$$

From (8) and (9), we find

(10) $$p_0r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 8r^2 + 4r$$

and

(11) $p_0r + p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 8r + 6.$

(10) less $(r-1)$ times (11) gives

(12) $p_0r - p_2(r-2) - 2p_3(r-3) - \cdots - p_{r-1} = 6r + 6.$

From (10), $p_0 \leqq (8+4)/r$; and from (12), $p_0 \geqq (6+6)/r$. Also p_0 is an integer; hence $p_0 = 8$ or 9.

If $p_0 = 8$, (10) and (11) give

(13) $p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 4r,$

(14) $p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 6.$

The maximum value that the sum of the squares of the multiplicities of the remaining multiple points can have occurs when there is one point of multiplicity 6. Then $4r = 36$ and $r = 9$. Hence $r \leqq 9$. Also $r \geqq 2$ in order that some p be > 0.

The new solutions of (13) and (14) are obtained by allowing r to take on the integral values from 2 to 9 inclusive. When $r = 2$, (13) reduces to $p_1 = 4$, and (14) to $p_1 = 6$. So we see that r cannot equal 2, from (13),

When $r = 3$ $\qquad 4p_1 + p_2 = 12,$ from (13),

and $\qquad 2p_1 + p_2 = 6,$ from (14).

Hence $\qquad 2p_1 = 6, \quad p_1 = 3, \quad n = 10.$

The pencil is C_{10}: $A^4 8B^3 3C^2$. Making use of (13) and (14), as above, with $r = 4, 5, 6, 7, 8, 9$, we find the following solutions:

$r = 4$; no solution.
$r = 5$; $p_1 = 1$, $p_2 = 0$, $p_3 = 1$, $n = 16$.
C_{16}: $A^6 8B^5 C^4 D^2$.
$r = 6$; no solution.
$r = 7$; no solution.
$r = 8$; no solution.
$r = 9$; $p_1 = p_2 = 0$, $p_3 = 1$, $n = 28$.
C_{28}: $A^{10} 8B^9 C^6$.

This completes the problem with $p_0 = 8$.

Putting $p_0 = 9$, reduces (10) and (11) to

$$p_1(r-1)^2 + p_2(r-2)^2 + p_3(r-3)^2 + \cdots + p_{r-1} = 4r - r^2 \quad \text{and}$$
$$p_1(r-1) + p_2(r-2) + p_3(r-3) + \cdots + p_{r-1} = 6 - r.$$

Since $4r - r^2 > 0$, $r < 4$. Also, $r > 1$. Proceeding as before

$$r = 2, \quad p_1 = 4, \quad n = 7.$$
$$C_7: \ A^3 9B^2 4C.$$
$$r = 3, \quad p_1 = 0, \quad n = 10.$$
$$C_{10}: \ A^4 9B^3 3C.$$

Suppose, as has been shown possible, that

$$r_1 = r_2 = r_3 = \cdots = r_s = r; \quad n \geqq 3r.$$

Using the inequality sign, (3) becomes

$$n(3r - n) - r(2r - 4) + 2r^2 > 0; \text{ i. e.}$$
$$n^2 - 3rn - 4r < 0.$$

By an exactly analogous consideration, it is seen that

$$n = 3r \text{ or } n = 3r + 1$$

Consider $n = 3r$; (1) and (2) give

(15) $$p_0 r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 \quad \text{and}$$

(16) $$p_0 r + p_1(r-1) + p_2(r-2) = \cdots = p_{r-1} = 9r + 4.$$

(Equation (15) less $(r-1)$ times (16) gives

(17) $$p_0 r - p_2(r-2) - p_3(r-3) - \cdots - p_{r-1} = 5r + 4.$$

Hence, from (15), $p_0 \leqq 9$; and, from (16), $p_0 \leqq (9 + 4)/r$. From (17),

$$p_0 \geqq (5 + 4)/r.$$

With $p_0 = 9$, (15) shows that all succeeding p's are 0; and (16) cannot be satisfied. So $p_0 = 6$, 7, or 8.

$$p_0 = 6; \text{ no solution.}$$
$$p_0 = 7, \quad \text{(15) and (16) reduce to}$$

(18) $$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 2r^2 \quad \text{and}$$

(19) $$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 2r + 4.$$

Proceeding as before, we find the following cases:

$r = 2, \quad p_1 = 8, \quad n = 6. \quad C_6: 7A^2 8B.$
$r = 3, \quad p_1 = 4, \quad n = 9. \quad C_9: 7A^3 4B^2 2C.$
$r = 4, \quad p_1 = 3, \quad p_2 = p_3 = 1, \quad n = 12. \quad C_{12}: 7A^4 3B^3 C^2 D.$

$r = 5$; $p_1 = 3$, $p_2 = p_3 = 0$, $p_4 = 2$, $n = 15$. $C_{15}: 7A^5 3B^4 2C$;
also

$p_1 = p_2 = 2$, $n = 15$. $C_{15}: 7A^5 2B^4 2C^3$.

$r = 6$; no solution.

$r = 7$, $p_1 = 2$, $p_2 = p_6 = 1$, $p_3 = p_4 = p_5 = 0$, $n = 21$.
$C_{21}: 7A^7 2B^6 C^5 D$.

$r = 8$; no solution.

$r = 9$, $p_1 = 1$, $p_2 = 2$, $n = 27$. $C_{27}: 7A^9 B^8 2C^7$.

Equation (18) less $(r-2)$ times (19) gives

(20) $$p_1(r-1) - p_3(r-3) - \cdots - p_{r-1} = 8. \quad \text{Hence } r \leq 9.$$

With $\nu = 8$, (15) and (16) become

(21) $$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = r^2 \quad \text{and}$$

(22) $$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = r + 4.$$

$p_1 = 0$ and $p_2 = 1$ reduce (21) and (22) to

$$p_3(r-3)^2 + \cdots + p_{r-1} = 4r - 4 \quad \text{and} \quad p_3(r-3) + \cdots + p_{r-1} = 6,$$

$$4r - 4 \leq 36, \quad r \leq 10, \quad r \geq 4.$$

When $4 \leq r \leq 9$, there is no solution.

$r = 10$, $p_3 = 0$, $p_4 = 1$, $n = 30$. $C_{30}: 8A^{10}B^8C^6$.

Taking $p_1 = 1$, (21) and (22) become

$$p_2(r-2)^2 + \cdots + p_{r-1} = 2r - 1 \quad \text{and} \quad p_2(r-2) + \cdots + p_{r-1} = 5$$

$$2r - 1 \leq 25 \quad \text{or} \quad r \leq 13. \quad \text{Also } r \geq 3.$$

$r = 3$, $p_2 = 5$, $n = 9$. $C_9: 8A^3B^2 5C$.

$r = 4$, $p_2 = 1$, $p_3 = 3$, $n = 12$. $C_{12}: 8A^4B^3C^2 3D$.

$r = 5$, $p_4 = 1$, $p_2 = 0$, $p_3 = 2$, $n = 15$. $C_{15}: 8A^5B^4 2C^2 D$.

$r = 6$, $p_3 = 1$, $p_5 = 2$, $p_2 = p_4 = 0$, $n = 18$. $C_{18}: 8A^6B^5C^3 2D$.

$r = 7$, $p_2 = p_3 = p_6 = 0$, $p_4 = p_5 = 1$, $n = 21$. $C_{21}: 8A^7B^6C^3D^2$.

$r = 8$; no solution.

$r = 9$, $p_2 = p_3 = p_4 = p_6 = p_7 = 0$,
$p_5 = p_8 = 1$, $n = 27$. $C_{27}: 8A^9B^8C^4D$.

$r = 10$, 11, or 12; no solution.

$r = 13$, $p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_9 = p_{10} = p_{11} = p_{12} = 0$,
$p_8 = 1$, $n = 39$. $C_{39}: 8A^{13}B^{12}C^5$.

Now take $p_1 = 2$. $\sum_{i=11}^{8} r_i = 6 - r$, whence $r < 6$. Also $\sum_{i=11}^{8} r_i^2 =$

$4r - r^2 - 2$. So there are no solutions. If $p_1 > 2$, $\sum r_i^2$ and $\sum r_i$ show that no solutions are possible.

Finally, if $n = 3r + 1$ while $r_1 = r_2 = r_3 = r$, (1) and (2) become

$$(23) \quad p_0 r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 + 6r + 1$$

and

$$(24) \quad p_0 r + p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 9r + 7.$$

Equation (23) less $(r-1)$ times (24) yields

$$(25) \quad p_0 r - p_2(r-2) - \cdots - p_{r-1} = 8r + 8.$$

We see from (24) that $p_0 \leq (9 + 7)/r$ and from (25) that $p_0 \geq (8 + 8)/r$. So $p_0 = 10, 11, 12$, or 16.

Taking $p_0 = 10$, (23) and (24) become

$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - r^2$ and
$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - r$; whence $r < 7$.
$r = 2$ or 3; no solutions.
$r = 4$, $p_1 = 1$, $n = 13$. $C_{13}: 10A^4B^3$.
$r = 5$; no solution.
$r = 6$, $p_1 = p_2 = p_3 = p_4 = 0$, $p_5 = 1$, $n = 19$. $C_{19}: 10A^6B$.

With $p_0 = 11$, we have from (23) and (24).

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - 2r^2$$

and $$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} + 7 - 2r.$$

So we note that r must be < 4.

$r = 2$; no solution.
$r = 3$, $p_1 = 0$, $p_2 = 1$, $n = 10$. $C_{10}: 11A^3B$.

When we put $p_0 = (12)$, (23) and (24) become

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - 3r^2$$

and $$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - 3r.$$

The last equation demands that $r < 3$.

$$r = 2, \quad p_1 = 1, \quad n = 7. \quad C_7: 12A^2B.$$

with $p_0 = 16$, (23) and (24) become

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 32r - 16 - 7r^2$$

and $p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - 7r$. Thus $p < 2$.

$$p_1 = p_2 = \cdots = p_n = 0, \quad n = 4. \quad C_4: 16A.$$

Here follows the set of arithmetically possible pencils of genus 3:

$C_4: 16A$
$C_5: A^3 16B$
$C_6: 7A^2 8B$
$C_7: 12A^2 B$
$C_7: A^3 9B^2 4C$
$C_8: A^4 12B^2$
$C_9: 7A^3 4B^2 2C$
$C_9: 8A^3 B^2 5C$
$C_{10}: A^4 8B^3 3C^2$
$C_{10}: A^4 9B^3 3C$
$C_{10}: 11A^3 B$
$C_{12}: 7A^4 3B^3 C^2 D$
$C_{12}: 8A^4 B^3 C^2 3D$
$C_{13}: 10A^4 B^3$
$C_{14}: A^6 10B^4$
$C_{15}: 7A^5 3B^4 2C$
$C_{15}: 7A^5 2B^4 2C^3$
$C_{15}: 8A^5 B^4 2C^2 D$
$C_{16}: A^6 8B^5 C^4 D^2$
$C_{18}: 8A^6 B^5 C^3 2D$
$C_{19}: 10A^6 B$
$C_{21}: 7A^7 2B^6 C^5 D$
$C_{21}: 8A^7 B^6 C^3 D^2$
$C_{26}: A^{10} 9B^8$
$C_{27}: 7A^9 B^8 2C^7$
$C_{27}: 8A^9 B^8 C^4 D$
$C_{28}: A^{10} 8B^9 C^6$
$C_{30}: 8A^{10} B^8 C^6$
$C_{39}: 8A^{13} B^{12} C^5$

Following the method employed with the pencils of curves of genus 2, the following families of involutions, based upon the pencil, $C_4: 16A$, have been found:

$\bar{C}$	Genus	Families of Involutions
$(C_4: 12A)$	—	$C_{4n}: 12A^n 4B^{n-1}(8n-8)C$
C_1	—	$C_{4n}: 16A^n(8n-3)B$
$C_3: 8A$	1	$C_{4n+3}: 8A^{n+1} 8B^n(8n-3)C$
$C_4: A^2 10B$	2	$C_{4n}: A^{n+1} 10B^n 5C^{n-1}(8n-10)D$
$C_5: A^3 13B$	3	$C_{4n+1}: A^{n+2} 13B^n 2C^{n-1}(8n-9)D$
$C_5: 2A^2 12B$	4	$C_{4n+1}: 2A^{n+1} 12B^n 2C^{n-1}(8n-7)D$
$C_6: 6A^2 8B$	4	$C_{4n+2}: 6A^{n+1} 8B^n 2C^{n-1}(8n-8)D$
$C_5: A^2 14B$	5	$C_{4n+1}: A^{n+1} 14B^n C^{n-1}(8n-5)D$
$C_6: 5A^2 10B$	5	$C_{4n+2}: 5A^{n+1} 10B^n C^{n-1}(8n-6)D$
$C_7: 10A^2 4B$	5	$C_{4n+3}: 10A^{n+1} 4B^n 2C^{n-1}(8n-7)D$
$C_7: A^3 7B^2 7C$	5	$C_{4n+3}: A^{n+2} 7B^{n+1} 7C^n D^{n-1}(8n-7)E$
$C_6: 4A^2 12B$	6	$C_{4n+2}: 4A^{n+1} 12B^n(8n-4)C$
$C_7: 9A^2 6B$	6	$C_{4n+3}: 9A^{n+1} 6B^n C^{n-1}(8n-5)D$
$C_7: 8A^2 8B$	7	$C_{4n+3}: 8A^{n+1} 8B^n(8n-3)C$

The remaining families of genus 3 are omitted chiefly on account of lack of space, but partly because the existence of some of the pencils has not been established. They can be found readily by the method used above.

4. *Geometric Considerations.* It remains to establish which of the pencils, listed in section 4, actually exist. In the cases in which there are at least three simple basis points, the two linearly independent members of the system are obtained by omitting three of the simple basis points, and determine by their intersection the three points omitted. In each of the other cases, a special investigation is necessary. As an example, the case of $C_{14}: A^6 10B^4$

is here considered. The remaining cases will be the subject of further investigations.

There are six linearly independent curves $C_7: A^3 8B^2$. Hence, for a general position of a point D, there are only three linearly independent curves $C_7: A^3 8B^2 D^2$. For special positions of D however, there are four linearly independent curves $C_7: A^3 8B^2 D^2$. Denote this web by $C_7: A^3 9B^2$. Through any point C in the plane pass three linearly independent curves $C_7: A^3 9B^2$; and one of these curves has a double point at C. Denote these curves by $C_7: A^3 9B^2 C^2$, $C_7': A^3 9B^2 C$, $C_7'': A^3 9B^2 C$, and let $C_7''': A^3 9B^2$ be a $C_7: A^3 9B^2$ not passing through C.

There are four linearly independent curves of order 14, $C_7 C_7'''$, $(C_7')^2$, $(C_7'')^2$, $C_7' C_7''$ having a double point at C. By a suitable linear combination of these, it is possible to get a C_{14} with a triple point at C. There were already two C_{14}'s having a triple point at C,—viz. $C_7 C_7'$ and $C_7 C_7''$. Since there are two degrees of freedom in the choice of C and of a linear combination of the three C_{14}'s, it is possible to find amongst them a C_{14} having a four-fold point at C. This C_{14}, together with $(C_7)^2$, define a pencil of curves $C_{14}: A^6 10B^4$.

PERSPECTIVITIES BETWEEN THE FUNDAMENTAL P-EDRA ASSOCIATED WITH THE ELLIPTIC NORM CURVE Q_p IN S_{p-1} WHERE P IS AN ODD PRIME.

By Bessie Irving Miller.

Associated with every elliptic norm curve of order p, Q_p in S_{p-1}, where p is an odd prime, is a set of $p+1$ fundamental p-edra. Each p-edron is composed of fixed points under an invariant subgroup, G_p, of the subgroup G_{p^2} of the G_{p^3} of collineations which leave the members of the family $F(\infty^1)$ on Q_p unaltered. The vertices and edges of the p-edra are the double points and constituent lines respectively of the degenerate members of the family, F.

The transformations on the coördinates of Q_p effected by the modular substitutions, S and T,

$$S:\ \begin{matrix} \omega_1' = \omega_1 + \omega_2 \\ \omega_2' = -\omega_2 \end{matrix} \qquad\qquad T:\ \begin{matrix} \omega_1' = -\omega_2 \\ \omega_2' = \omega_1, \end{matrix}$$

are given by Klein and Fricke in the form

$$S:\ X_a' = \epsilon^{-a(p-a)/2} X_a, \qquad\qquad T:\ X_a' = \sum_{\beta=0}^{p-1} \epsilon^{-a\beta} X_a,$$
$$(\epsilon = e^{2\pi i/p}).$$

These transformations at the most permute the p-edra.

There are $p+1$ p-edra, for the cyclic G_p's are generated by S_{01} and $S_{10}S_{01}{}^i$ $(i = 0, \cdots, p-1)$, where S_{01} and S_{10} are of the form

$$S_{01}:\ u' = u + \omega_2/p, \qquad\qquad S_{10}:\ u' = u + \omega_1/p.$$

Since the p-edra are permuted by S and T the simplest method of securing them explicitly is to let P_∞ be the reference p-edron, secure another p-edron, P_0, by operating with T on P_∞, and then operate on P_0 with S^i $(i = 1, \cdots, p-1)$ to obtain the remaining p-edra, P_i $(i = 1, \cdots, p-1)$.

The vertices, V_a $(\alpha = 0, \cdots, p-1)$ of P_∞ have all coördinates equal to zero, except $X_a = 1$. Since P_0 is obtained from P_∞ by operating with T, the coördinates of V_a of P_0 have the form

$$X_\beta = \epsilon^{-a\beta} \qquad\qquad (\alpha, \beta = 0, 1, \cdots, p-1) \tag{1}$$

The vertex V_0 of P_0 has all of its coördinates equal to one, since $\alpha = 0$, and the first coördinate of every vertex is one since $\beta = 0$. The exponents of ϵ in X_β and $X_{p-\beta}$ are congruent (mod. p). Since the number of coördinates is odd,

it is always possible to isolate one and write $(p-1)/2$ relations between the others, taking them by pairs. The exponents of ϵ in X_β of V_α $(\alpha \neq 0)$, and $V_{p-\alpha}$, $(\alpha \neq 0$, are congruent (mod. p).

When S^i $(i=1, \cdots, p-1)$ is applied to P_0 to give P_i, then the coördinates of the vertices V_α or P_i are given by

$$X' = \epsilon^{-\alpha\beta-\beta i(p-\beta)/2} X_\beta, \tag{2}$$
$$(i=1, \cdots, p-1;\ \beta=0, 1, \cdots, p-1).$$

The first coördinate of every vertex equals one since $\beta=0$. In every vertex except V_0 there is a second coördinate equal to one, for the condition that

$$X_\beta = 1$$

is that

$$-\alpha\beta - \beta i(p-\beta)/2 \equiv 0 \pmod{p},$$

or

$$\beta \equiv 0 \pmod{p} \text{ and } i\beta \equiv 2\alpha \pmod{p}.$$

The congruence

$$i\beta \equiv 2\alpha \pmod{p} \tag{3}$$

for every i and every α can be solved for β, where $i=1, \cdots, p-1$, and $\alpha = 1, \cdots, p-1$.

In general two coördinates of V_α of P_i are equal, i. e.,

$$X_{\beta_1} = X_{\beta_2}$$

if

$$-\alpha\beta_1 - \beta_1 i(p-\beta)/2 \equiv -\alpha\beta_2 - \beta_2 i(p-\beta_2)/2 \pmod{p}$$

or

$$(\beta_2-\beta_1)\alpha + (ip/2)(\beta_2-\beta_1) \equiv (i/2)(\beta_2^2-\beta_1^2) \pmod{p}.$$

Since $\beta_2 \neq \beta_1$,

$$i(\beta_1+\beta_2) \equiv 2\alpha \pmod{p} \tag{4}$$

is the congruence which must be satisfied. The congruence (4) includes the congruence (3). Hence when β has been determined from (3), it is possible to secure β_1 and β_2 by taking numbers less than p whose sum equals β (mod. p). There are then $(p-1)/2$ pairs of equal coördinates in any one vertex V_i $(i=1, \cdots, p-1)$ of P_i since there are $(p-1)/2$ solutions of the congruence

$$\beta_1 + \beta_2 \equiv \beta \pmod{p}.$$

Moreover in every P_{i+1} there will be a vertex having the same pairs of equal ocördinates as that exhibited by any particular vertex of P_i, for if in the congruence (3′)

$$(3') \qquad (i+1)\beta \equiv 2\alpha \pmod{p}$$

β is fixed, there is one value of α which will satisfy (3′). Consequently the congruence (4) also can be satisfied.

By means of the relations established above it can be proved that if we choose a vertex of P_1 and write down the $(p-1)/2$ equalities between coördinates, there will be one vertex of each P_i, $(i=2,\cdots,p-1)$, whose coördinates satisfy the same relations. Hence there is a space $S_{(p-1)/2}$ lying on $p-1$ vertices of P_i, $(i=1,\cdots,p-1)$, one vertex from each P_i. Also one vertex of P_∞ will lie on the $S_{(p-1)/2}$. Since there are p vertices of P_1 there are p spaces $S_{(p-1)/2}$. But V_0 of P_0 is the point $(1, 1, 1, \cdots, 1)$. Therefore it lies on every space $S_{(p-1)/2}$ determined above. Hence the p-edra P_∞, P_i, $(i=1,\cdots,p-1)$, are perspective with V_0 or P_0 as the center of perspectivity, where corresponding vertices of the p perspective p-edra are joined by spaces $S_{(p-1)/2}$.

Because of the group properties under which the p-edra are formed, symmetry requires that a relation true for one vertex of P_0 be matched by a similar relation for every other vertex of P_0. Moreover since the p-edra are interchangeable under the modular transformations, any one of the p-edra might have been isolated to furnish the centers of perspectivity.

Therefore the following theorem can be stated.

If the vertices of any one of the $p+1$ p-edra associated with Q_p in S_{p-1}, where p is an odd prime, are chosen as centers of perspectivity, there are p sets of p spaces $S_{(p-1)/2}$ which join the corresponding vertices of the remaining p p-edra so that on each $S_{(p-1)/2}$ there is one vertex from each p-edron, and so that each set of p spaces $S_{(p-1)/2}$ lies on a vertex chosen as center of perspectivity. On every vertex chosen as center there is one set of p spaces $S_{(p-1)/2}$.

A simple but pretty illustration of the theory is given when $p=3$. Then the four flex triangles are the reference triangle, T_∞, and the triangles T_0, T_1, T_2.

T_0	T_1	T_2
$(1, 1, 1)$	$(1, \epsilon^{-1}, \epsilon^{-1})$	$(1, \epsilon^{-2}, \epsilon^{-2})$
$(1, \epsilon^{-1}, \epsilon^{-2})$	$(1, \epsilon^{-2}, 1)$	$(1, 1, \epsilon^{-1})$
$(1, \epsilon^{-2}, \epsilon^{-1})$	$(1, 1, \epsilon^{-2})$	$(1, \epsilon^{-1}, 1)$.

It is easy to see that if V_0 of T_0 is the center of perspectivity the lines joining corresponding vertices of T_1, T_2, T_∞ are

$$X_0 = X_1, \quad X_1 = X_2, \quad X_2 = X_0.$$

If V_1 of T_0 is the center of perspectivity the lines are

$$X_0 = \epsilon^{-2} X_1, \quad X_1 = \epsilon^{-2} X_2, \quad X_2 = \epsilon^{-2} X_0.$$

If V_2 of T_0 is the center of perspectivity the lines are

$$X_0 = \epsilon^{-1} X_1, \quad X_1 = \epsilon^{-1} X_2, \quad X_2 = \epsilon^{-1} X_0.$$

In an analogous fashion the equations of the planes joining corresponding vertices of the fundamental pentahedra associated with Q_5 in S_4, of the spaces S_3's joining corresponding vertices of the fundamental heptahedra associated with Q_7 in S_6, of the spaces S_5's joining corresponding vertices of the henahedra associated with Q_{11} in S_{10}, can be written down. If a vertex of P_0 is chosen to be the center of perspectivity, the choice of a vertex of P_1 then determines which coördinate is to be isolated and which $(p-1)/2$ relations between the remaining coördinates are to be used.

When $p = 3$, there are 3 lines on every vertex, and 4 vertices on every line. This is a different type of perspectivity from that usually mentioned, since there are 2 points on each line in excess of the 2 required to determine the line. Corresponding statements can be made for the other values of n under discussion, but the details are of little significance at present for $p > 7$ since little of the geometry of Q_p, $(p > 7)$, is as yet known.

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UNIVERSITY OF ILLINOIS.

ON THE PROJECTIVE DIFFERENTIAL GEOMETRY OF CONJUGATE NETS.

By M. M. SLOTNICK.*

The analytic theory of the projective differential geometry of a conjugate net on a surface in ordinary space suffers a lack of elegance and symmetry in the two parameters. The method developed in the present paper is intended to overcome this disadvantage, and, in brief form, discusses the elementary theorems of conjugate nets. It will be noticed that, for the most part, the proofs are simple, direct, and not lacking in elegance as far as the symmetry in the parameters is concerned.

A special type of net—the net A—is introduced. It is hoped that the methods will yield further results on such nets as well as on the general and other special types of nets.

1. In a three-dimensional projective space, referred to a system of homogeneous point coördinates, we shall consider a non-developable surface S defined parametrically by the four equations

$$x_i = x_i(u, v), \qquad (i = 1, 2, 3, 4),$$

which we shall indicate by the single equation

$$x = x(u, v).$$

The parametric curves on this surface shall form a conjugate system; i. e., a net N.

Let z represent the coördinates of an arbitrary point on the axis of the net corresponding to the point of the net with the coördinates x. The four functions $x(u, v)$ can then be chosen, by using a suitable factor of proportionality,† to satisfy three equations of the forms

1.1 $$x_{uu} = Ax_u + Mx + ez,$$

1.2 $$x_{uv} = (\log a)_v x_u + (\log b)_u x_v,$$

1.3 $$x_{vv} = Bx_v + Nx + gz.$$

The asymptotic lines of S are defined by

1.4 $$edu^2 + gdv^2 = 0,$$

and inasmuch as the surface is non-developable, $eg \neq 0$.

* National Research Fellow.

† L. P. Eisenhart, *Transformations of Surfaces*, Princeton (1923), p. 72. Further references to this treatise will be indicated by *T. S.*

The coördinates of every point of the space can be written as linear combinations of x_u, x_v, x, z, since these four points are non-coplanar. For the present, we write

$$1.5 \qquad \begin{aligned} z_u &= \alpha_1 x_u + \alpha_2 x_v + \alpha_3 x + \alpha_4 z, \\ z_v &= \beta_1 x_u + \beta_2 x_v + \beta_3 x + \beta_4 z. \end{aligned}$$

The coördinates of the focal points of the axis congruence are $z + \rho x$, where ρ is found to satisfy the equation

$$1.6 \qquad \rho^2 + \rho(\alpha_1 + \beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1) = 0.$$

We shall now choose the point z as the harmonic conjugate of x with respect to these two focal points, and, as a result,

$$1.7 \qquad \alpha_1 = -\beta_2 = r.$$

The point z, so chosen, will be called the pivotal point* of the axis.

Here two cases may arise: either $r \not\equiv 0$, or $r \equiv 0$. If $r \not\equiv 0$ the coördinates z of equations (1.5) may be replaced by rz. We shall then have $\alpha_1 \equiv -\beta_2 \equiv 1$. If, however, $r \equiv 0$, then $\alpha_1 \equiv -\beta_2 \equiv 0$. Both of these cases may be discussed simultaneously by introducing a symbol δ, whose significance is such that it is either equal to unity or equal to zero through the discussion.

2. Equations (1.1), (1.2), (1.3), and (1.5), in which $\alpha_1 \equiv -\beta_2 \equiv \delta$, form a completely integrable system. The conditions of integrability of the first three of these reduce to:

$$2.1 \qquad \begin{aligned} &\text{a.} & A_v + e\beta_1 &= (\log a)_{uv} + (\log a)_v(\log b)_u, \\ &\text{b.} & A(\log b)_u + M - e\delta &= (\log b)_{uu} + (\log b)_u^2, \\ &\text{c.} & M_v + e\beta_3 &= M(\log a)_v, \\ &\text{d.} & \beta_4 &= [\log(a/e)]_v, \end{aligned}$$

and

$$2.2 \qquad \begin{aligned} &\text{a.} & B(\log a)_v + N + g\delta &= (\log a)_{vv} + (\log a)_v^2, \\ &\text{b.} & B_u + g\alpha_2 &= (\log b)_{uv} + (\log a)_v(\log b)_u, \\ &\text{c.} & N_u + g\alpha_3 &= N(\log b)_u, \\ &\text{d.} & \alpha_4 &= [\log(b/g)]_u. \end{aligned}$$

The conditions of integrability for equations (1.5), similarly, are:

$$2.3 \qquad \begin{aligned} &\text{a.} & \delta(\log ae)_v + \beta_1[\log(b/g)]_u &= \beta_{1u} + \beta_1 A + \beta_3, \\ &\text{b.} & \delta(\log bg)_u - \alpha_2[\log(a/e)]_v &= -\alpha_{2v} - \alpha_2 B - \alpha_3, \\ &\text{c.} & \alpha_2 N + \alpha_{3v} + \beta_3[\log(b/g)]_u &= \beta_1 M + \beta_{3u} + \alpha_3[\log(a/e)]_v, \\ &\text{d.} & \alpha_2 g + [\log(b/g)]_{uv} &= \beta_1 e + [\log(a/e)]_{uv}. \end{aligned}$$

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* This is the point to which Green calls attention, cf. G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 292, (21).

The first of these equations, taken with (2.1, a) and (2.2, b), indicate that we may write

i. e., $\quad A + [\log(b^2/g)]_u = \phi_u/\phi, \qquad B + [\log(a^2/e)]_v = \phi_v/\phi,$

(2. 4) $\quad A = [\log(\phi g/b^2)]_u, \qquad B = [\log(\phi e/a^2)]_v,$

where ϕ is some function of u and v.

Moreover, equations (2. 1, b) and (2. 2, a) may now be written:

(2. 5)
$$\begin{aligned} M &= e\delta + (\log b)_{uu} + (\log b)_u[\log(b^3/\phi g)]_u, \\ N &= g\delta + (\log a)_{vv} + (\log a)_v[\log(a^3/\phi e)]_v, \end{aligned}$$

and equations (2. 1, a) and (2. 2, b):

(2. 6)
$$\begin{aligned} e\beta_1 &= [\log(ab^2/\phi g)]_{uv} + (\log a)_v(\log b)_u, \\ g\gamma_2 &= [\log(a^2b/\phi e)]_{uv} + (\log a)_v(\log b)_u. \end{aligned}$$

3. We shall now consider the conjugate net from the point of view of its tangential coördinates. The quadruple of ordered cofactors of a_1, a_2, a_3, a_4, respectively, in the determinant

$$|\, a\; b\; c\; d \,|$$

will be indicated by (bcd). Accordingly, the tangential coördinates of the surface S at the point x are

$$\xi = (x_u x_v x)/\psi,$$

where $1/\psi$ is, for the present, an arbitrary factor of proportionality.

Using the equations of the preceding section, we find that the tangential equation* of the net N is

(3. 1)
$$\xi_{uv} = [\log(e\phi/a\psi)]_v \xi_u + [\log(g\phi/b\psi)]_u \xi_v,$$

where $\psi\ (\neq 0)$ is now chosen as an arbitrary solution of the equation

(3. 2)
$$\begin{aligned} \psi_{uv} = {} & [\log(e\phi/a)]_v \psi_u + [\log(g\phi/b)]_u \psi_v \\ & + \{(\log ab)_{uv} + (\log a)_v(\log b)_u - [\log(g\phi/b)]_u[\log(e\phi/a)]_v\}\psi. \end{aligned}$$

The invariants of (3. 1); i. e., the tangential invariants † of the net N, are found to be

(3. 3)
$$\begin{aligned} \mathbb{H} &= [\log(a^2b/\phi e)]_{uv} + (\log a)_v(\log b)_u, \\ \mathbb{K} &= [\log(ab^2/\phi g)]_{uv} + (\log a)_v(\log b)_u. \end{aligned}$$

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* P. S., p. 128.

† P. S., p. 16 and p. 128.

These equations and equations (2. 6) indicate that

3. 4 $$\alpha_2 = \mathbf{H}/g, \qquad \beta_1 = \mathbf{K}/e.$$

As a first result, we note that the condition that *the u-curves be plane curves;* i. e., $(x\, x_u\, x_{uu}\, x_{uuu}) = 0$ *reduces to* $\mathbf{H} = 0$. Similarly, *the v-curves are plane if and only if* $\mathbf{K} = 0$.

4. The values of α_3 and β_3 are evaluated from (2. 2, c) and (2. 1, c), using (2. 5), thus obtaining:

4. 1 $$\begin{aligned} \alpha_3 &= -\,\delta[\log(b/g)]_u + (H/g)\,[\log(a^4H/\phi e)]_v - (\mathbf{H}/g)\,(\log a)_v, \\ \beta_3 &= \quad \delta[\log(a/e)]_v + (K/e)\,[\log(b^4K/\phi g)]_u - (\mathbf{K}/e)\,(\log b)_u. \end{aligned}$$

Here H and K are the point invariants of the net*; i. e.,

4. 2 $$H = -(\log a)_{uv} + (\log a)_v(\log b)_u, \quad K = -(\log b)_{uv} + (\log a)_v(\log b)_u.$$

The equations of compatibility (2. 3, a, b) can now be written

4. 3 $$\begin{aligned} &\text{a.}\quad \delta(\log e^2)_v + (\mathbf{K}/e)\,[\log(b^4e/\phi g^2\mathbf{K})]_v - (K/e)\,[\log(b^4K/\phi g)]_u = 0, \\ &\text{b.}\quad \delta(\log g^2)_u - (\mathbf{H}/g)[\log(a^4g/\phi e^2\mathbf{H})]_u + (H/g)\,[\log(a^4H/\phi e)]_v = 0. \end{aligned}$$

It is also well to point out that equations (4. 1) have the alternative forms:

4. 4 $$\begin{aligned} \alpha_3 &= -\,\delta[\log(bg)]_u + (\mathbf{H}/g)\,[\log(a^3g/\phi e^2\mathbf{H})]_v, \\ \beta_3 &= \quad \delta[\log(ae)]_v + (\mathbf{K}/e)\,[\log(b^3e/\phi g^2\mathbf{K})]_u. \end{aligned}$$

Finally, equations (1. 5), in their evaluated forms are:

4. 5 $$\begin{aligned} z_u &= \delta x_u + (\mathbf{H}/g)x_v + \alpha_3 x + [\log(b/g)]_u z, \\ z_v &= (\mathbf{K}/e)x_u - \delta x_v + \beta_3 x + [\log(a/e)]_v z. \end{aligned}$$

5. Consider an equation of the form (1. 2) and three functions e, g, and ϕ of u and v, satisfying the relations (4. 3) and (2. 3, c), where the functions involved are those defined in the preceding sections in terms of a, b, e, g, and ϕ. Equation (1. 2) and the equation

$$x_{vv} = (g/e)x_{uu} - (gA/e)x_u + Bx_v + (N - Mg/e)x$$

are then compatible.† Accordingly, a surface is defined in the three-dimen-

* *T. S.*, p. 58.

† The full proof of the facts here stated is contained in *T. S.*, pp. 100-103. The equations of compatibility in that text [p. 103, (22)] are fulfilled by virtue of our equations (4. 3) and (2. 3, c). To reconcile our functions with those of Eisenhart, it is to be noted that

$$r = g/e, \qquad a' = -\,gA/e, \qquad b' = B, \qquad c' = N - Mg/e,$$
$$a = a, \qquad b = b, \qquad c = 0.$$

sional space to within a projectivity, the parametric curves of which form a conjugate system, having (1.2) as its point equation. The pivotal point of the axis congruence associated with the net has coördinates of the form

$$z = (x_{uu} - Ax_u - Mx)/e = (x_{vv} - Bx_v - Nx)/g.$$

6. Let the tangential coördinates of the plane through a ray* of the net and the corresponding pivotal point be indicated by ζ. The point coördinates of the two Laplace transforms of the net are

6.1 $$x_{-1} = x_u - (\log b)_u x, \qquad x_1 = x_v - (\log a)_v x.$$

Accordingly,

6.2 $$\zeta = (x_{-1}\ x_1\ z)/\psi.$$

Using the relations of the preceding sections we find that

6.3 $$\begin{aligned}\zeta_u &= -\delta\xi_u - (H/g)\xi_v \\ &\quad + \{\delta[\log(\phi g^2/b\psi)]_u + (H/g)[\log(a^3H/\psi)]_v\}\xi + [\log(\phi/b\psi)]_u\zeta, \\ \zeta_v &= -(K/e)\xi_u + \delta\xi_v \\ &\quad + \{-\delta[\log(\phi e^2/a\psi)]_v + (K/e)[\log(b^3K/\psi)]_u\}\xi + [\log(\phi/a\psi)]_v\zeta.\end{aligned}$$

The equations dual to (1.1) and (1.3), the values of A, B, M, N being those given in § 2, are respectively:

6.4 $$\begin{aligned}\xi_{uu} &= [\log(eb^2\phi/\psi^2)]_u\xi_u \\ &\quad + \{e\delta + [\log(g\phi/b\psi]_{uu} + [\log(g\phi/b\psi]_u[\log(g\phi/eb^3)]_u\}\xi - e\zeta\end{aligned}$$

and

6.5 $$\begin{aligned}\xi_{vv} &= [\log(ga^2\phi/\psi^2)]_v\xi_v \\ &\quad + \{-g\delta + [\log(e\phi/a\psi)]_{vv} + [\log(e\phi/a\psi)]_v[\log(e\phi/ga^3)]_v\}\xi - g\zeta.\end{aligned}$$

All these equations and (3.1) are compatible by virtue of the relations (2.1), (2.2) and (2.3).

The plane which is the harmonic conjugate of the tangent plane to the surface with respect to the two focal planes of the corresponding ray of the net shall be termed the pivotal plane of the net. The tangential coördinates of these focal planes are of the form $\zeta + \rho\xi$ where ρ is determined from the fact that $\xi, \zeta, (\zeta + \rho\xi)_u, (\zeta + \rho\xi)_v$ are linearly dependent. With the aid of (6.3), this condition reduces to

$$\rho^2 - (\delta + HK/eg) = 0.$$

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* E. J. Wilczynski, *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 317.

Hence the coördinates of the pivotal plane are ζ; i. e., *the pivotal plane of a net passes through the corresponding pivotal point.*

7. The following equations are found to hold for the two Laplace transforms of the net:

$$
7.1\quad
\begin{aligned}
&\text{a. } x_{-1u} = [\log(g\phi/b^3)]_u x_u + \{e\delta - (\log b)_u[\log(g\phi/b^3)]_u\}x + ez \\
&\qquad\quad = [\log(g\phi/b^3]_u x_{-1} + e(\delta x + z), \\
&\text{b. } x_{-1v} = (\log a)_v x_u - (\log b)_{uv} x, \\
&\text{c. } x_{-1uv} = \{K - H + (\log a)_v[\log(g\phi/b^2)]_u\}x_u \\
&\qquad\qquad\qquad\qquad + [M(\log a)_v - (\log b)_{uuv}]x + e(\log a)_v z \\
&\qquad\quad = (\log a)_v x_{-1u} + [\log(bK)]_u x_{-1v} + [K - H - (\log a)_v(\log K)_u]x_{-1}.
\end{aligned}
$$

$$
7.2\quad
\begin{aligned}
&\text{a. } x_{1u} = (\log b)_u x_v - (\log a)_{uv} x, \\
&\text{b. } x_{1v} = [\log(e\phi/a^3)]_v x_v + \{-g\delta - (\log a)_v[\log(e\phi/a^3)]_v\}x + gz \\
&\qquad\quad = [\log(e\phi/a^3)]_v x_1 - g(\delta x - z), \\
&\text{c. } x_{1uv} = \{H - K + (\log b)_u[\log(e\phi/a^2)]_v\}x_v \\
&\qquad\qquad\qquad\qquad + [N(\log b)_u - (\log a)_{uvv}]x + g(\log b)_u z \\
&\qquad\quad = [\log(aH)]_v x_{1u} + (\log b)_u x_{1v} + [H - K - (\log b)_u(\log H)_v]x_1.
\end{aligned}
$$

The second part of equation (7. 1, a) indicates that the tangent to the u-curve of the minus first Laplace transform of the net intersects the axis in the point whose coördinates are $\delta x + z$. Similarly from (8. 2, b), the tangent to the v-curve of the first Laplace transform intersects the axis in the point $\delta x - z$. Thus, *the nets characterized by $\delta = 0$ are those for which each tangent to a u-curve of the minus first Laplace transform of the net, the tangent to the v-curve of the first Laplace transform at the corresponding point and the corresponding axis of the net are concurrent in the pivotal point.**

For the general case ($\delta = 1$), *the focal points of the axis of a net, the point in which the tangent to the u-curve of its minus first Laplace transform intersects that axis and the point in which the tangent to the v-curve of the first Laplace transform intersects the axis, the points of the axis which lie in the focal planes of the corresponding ray, are pairs of points in an involution, the double points of which are the point of the net and the corresponding pivotal point.*†

8. The equations of the developables of the axis congruence of the net, $(x\, z\, dx\, dz),= 0$ reduce to

* Such nets have been called *harmonic* by Wilczynski; cf. *American Journal of Mathematics*, Vol. 42 (1920), p. 215.

† Cf. the end of § 6, above, and also G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 306.

$$(H/g)\,du^2 - 2\delta\,du\,dv - (K/e)\,dv^2 = 0, \tag{8.1}$$

[illegible] of equations (4.5).

The curves on the surface of the net defined by this equation have been called the *axis curves* associated with the net. They are obviously the curves of intersection of the surface with these developables.

Recalling equation (1.4) as defining the asymptotic lines of the surface, we see that *the axis curves associated with a conjugate net form a conjugate system if and only if the net has equal tangential invariants. The axis congruence is then conjugate* to the surface. The axis curves coincide with the parametric net if and only if $\delta = 1$ and the curves of the net are plane*; in this case the Laplace transforms of the net are developable†; the axis curves are indeterminate if and only if $\delta = 0$ and both families of curves of the net are plane.*§

If the net has equal tangential invariants, it follows from (3.3) that

$$[\log(ag/be)]_{uv} = 0, \tag{8.2}$$

and conversely. Similarly, the net has equal point invariants if and only if

$$[\log(a/b)]_{uv} = 0. \tag{8.3}$$

Hence, *if a net has two of the following properties it has the third also: equal point invariants, equal tangential invariants, isothermal-conjugate.*[¶]

The equation of the developables of the ray congruence is obtained from the fact that for them

$$(x_1\ x_{-1}\ dx_1\ dx_{-1}) = 0.$$

We have the alternative method, however, dual to the method used above:

$$(\xi\ \zeta\ d\xi\ d\zeta) = 0.$$

These equations reduce to

$$(H/g)\,du^2 - 2\delta\,du\,dv - (K/e)\,dv^2 = 0. \tag{8.4}$$

The curves of the surface of the parametric net defined by this equation are called the *ray curves* of the net. The theorem of Wilczynski,‖ dual to the

* Guichard, *Annales de l'École Normale Supérieure*, 3°, Vol. 14 (1897), p. 178.

† Cf. end of § 3, above, and G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 304.

‡ Cf. § 10.

§ Cf. footnote of G. M. Green, *l. c.*, p. 311.

¶ E. S., p. 150.

‖ *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 318.

first part of the last theorem, follows from equations (8.4) and (1.4): *The ray curves associated with a conjugate net form a conjugate system if and only if the net has equal point invariants.*

Comparing equations (8.1) and (8.4), we note that *a necessary and sufficient condition that the axis curves and the ray curves associated with a net coincide is that each of the point invariants of the net be equal to the corresponding tangential invariant:*

$$H = \mathbf{H}, \qquad K = \mathbf{K}.$$

This last condition is equivalent, by virtue of (3.3) and (4.2), to

8.5 $$[\log(a^3 b/\phi e)]_{uv} = 0, \qquad [\log(a b^3/\phi g)]_{uv} = 0.$$

We shall call a conjugate net for which the associated axis and ray curves coincide a net A. Equations (8.2), (8.3) and (8.5) indicate that *if a net A has any one of the following four properties it has the other three also: equal point invariants, equal tangential invariants, isothermal-conjugate, the curves which are both the axis and ray curves of the net A form a conjugate system.*

A net A of this type is characterized by the fact that the four point and tangential invariants of it are all equal.

9. The focal points of the ray congruence associated with the net N have coördinates of the form $\sigma x_1 + \tau x_{-1}$, where σ/τ satisfies the relation:

9.1 $$\tau^2 K/g - 2\delta\sigma\tau - \sigma^2 H/e = 0.$$

The tangents to the ray curves, (8.4), *will pass through these focal points if and only if $H = K$; i. e., if and only if the net has equal point invariants; the ray curves will then form a conjugate system.**

Guichard † has defined a conjugate net and a congruence of lines as *harmonic* to one another if the focal points of the lines of the congruence lie on the tangents to the curves of the net at corresponding points. Accordingly, the last result may be stated thus: *the ray curves associated with a net form a conjugate system and are harmonic to the ray congruence if and only if the net has equal point invariants.*

We also conclude readily, from equations (8.1) and (9.1) that *the tangents to the axis curves of a net pass through the focal points of the*

* *T. S.*, p. 124, problems 9 and 10.

† Guichard, *l. c.*

corresponding ray of the net if and only if each of the point invariants of the net is equal to the opposite tangential invariant:

$$H = \boldsymbol{K}, \qquad K = \boldsymbol{H}.$$

In view of (3. 3) and (4. 2), this condition reduces to

9. 2 $$[\log(a^2b^2/\phi e)]_{uv} = 0, \qquad [\log(a^2b^2/\phi g)]_{uv} = 0.$$

Accordingly, *a necessary condition that the tangents to the axis curves of a net pass through the focal points of the corresponding ray is that the net be isothermal-conjugate.*

Again, *if the axis curves of a net A, which are also the ray curves of the net A, form a conjugate system, that conjugate system is harmonic to the ray congruence of the net A and is conjugate to the axis congruence of the net A.*

10. Using the methods of § 7 to compute x_{1uu} and x_{1vv}, we find that the asymptotic lines of the surface of the first Laplace transform of the net N are defined by

10. 1 $$eHdu^2 + g\boldsymbol{K}dv^2 = 0,$$

and those of the surface of the minus first Laplace transform by

10. 2 $$e\boldsymbol{H}du^2 + gKdv^2 = 0$$

We shall disregard the case $HK \equiv 0$ (cf. *T. S.*, p. 73).

It is well to note in the last two equations that if $\boldsymbol{H} = 0$ or $\boldsymbol{K} = 0$, the corresponding Laplace transform lies on a developable surface.

The developables of the ray congruence associated with the net, defined by equations (8. 4), will intersect the surface of the first Laplace transform of the net in a conjugate system if and only if $K = \boldsymbol{K}$, and these will intersect the surface of the minus first Laplace transform in a conjugate net if and only if $H = \boldsymbol{H}$. This follows from equations (10. 1) and (10. 2). Thus, *a necessary and sufficient condition that the developables of the ray congruence of a conjugate net intersect the surfaces of the two Laplace transforms of the net in conjugate systems is that the net be a net A. The asymptotic lines on the two Laplace transforms will then correspond.*

The situation here is that *the surfaces of the two Laplace transforms are mapped upon one another by a fundamental transformation.*[*] The axis congruence associated with the net will then be harmonic, in the sense of

[*] *T. S.*, Chapter 2.

Guichard, to the conjugate systems on the surfaces of the Laplace transforms defined by (8.4). The two focal surfaces of the axis congruence of the net then carry Levy transforms of these conjugate systems.*

11. Tzitzéica † has defined an R net as a conjugate net such that the tangents to both families of curves of the net form W congruences.

Comparing equation (1.4) with equations (10.1) and (10.2), we conclude that *a necessary and sufficient condition that a conjugate net be a net R is that each of the point invariants of the net be equal to the opposite tangential invariant:*

$$H = \mathbf{K}, \qquad K = \mathbf{H}.$$

The similar conditions of § 9 indicate that *a necessary and sufficient condition that a net be a net R is that the tangents to the associated axis curves of the net pass through the focal points of the corresponding ray.* Moreover, *an R net is isothermal-conjugate.*‡

A theorem due to Demoulin § also follows at once: *If the tangents to the curves of either family of an isothermal-conjugate net form a W congruence, it is a net R.*

Again, *if a net R has any one of the following three properties, it has the other two also: equal point invariants, equal tangential invariants, it is a net A.*

Finally, *a net A which is isothermal-conjugate is a net R.*

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* *l. c.* It is evident that the fundamental transformation existing between the surfaces of the Laplace transforms of a net A will be such that the product of its harmonic and its conjugate invariant will be unity (cf. Author, "A Contribution to the Theory of Fundamental Transformations of Surfaces, *Transactions of the American Mathematical Society*, Vol. 30, 18).

† *Comptes Rendus*, Vol. 152 (1911), p. 1077.

‡ Tzitzéica, *l. c.*

§ *Comptes Rendus*, Vol. 153 (1911), p. 592.

ADMISSIBLE NUMBERS IN THE THEORY OF GEOMETRICAL PROBABILITY.*

By A. H. Copeland.

Geometry is concerned with non-denumerable aggregates of points. On the other hand, probability, from the point of view of its statistical definition, has an essentially denumerable character. Thus in geometrical probability we apply an analysis which is concerned with denumerable aggregates, to a subject which is concerned with non-denumerable aggregates. As a result we get certain inconsistencies in our assumptions. Fortunately these inconsistencies are not serious, and it is possible to obtain a set of assumptions which are both consistent and useful.†

The assumptions made in the case of a simple event have been shown consistent by proving the existence of admissible numbers.‡ In geometrical

* This paper was presented to the Society Sept. 7, 1928. It is based on a paper by the author, entitled, "Admissible Numbers in the Theory of Probability," *American Journal of Mathematics*, Vol. 50 (1928), pp. 535-552. The reader is referred to this memoir for definitions and notation.

† For other discussions of the foundations of the theory of probability see von Mises, "Grundlagen der Wahrscheinlichkeitsrechnung," *Mathematische Zeitschrift*, Vol. 5 (1919), pp. 52-99; Lomnicki, "Nouveaux fondéments du calcul des probabilités," *Fundamenta Mathematicae*, Vol. 4 (1923), pp. 34-71; Steinhaus, "Les probabilités dénombrables et leur rapport à la théorie de la mesure," *Fundamenta Mathematicae*, Vol. 4 (1923), pp. 286-310; Dodd, "Probability as Expressed by Asymptotic Limits of Pencils of Sequences," *Bulletin of the American Mathematical Society*, Vol. 36, No. [illegible] (1930), pp. 299-305; Borel, "Traité du calcul des probabilités," Chapitre I. The *nombres normaux* of Borel are members of the set, $A(1/2)$. That is, the set of admissible numbers includes the set of normal numbers as a sub set.

‡ See the author's memoir cited above. The statistical definition of probability has been criticised by T. C. Fry (*Probability and Its Engineering Uses*, pp. 88-9[illegible]). Briefly Fry's argument is as follows. Let us suppose that the ratio, $p_n(x)$, of the number of successes to the number of trials of a given event, x, approaches the probability, p, as the number of trials is indefinitely increased. That is we assume that if we have given an arbitrary positive number, ϵ, we can find a number, N, such that $|p_n(x)-p|<\epsilon$ whenever $n \geqq N$. For definiteness let us choose $p=1/2$, and $\epsilon=1/4$. Then there must exist an N such that $|p_n(x)-1/2|<1/4$ whenever $n \geqq N$. Let us make N trials of the given event. If the result of the experiment is such that $p_n(x)>1/2$, then since the trials are independent, there is a finite probability that the next N trials will all be successes. If $p_n(x) \leqq 1/2$, then it is possible for the next N trials all to be failures. In either case it is easily seen that $|p_n(x)-1/2| \geqq 1/4$ when $n=2N$. So far Fry's reasoning is correct. But he concludes that the statistical definition of probability is inconsistent with the postulate that the trials of a given event are all independent. This conclusion is not justified.

probability we are confronted with the problem of proving the existence of a set of related admissible numbers having the power of the continuum.

Let $\pi(E)$ be the probability that a point, P, of an n-dimensional continuum, belong to a given set, E. The function, $\pi(E)$, is necessarily additive and we will assume further that it is absolutely additive and absolutely continuous. These restrictions are satisfied in most of the cases that arise. Finally $0 \leqq \pi(E) \leqq 1$, and there exists a domain, Δ, such that $\pi(\Delta) = 1$. For example in the case of the normal law of error the domain, Δ, is the infinite interval, $(-\infty, +\infty)$, and the probability that the error of a given measurement be one of a set of values, E, is $\pi(E) = k/(\pi)^{1/2} \int_E e^{-k^2x^2} dx$.

We shall now investigate the admissibility of the event histories associated with the sets, E. Let $x(E)$ represent the event history associated with E. Then the function, $x(E)$, must satisfy the following restrictions.

(a) $x(E_1) \cdot x(E_2) = 0$ whenever $E_1 \cdot E_2 = 0$.

(b) If $E_1, E_2, E_3, \cdots$ is any sequence (finite or infinite) of mutually exclusive point sets, then the n-th digit of $x(E_1 + E_2 + E_3 + \cdots)$ is 1 if and only if the n-th digit of one of the numbers, $x(E_1)$, $x(E_2), \cdots$, is 1.

(c) $x(\Delta) = 1.$*

Condition (a) demands that events corresponding to mutually exclusive sets, be, themselves, mutually exclusive. Condition (b) demands that the function $x(E)$ be absolutely additive. It further specifies the mode of representation in certain cases where this representation is ambiguous. The interpretation of this condition is immediate.

The above conditions contain no reference to the function, $\pi(E)$, and no reference to the admissibility of the numbers, $x(E)$. We should expect, in fact, the further condition that every number, $x(E)$, must belong to the set, $A[\pi(E)]$. Moreover we should expect the numbers $(r_1/n)x(E_1)$, $(r_2/n)x(E_2), \cdots (r_k/n)x(E_k)$, to be independent for all sets, $E_1, E_2, \cdots E_k$,

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As stated above, I have proved that no such inconsistency can arise in the case of a simple event. Fry's reasoning merely shows that the choice of N depends upon x. That is, $p_n(x)$ does not approach its limit uniformly with regard to x. This has nothing to do with the existence of the limit for a given x.

One important conclusion can be drawn from Fry's reasoning. Namely that we can never in any physical situation, know the value of N. Hence in order to consider questions of consistency in the theory of probability we are forced to depend on some such device as that of admissible numbers.

* The number, 1, admits of two representations. For the sake of brevity we write, 1, whereas the representation to which we refer in this case is, .111, 111, 111, $\cdots$.

and for all sets of positive integers, $r_1, r_2, \cdots, r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n. If these conditions, together with conditions, (a), (b), (c), could all be satisfied then the fundamental assumptions of geometrical probability would be consistent when applied to arbitrary sets, E. It turns out that these conditions cannot all be satisfied unless we confine ourselves to sets, E, whose frontier points are of measure zero. However, this restriction upon the point sets is so light that the assumptions of geometrical probability are satisfied in all of the interesting cases.

We are now in a position to state the fourth condition which we shall place upon the function, $x(E)$.

$$(d) \qquad p[(r_1/n)x(E_1)\cdot(r_2/n)x(E_2)\cdots(r_k/n)x(E_k)] = \pi(E_1)\cdot\pi(E_2)\cdots\pi(E_k)$$

for all sets, $E_1, E_2, \cdots, E_k$, whose frontier points are of measure zero, and for all sets of positive integers, $r_1, r_2, \cdots r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n.

Condition (d) demands that every number, $x(E)$, belong to the set $M[\pi(E)]$, and that every set of numbers, $(r_1/n)x(E_1)$, $(r_2/n)x(E_2), \cdots (r_k/n)x(E_k)$, be independent provided the frontier points of the sets, $E, E_1, E_2, \cdots E_k$, are of measure zero. In order to see why we can include only those sets, E, whose frontier points are of measure zero, let us investigate to what extent the function, $x(E)$, is restricted by conditions, (a), (b), (c).

THEOREM 1. *A necessary and sufficient condition that a function, $x(E)$, satisfy conditions, (a), (b), (c), is that there exist a denumerable set, $D: (P_1, P_2, P_3, \cdots)$ such that $D < \Delta$ and*

$$(1) \qquad x(E) = \cdot\, \phi_E(P_1), \phi_E(P_2), \phi_E(P_3), \cdots$$

*where $\phi_E(P)$ is the fundamental function of the set, E.**

The condition is obviously sufficient so we shall concern ourselves with proving that it is necessary. We shall prove the theorem first for an n-dimensional continuum in which Δ is the region defined by the inequalities, $0 \leq y_i < 1$, where $i = 1, 2, 3, \cdots n$ and where $y_1, y_2, \cdots, y_n$ are the coordinates of a point, P, in Δ. We shall cover Δ with a net consisting of an infinite set of lattices, $G_1, G_2, G_3, \cdots$. To construct the lattice, G_1, we

* See de la Vallée Poussin, "Sur l'intégral de Lebesgue," *Transactions of the American Mathematical Society* (1915).

decompose Δ into 2^n meshes, $m_{1,1}, m_{1,2}, m_{1,3}, \cdots m_{1,2^n}$, where the mesh, $m_{1,j}$, is the region, $a_{ij}/2 \leqq y_i < (a_{ij}+1)/2$ and where each a_{ij} has the value 1 or 0. The numbers, a_{ij}, are further defined so that the meshes, $m_{1,j}$, are non-overlapping and together include all of the points of Δ. To form the lattice, G_2, we decompose each of the meshes, $m_{1,j}$, into 2^n meshes, $m_{2,k}$, in the same manner in which Δ was decomposed to form the lattice, G_1. Thus G_2 contains 2^{2n} meshes. The lattices, G_3, G_4, etc. are formed in a similar manner. We shall assume that the meshes are so numbered that $m_{i-1,j}$ includes $m_{i,(j-1)2^n+1}, m_{i,(j-1)2^n+2}, \cdots m_{i,j2^n}$.

It follows from conditions, (a), (b), (c), that for a given i, one and only one of the numbers, $x(m_{i,j})$, has its first digit equal to one. Let this number be $x(m_{i,k_i})$. It follows from condition (a) that $m_{i+1,k_{i+1}} < m_{i,k_i}$. Thus if we let i become infinite we obtain a limiting set of the sequence, m_{i,k_i}. This set consists of a single point, P_1. By conditions (b), (c) we see that the first digit of $x(\Delta - P_1)$ must be zero and the first digit of $x(P_1)$ must be one. In a like manner we can find a point P_2 such that the second digit of $x(\Delta - P_2)$ is zero and the second digit of $x(P_2)$ is one. By continuing this process we obtain a denumerable set, D, consisting of the points, $P_1, P_2, P_3, \cdots$, and such that $x(\Delta - D) = 0$ and $x(D) = 1$. Thus $x(E)$ is given by equation, (1).

Next let us consider the case in which Δ includes all space. This case can be reduced to the above by means of the set of transformations, $T: y_i' = F(y_i)$, where $F(y)$ is monotone and continuous and such that $F(-\infty) = 0$ and $F(+\infty) = 1$ and where the inverse of $F(y)$ is continuous in the interval $0 < y' < 1$. In particular we can take $F(y) = 1/2 + y/2(1+y^2)^{1/2}$. If $x'(E') = x(E)$ where E' is the transformed set, E, then conditions, (a), (b), (c), are satisfied by $x'(E')$ provided they are satisfied by $x(E)$.* Hence this case is reduced to the above.

Finally if Δ is an arbitrary region then this case can be reduced to the preceding one by extending the definition of $x(E)$ outside of Δ by means of the equation, $x(E) = x(E \cdot \Delta)$.

Theorem 1 admits of a simple physical interpretation. The number, $x(E)$, represents the history of some imaginary event which succeeds or fails on the n-th trial according as P_n belongs or fails to belong to E. That is, P_n is the point obtained on the n-th trial of some imaginary physical ex-

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* For sets, E' containing points with one or more coördinates equal to zero, $x'(E')$ is defined by the equation, $x'(E') = x'(E' \cdot \Delta')$, where Δ' is the region $0 \leqq y_i' < 1$. This extension of the definition is necessary since we shall exclude regions Δ having points at infinity.

... it. Thus we might have forseen the existance of the set, D, from ... considerations.

... can now see why it is possible to include only those sets ... frontier points are of zero measure. For let Δ be the unit interval, $0 \leq y < 1$, and let $\pi(E)$ be the Lebesgue measure of the set, E. Then if E ... is an interval of length $\epsilon < 1$, we have the equations, $\pi(E) = \epsilon$, ... 1. Hence $x(E)$ is not an element of $A[\pi(E)]$. It will be observed that the frontier points of E are of measure, $1 - \epsilon > 0$.

I. Sets Whose Frontier Points Are of Measure Zero.

It turns out that our work will be greatly simplified if we replace ... (...) by a somewhat lighter condition which we shall refer to as ... Condition (d′) is the same as (d) except that we include only those ... $E_1, E_2 \cdots E_{\Sigma}$, which consist of finite sums of meshes. We shall also have ... to refer to a still lighter condition, (d″), which differs from (d′) ... that the meshes, $m_{i,j}$, which constitute the sets, $E_1, E_2, \cdots E$ ... s, where s is any given positive integer. We shall prove ... function satisfies conditions (a), (b), (c) and (d′), it will automatically satisfy (e). In order to show this we shall make use of the following theorem.

Theorem 2. *If $E < \Delta$ where Δ is the region, $0 \leqq y_i < 1$, ... necessary and sufficient condition that the frontier points, f, of E ... measure zero, is that given any positive number, ϵ, there exist two sets, E_1 and E_2, such that $E_1 < E < E_2$ and $m(E_2 - - E_1) < \epsilon$ and such that E_2 consists of a finite sum of meshes and E_1 is either null or else consists of a finite sum of meshes.*

The set, f, is closed and all of its points lie within or on the boundary of Δ. If $m(f) = 0$, then there exists an open set, 0, such that $f < 0$ and $m(0) < \epsilon$, where ϵ is an arbitrary positive number. If P is any point of $f \cdot \Delta$ then P lies in a mesh which in turn lies entirely within 0. If P is any other point of f then P lies on the boundary of a mesh which lies entirely within 0. It follows from the Heine-Borel theorem that there exists a set, $e < 0$, consisting of a finite sum of meshes and such that f is included in e plus those boundary points of e which are also boundary points of Δ.

Every interior point of E can be enclosed in a mesh which contains no points of f, and we have already obtained a law whereby the points of f can be enclosed in a finite sum of meshes plus certain boundary points of those meshes. Thus, applying the Heine-Borel theorem again we see that there exists a set, E_1, which is either null or else consists of a finite sum of

meshes no one of which includes points of f, and such that $E_1 + e > E$. Let $E_2 = E_1 + e$. Then $E_1 < E < E_2$ and $m(E_2 - E_1) = m(e) \leqq m(0) < \epsilon$. Moreover E_2 consists of a finite sum of meshes and E_1 is either null or else consists of a finite sum of meshes.

We shall now prove the converse. If we delete from E_1, the frontier points of E_1 we obtain a set 0_1, which differs from E_1 by a set of measure zero. Similarly if we add to E_2 the frontier points of E_2 we obtain a set F_2, which differs from E_2 by a set of measure zero. Then $F_2 - 0_1 > f$, and since the quantity, $m(F_2 - 0_1)$, can be made arbitrarily small by a proper choice of the sets, E_1 and E_2, it follows that $m(f) = 0$.

THEOREM 3. *If $\pi(E)$ is an absolutely additive absolutely continuous function defined in $\Delta: 0 \leqq y_i < 1$, and if $x(E)$ is a function which satisfies conditions (a), (b), (c), (d′) with respect to $\pi(E)$, then $x(E)$ satisfies conditions (a), (b), (c), (d), with respect to $\pi(E)$.*

If $E_1, E_2, \cdots E_k$ are any sets such that they all lie in Δ and their frontier points are of measure zero, then given any positive number ϵ we can find sets, $E_1', E_2', \cdots E_k', E_1'', E_2'', \cdots E_k''$, consisting of finite sums of meshes and such that $E_i' < E_i < E_i''$ and

$$\begin{aligned}&\pi(E_1'') \cdot \pi(E_2'') \cdots \pi(E_k'') - \epsilon/2 \\ &\leqq \pi(E_1) \cdot \pi(E_2) \cdots \pi(E) \leqq \pi(E_1') \cdots \pi(E_k') + \epsilon/2.\end{aligned}$$

Since $x(E)$ satisfies condition (d′) it follows that if we have given any set of positive integers, $r_1, r_2, \cdots r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n, then we can select a number, s_0, such that

$$\pi(E_1') \cdot \pi(E_2') \cdots \pi(E_k') - \epsilon/2 \leqq p_s\,[(r_1/n)x(E_1') \cdots (r_k/n)x(E_k')]$$

and

$$p_s\,[(r_1/n)x(E_1'') \cdots] \leqq \pi(E_1'') \cdots \pi(E_k'') + \epsilon/2$$

whenever $s > s_0$. Moreover since $x(E)$ satisfies conditions (a), (b), (c) it is defined by equation (1) and hence

$$p_s[(r_1/n)x(E_1') \cdots] \leqq p_s[(r_1/n)x(E_1) \cdots] \leqq p_s[(r_1/n)x(E_1'') \cdots].$$

Combining these inequalities we get the relation

$$\pi(E_1) \cdots \pi(E_k) - \epsilon < p_s[(r_1/n)x(E_1) \cdots] < \pi(E_1) \cdots \pi(E_k) + \epsilon$$

whenever $s > s_0$. Therefore $x(E)$ satisfies (a), (b), (c), (d).

II. ADMISSIBLY ORDERED SETS.

We shall say that a denumerable set, D, is admissibly ordered with respect to a given function, $\pi(E)$, provided the corresponding function, $\rho(E)$, satisfies conditions, (a), (b), (c), and (d).

THEOREM 1. *There exists a set, D, which is admissibly ordered with respect to the function, $m(E)$, (the Lebesgue measure of E) defined in the interval, $\Delta: 0 \leqq y < 1$.*

We shall show that conditions, (a), (b), (c), (d″), can be satisfied. We shall then prove that the restriction, $i \leqq s$, of condition (d″), can be removed.

Let us define a function, $X_s(E)$, as follows. Let X be any member of [illegible] (1/2) and let $X_s(m_{11})=(1/s)X$ and $X_s(m_{12})=1-(1/s)X$. In general let

$$X_s(m_{i,2j-1})=X_s(m_{i-1,j}) \cdot (i/s)X$$

and

$$X_s(m_{i,2j})=X_s(m_{i-1,j}) \cdot [1-(i/s)X]$$

where $1 < i \leqq s$. Then $X_s(m_{i,j}) \cdot X_s(m_{i,j'})=0$ if $j \neq j'$ and

$$X_s(m_{i-1,j})=X_s(m_{i,2j-1})+X_s(m_{i,2j}).$$

Hence we will introduce no contradiction in our notation if we define $X_s(\Sigma m_{i,j})$ to be equal to $\Sigma X_s(m_{i,j})$ where the meshes, $m_{i,j}$, which appear in the summation, are mutually exclusive. Thus conditions, (a), (b), are satisfied by the function, $X_s(E)$, for all sets, E, consisting of sums of meshes, $m_{i,j}$ $(i \leqq s)$. Moreover condition (d″) is satisfied since every number, $X_s(m_{i,j})$, can be written as the product of i numbers each of which is of the form $(k/s)X$ or $1-(k/s)X$.

Corresponding to the function, $X_s(E)$, we can define a set, $D_s:(P_1^s, P_2^s, P_3^s, \cdots)$, as follows. The point, P_n^s, is an arbitrary point of that mesh, $m_{s,j}$, for which the n-th digit of $X_s(m_{s,j})$ is equal to 1. The function, $\rho_s(E)$, can now be defined for all sets, E, by means of equation 1. This function satisfies conditions, (a), (b), (c).

To remove the restriction, $i \leqq s$, we shall define a new set, $D:(P_1, P_2, P_3, \cdots)$. Points P_{ν_s+1} to $P_{\nu_{s+1}}$ of D are the same as points, P_1^s to $P_{X_s}^s$, of D_s, where $\nu_s = X_1 + X_2 + \cdots X_{s-1}$. We shall show that the integers, $X_1, X_2, \cdots$, can be chosen so that the set, D, will be admissibly [illegible]

See Theorem 16 of the memoir previously cited.

Let $\epsilon_1, \epsilon_2, \cdots \epsilon_s, \cdots$ be a decreasing sequence of positive numbers having the limit zero. We can choose two sets of integers, $M_1, M_2, M_3, \cdots M_s, \cdots$ and $N_1, N_2, \cdots$, such that

$$\text{(e)} \quad | p_N[(r_1/n)X_{s+1}(E_1) \cdot (r_2/n)X_{s+1}(E_2) \cdots (r_k/n)X_{s+1}(E_k)] - m(E_1) \cdot m(E_2) \cdots m(E_k) | < \epsilon_s/3 \quad \text{if} \quad N \geqq M_s/n$$

$$\text{(f)} \quad | p_N[(r_1/n)X_s(E_1) \cdot (r_2/n)X_s(E_2) \cdots (r_k/n)X_s(E_k) - m(E_1)m(E_2) \cdots m(E_k) | + (\nu_s + M_s)/N_s < \epsilon_s/3 \text{ if } N \geqq N_s/n.$$

The numbers, M_s, and N_s, are so chosen that conditions, (e) and (f), hold for every set of positive integers, $r_1, r_2, \cdots r_k, n$, such that $n \leqq s$ and the numbers, r_i, are all distinct and less than or equal to n, and for all sets, $E_1, E_2, \cdots E_k$, consisting of sums of meshes, $m_{i,j}$, such that $i \leqq s$. At the same time the numbers, N_s, are so chosen that ν_s/n is an integer if $n \leqq s$.

Since digits $\nu_s + 1$ to ν_{s+1} of $x(E)$ are the same as digits 1 to N_s of $X_s(E)$ it follows that

$$\text{(g)} \quad | p_N[(r_1/n)x(E_1) \cdots (r_k/n)x(E_k)] - m(E_1) \cdot m(E_2) \cdots m(E_k) | < \epsilon_s \quad \text{if} \quad \nu_{s+1}/n \leqq N \leqq \nu_{s+2}/n.^*$$

Moreover the restrictions on i and n are no longer necessary, for if we select first the sets, $E_1, E_2, \cdots E_k$, consisting of finite sums of meshes, $m_{i,j}$, and next the numbers, $r_1, r_2, \cdots r_k, n$, then condition (g) holds for every s which is at the same time greater than n and greater than the largest subscript, i. Hence

$$p[(r_1/n)x(E_1) \cdots (r_k/n)x(E_k)] = m(E_1) \cdot m(E_2) \cdots m(E_k).$$

It follows from theorem 3 that D is admissibly ordered.

THEOREM 5. *Given a denumerable set, D, which is everywhere dense in the domain, $\Delta: 0 \leqq y_i < 1$, and given an absolutely additive absolutely continuous function, $\pi(E)$, such that $\pi(\Delta) = 1$ and $\pi(E) > 0$ if $m(E) > 0$ (where $E < \Delta$), then the set, D, can be admissibly ordered with respect to the function, $\pi(E)$.*

Let D consist of the points, $P_1, P_2, P_3, \cdots$, and let $D'': P_1'', P_2'', \cdots$ be the reordered set, D. We have to show that the reordering can be accomplished in such a manner that the function, $x(E) = \cdot\, \phi_E(P_1''), \phi_E(P_2'') \cdots$, satisfies condition (d′).

Let Δ' be the domain, $0 \leqq y < 1$, and let $D': P_1', P_2' \cdots$ be a set which is admissibly ordered with respect to the function, $m(E')$. We shall set up

* Compare theorem 11 of the memoir cited.

a correspondence between the meshes $m_{i,j}$, and a set of half open intervals, $m'_{i,j}$, which are included in Δ'. The interval, $m'_{i,j}$, is given by the inequalities,

$$\pi(m_{i1}+m_{i2}+\cdot\cdot\cdot m_{i,j-1}) \leqq y < \pi(m_{i1}+\cdot\cdot\cdot m_{i,j}), \quad \text{if} \quad j>1.$$

The interval, $m'_{i,1}$, is given by the inequalities, $0 \leqq y < \pi(m_{i,1})$. Then the interval, $m'_{i-1,j}$, will contain the intervals, $m'_{i,(j-1)2^n+1}$, $m'_{i,(j-1)2^n+2}, \cdot\cdot\cdot, m'_{i,j2^n}$. Thus the intervals will cover Δ' in the same manner that the meshes cover Δ.

Every point of D' is included in one of the intervals, $m'_{1,j}$. Let m'_{1,j_1} be the interval which includes the point, P_1'. We shall select from those points of D which lie in m_{1,j_1}, that one whose subscript is least. We shall relabel this point, P_1''. We shall continue this process with the points, $P_2', P_3', \cdot\cdot\cdot$, setting up a correspondence in each case with that point of D which lies in the proper interval and which has the least subscript of those points not already assigned. We finally reach a point, P_{n_1}', such that at least one of the points, $P_1', P_2', \cdot\cdot\cdot P_{n_1}'$, lies in each of the intervals, $m'_{1,j}$. One of the n_1 relabeled points must be the point, P_1.

We shall select a number, n_2, so that at least one of the points, P'_{n_1+1}, $P'_{n_1+2}, \cdot\cdot\cdot P'_{n_2}$, lies in each of the intervals, $m_{2,j}$. Using these intervals we shall relabel n_2-n_1 more of the points of D in the manner described above. We shall call the relabeled points, $P''_{n_1+1}, P''_{n_1+2}, \cdot\cdot\cdot P''_{n_2}$. We have now relabeled the point, P_2. This process is continued indefinitely. By the time the assignment of points has been completed for the k-th lattice, the point, P_k, has been relabeled.

We shall now show that the reordering of D has been accomplished in such a manner that $x(E)$ satisfies condition (d′) and hence satisfies (d). Let $E_1, E_2, \cdot\cdot\cdot E_k$ be any sets consisting of finite sums of meshes. Let $E_1', E_2', \cdot\cdot\cdot E_k'$ be corresponding linear sets, the correspondence being defined in terms of the correspondence which we have already established between meshes and half open intervals. Then at most a finite number of the digits of $x(E_i)$ differ from the corresponding digits of

$$x'(E_i') = \cdot\, \phi_{E_i''}(P_1'),\ \phi_{E_i'}(P_2') \cdot\cdot\cdot .$$

Therefore

$$\begin{aligned} p[(r_1/n)x(E_1)\cdot(r_2/n)x(E_2)\cdot\cdot\cdot(r_k/n)x(E_k)] \\ &= p[(r_1/n)x'(E_1')\cdot(r_2/n)x'(E_2')\cdot\cdot\cdot] \\ &= \pi(E_1)\cdot\ \pi(E_2)\cdot\cdot\cdot\pi(E_k). \end{aligned}$$

Hence D has been admissibly ordered with respect to $\pi(E)$.

In theorem 5, the restriction, $\pi(E)>0$ if $m(E)>0$, was made in order that D could be reordered without leaving out any of its points. Let us see

to what extent this restriction could be removed. Let Δ be the region $0 \leqq y_i < 1$, and let us define a set, Δ_0, by means of its complement with respect to Δ. The set $C\Delta_0$ will consist of all of the meshes, $m_{i,j}$, for which $\pi(m_{i,j}) = 0$. It is easily seen that $\pi(C\Delta_0) = 0$ and $\pi(\Delta_0) = 1$. Let D be a denumerable set such that $D < \Delta_0$ and every point of Δ_0 is a limit point of points of D. Obviously D can be admissibly ordered with respect to $\pi(E)$. We shall call D the skeleton set of the function, $\pi(E)$.

Next let us consider an arbitrary absolutely additive absolutely continuous non-negative function, $\pi(E)$, such that $\pi(\Delta) = 1$. The most general case is that in which Δ includes all space. Let us apply transformations, T, to the sets, E, and let $\pi'(E')$ be the transformed function $\pi(E)$. We can define a skeleton set, D', for the function, $\pi'(E')$, the inverse of the transformations, T, carry D' into a set, D, which we shall call the skeleton set of the function $\pi(E)$. The following theorem is now obvious.

THEOREM 6. *If $\pi(E)$ is an absolutely additive absolutely continuous non-negative function such that $\pi(\Delta) = 1$, where Δ includes all space, then the skeleton set, D, of $\pi(E)$ can be admissibly ordered with respect to $\pi(E)$.*

The admissibly ordered skeleton set, D, characterizes the function, $\pi(E)$. In fact we have the equation,

$$\pi(E) = \lim_{n \to \infty} \sum_{k=1}^{n} \phi_E(P_k)/n$$

which holds for all sets whose frontier points are of zero measure.

We have now proved that the fundamental assumptions of geometrical probability are valid when applied to sets whose frontier points are of measure zero, but that they lead to inconsistencies if applied to arbitrary sets.

UNIVERSITY OF MICHIGAN.

CONTINUOUS CURVES WITHOUT LOCAL SEPARATING POINTS.*

By G. T. Whyburn.

1. In this paper it will be shown that every pair of points a and b of a continuous curve M which has no local separating point lie together on a subcontinuum T of M which is the sum of c ($=$ the power of the continuum) independent simple continuous arcs from a to b. It follows at once from this result that every continuous curve which has no local separating point contains continua that are not locally connected or, in other words, that every continuous curve all of whose subcontinua are continuous curves has local separating points.

We use the term *continuous curve* to designate any locally compact, metric, separable, connected and locally connected space. Any connected open subset of such a space is called a *region;* and a point which is a cut point of at least one region in the space is called a *local separating point* of the space.

2. Lemma. *Let R be any compact region in a continuous curve M, and let N be a closed subset of $M - R$ such that $\bar{R} \cdot N$ is totally disconnected. Then there exists a compact region G containing R but containing no point of N and such that* (1) $\bar{G} \cdot N = \bar{R} \cdot N$, (2) *$G + \bar{R} \cdot N$ contains a compact continuous curve H which contains $\bar{R} \cdot N$ and is such that $H - \bar{R} \cdot N$ is connected and contains R, and* (3) *each point of $\bar{R} \cdot N$ is accessible from $H - \bar{R} \cdot N$ and hence also from G.*

Proof. Let K_1 denote the set of all points of $\bar{R}$ at a distance > 1 from the point set $\bar{R} \cdot N$; and for each integer $n > 1$, let K_n denote the set of all points x of $\bar{R}$ such that $1/n \leq \rho(x, \bar{R} \cdot N) \leq 1/(n-1)$.

A simple application of the Borel Theorem proves the existence, for each positive integer n, of a finite number of compact continua $C_1{}^n$, $C_2{}^n$, $\cdots$, $C_m{}^n$ each containing a point of R and whose sum C^n contains K_n in its interior (rel. M) but contains no point of N and no point whose distance from K_n is greater than $1/4n$. For each i, $1 < i \leq m$, let t_i be an arc in R joining a point $C_i{}^n$ to a point of $C_1{}^n$. Add all these arcs t_i to C^n and call the point set thus obtained D_n. Then D_n is a compact continuum which contains K_n but contains neither a point of N nor any point of $M - R$ whose distance from K_n is greater than $1/4n$. By a theorem due to Ayres

* Presented to the American Mathematical Society, Sept. 10, 1930.

and the author,* M contains, for each n, a compact continuous curve H_n containing D_n but which contains neither a point of N nor any point of $M-R$ whose distance from K_n is greater than $1/2n$.

Let $H_0 = \sum_1^\infty H_n$, and let $H = H_0 + \bar{R}\cdot N$. For each n, let G_n be a compact region containing H_n but containing no points or boundary points in N and containing no point whose distance from H_n is greater than $1/2n$. Let $G = \sum_1^\infty G_n$. Then the sets G and H have the desired properties.

2. THEOREM. *If a and b are any two points of a continuous curve M having no local separating point, then there exists a subcontinuum T of M such that*

$$T = \sum_{0 \leq x \leq 1} axb,$$

where, for each x, axb is an arc from a to b and where, for $x \neq y$, $0 \leq x, y \leq 1$, $axb \cdot ayb = a + b$.

We shall first prove the existence of a continuum T satisfying all the conditions except that the sets axb, $(0 \leq x \leq 1)$, are compact continua but not necessarily simple arcs; and then we shall give the modifications in this argument which are necessary to insure that the continua $[axb]$ will be arcs from a to b.

Since no point separates a and b in M, there exist † in M two independent ‡ arcs T_0 and T_1 from a to b. There exist in M two compact regions R_0 and R_1 containing the open arcs $T_0-(a+b)$ and $T_1-(a+b)$ respectively and such that $\bar{R}_0 \cdot \bar{R}_1 = a+b$. Now, applying the lemma in turn to the regions R_0 and R_1, using for the closed set N in the lemma first the set $\bar{R}_1$ and then the set $\bar{G}_0$, we obtain two compact regions G_0 and G_1 such that (1) $\bar{G}_0 \cdot \bar{G}_1 = a+b$, (2) $\bar{G}_0$ and $\bar{G}_1$ contain continuous curves H_0 and H_1 respectively both of which contain $a+b$ and such that $H_0-(a+b)$ and $H_1-(a+b)$ are connected sets which contain R_0 and R_1 respectively.

Since $H_0 \supset R_0$, clearly no point can separate a and b in the continuous curve H_0. Hence there exist in H_0 two independent arcs T_{00} and T_{01} from a to b. There exist in G_0 two regions R_{00} and R_{01} containing the open arcs $T_{00}-(a+b)$ and $T_{01}-(a+b)$ respectively and such that $\bar{R}_{00} + \bar{R}_{01} \subset G_0 + a + b$ and $\bar{R}_{00} \cdot \bar{R}_{01} = a+b$. Then, applying the lemma just as in

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* See *Bulletin of the American Mathematical Society*, Vol. 34 (1928), p. 350.

† See the author's paper in the *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 308, Theorem III, and a paper by W. L. Ayres in this Journal, Vol. 51 (1929), p. 590, where the author's theorem is extended to the more general space required in the present application.

‡ Two arcs are said to be independent if they have at most their end points in common.

the preceeding paragraph, we obtain regions G_{00} and G_{01} in R_0 and continuous curves H_{00} and H_{01} such that, $\bar{G}_{00} \cdot \bar{G}_{01} = (a+b)$, $R_{00} \subset H_{00} - (a+b) \subset H_{00} \subset \bar{G}_{00}$, and $R_{01} \subset H_{01} - (a+b) \subset H_{01} \subset \bar{G}_{01}$. Similarly in H_1 we get the arcs T_{10} and T_{11} from a to b and in G_1 we get regions R_{10}, R_{11}, G_{10} and G_{11} and continuous curves H_{10} and H_{11} satisfying similar conditions. Continue this process indefinitely.

At each stage n, let G_n denote the sum of the 2^n closed regions $\bar{G}_{a_1,a_2 \ldots a_n}$ constructed at that stage, i. e., let

$$G_n = \sum_{a_i=0,1} \bar{G}_{a_1,a_2, \ldots a_n}.$$

Let $T = \prod_1^\infty G_n$. Then T is the desired continuum. For let x be any real number, $0 \leq x \leq 1$. Write $x = .a_1a_2a_3 \cdots$, where for each i, a_i is either 0 or 1.* Now the point set $axb = \prod_1^\infty \bar{G}_{a_1a_2 \ldots a_i}$ is a subcontinuum of M; and if x and y are distinct numbers between 0 and 1, it is readily seen that $axb \cdot ayb = a + b$.

3. I shall now indicate the modifications necessary in the construction of the sets $[G_{a_1a_2 \ldots a_n}]$ in order to insure that the continua $[axb]$ will be simple arcs from a to b. For simplicity, I shall define only the sets G_0, G_{00}, $G_{000}, \cdots$ so that the product $\bar{G}_0 \cdot \bar{G}_{00} \cdot G_{000} \cdots$ will be an arc from a to b, since obviously the construction is the same for all the other sets $[G_{a_1a_2 \ldots a_n}]$, it being understood that for each n, the 2^n sets $[G_{a_1a_2 \ldots a_n}]$ are constructed so that if G_1 and G_2 are any two of these sets, then $\bar{G}_1 \cdot \bar{G}_2 = a + b$.

With the aid of the arc T_0 it is easily seen that there exists a compact simple chain R_0 of regions $V_1, V_2, V_3, \cdots V_n$ all of diameter < 1 such that $\bar{V}_1 \cdot T_1 = a$, $\bar{V}_n \cdot T_1 = b$, $\bar{V}_i \cdot T_1 = 0$ for $1 < i < n$, $V_i \cdot V_j \neq 0$ for $i \neq j$ if and only if $|i - j| = 1$, and $\bar{V}_i \cdot \overline{(R_0 - V_{i-1} - V_i - V_{i+1})} = 0$ for $i \leq i \leq n$, where $V_0 = V_{n+1} = 0$. Now, applying the lemma to the region V_1, we can obtain a compact region U_1 of diameter < 1 such that $\bar{U}_1 \cdot T_1 = a$, $\bar{U}_1 \cdot \overline{(R_0 - V_1 - V_2)} = 0$, and such that $U_1 + a$ contains a compact continuous curve H_1 having the property that $H_1 - a$ is connected and contains V_1. There exist distinct points x and y in the set $H_1 \cdot V_1 \cdot V_2$. Since $H_1 \supset V_1$ and M has no local separating point, it follows that no point can separate either x or y from a in H_1. Therefore † there exist arcs ax and ay in H_1 such that $ax \cdot ay = a$. On the arcs ax and ay, in the orders a, x and a, y, let x_1 and y_1 denote respectively the first points belonging to $\bar{V}_2$. Now,

* That is, x is expressed according to the dyadic number system.

† See W. L. Ayres, *loc. cit.*

applying the lemma to V_2, we can obtain a compact region U_2 of diameter < 1 containing V_2 and such that $\bar{U}_2 \cdot (T_1 + ax_1 + ay_1) = x_1 + y_1$, $\bar{U}_2 \cdot \overline{(R_0 - V_{2-1} - V_2 - V_{2+1})} = 0$ and such that $\bar{U}_2$ contains a compact continuous curve H_2 having the property that $H_2 - (x_1 + y_1)$ is connected and contains V_2. Let w and z be two points belonging to the set $H_2 \cdot V_2 \cdot V_3$. Then, just as before, no point can separate any two of the points x_1, y_1, w and z in H_2, and accordingly there exist two mutually exclusive arcs in H_2 joining the sets $x_1 + y_1$ and $w + z$. The two possible cases here are alike, so we shall suppose there exist mutually exclusive arcs x_1w and y_1z in H_2. On these two arcs, in the orders x_1, w, and y_1, z, let x_2 and y_2 respectively denote the first points belonging to $\bar{V}_3$. Then apply the lemma to V_3, and so on. Continuing this process for $n - 1$ steps, we obtain regions U_1, U_2, $\cdots, U_{n-1}$ and arcs ax_1, x_1x_2, $x_2x_3, \cdots x_{n-2}x_{n-1}$, ay_1, y_1y_2, $y_2y_3, \cdots y_{n-2}y_{n-1}$. Applying the lemma to V_n we can obtain, just as before, a region U_n and two arcs $x_{n-1}b$ and $y_{n-1}b$ satisfying similar conditions. Set $G_0 = \sum_1^n U_m$, $T_{00} = ax_1 + x_1x_2 + \cdots + x_{n-1}b$, and $T_{01} = ay_1 + y_1y_2 + \cdots + y_{n-1}b$. Then clearly G_0 is a simple chain of regions with links U_i of diameter < 1 such that $\bar{U}_1 \supset a$, $\bar{U}_n \supset b$, and T_{00} and T_{01} are arcs from a to b such that $T_{00} \cdot T_{01} = a + b$ and $x_ix_{i+1} + y_iy_{i+1} \subset U_{i+1}$ (where $0 \leq i < n$, $x_0 = y_0 = a$, $x_n = y_n = b$).

Now with the aid of the arc $T_{00} = ax_1 + \cdots + x_{n-1}b$ it is easily seen that there exists within G_0 a simple chain R_{00} of regions Q_1, $Q_2, \cdots Q_m$ all of diameter $< 1/2$ such that $\bar{Q}_1 \supset a$, $\bar{Q}_m \supset b$, etc., and such that R_{00} is the sum of n simple chains $C_1 = Q_1 + Q_2 + \cdots Q_{m_1}$, $C_2 = Q_{m_1} + Q_{m_1+1} + \cdots + Q_{m_2}, \cdots$, $C_n = Q_{m_{n-1}+1} + \cdots + Q_m$, where for each i, $1 \leq i \leq n$, $C_i \subset U_i$. Then by the same method as used above in the case of the chain R_0, we can define a chain G_{00} with links S_1, $S_2, \cdots S_m$ all of diameter $< 1/2$, having the same properties as stated for the chain R_{00} and in addition the property that $G_{00} + a + b$ contains two independent arcs T_{000} and T_{001} from a to b each of which is the sum of m arcs ar_1, r_1r_2, $r_2r_3, \cdots r_{m-1}b$, where for each i, $(0 \leq i < n)$, $r_ir_{i+1} \subset S_i$, where $r_0 = a$ and $r_n = b$.

Repeating this process, using T_{000}, we obtain a chain G_{000}, and so on. Continuing this process indefinitely, we obtain a sequence G_0, G_{00}, $G_{000}, \cdots$ of simple chains of regions having all the properties necessary to insure that the sets of points $axb = \bar{G}_0 \cdot \bar{G}_{00} \cdot \bar{G}_{000} \cdots$ will be a simple continuous arc from a to b. The proof that this is the case is almost identically the same as that given in a proof by R. L. Moore.*

THE JOHNS HOPKINS UNIVERSITY.

* See "On the Foundations of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 138.

ON THE LIBRATION POINTS OF THE RESTRICTED PROBLEM OF THREE BODIES.

By Monroe Martin.

Introduction.

In the restricted problem of three bodies if the constant distance between the two finite masses be taken as the unit of distance, if the unit of time be so chosen that the gravitational constant is unity, and if the unit of mass be taken as the sum of the masses of the two finite bodies, the equations of motion for the infinitesimal mass become †

$$(1) \qquad \ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y,$$

where

$$(2) \qquad \Omega(x, y) = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{|[(x+\mu)^2 + y^2]^{1/2}|} + \frac{\mu}{|[(x+\mu-1)^2 + y^2]^{1/2}|}, \qquad (0 < \mu < 1)$$

in which μ is the mass of that one of the two finite bodies (which lie on the x-axis and have the coördinates $x = 1 - \mu$ and $x = -\mu$) which is situated at $x = 1 - \mu$, the origin of the coördinate system being the center of mass. As is well known there exist for $0 < \mu < 1$ ‡ five and only five points of zero force called libration points whose coördinates satisfy the equation

$$(3) \qquad \operatorname{grad} \Omega = 0,$$

that is

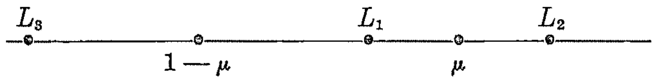
$$(4) \qquad \begin{aligned} \Omega_x(x, y) &\equiv x - \frac{(1-\mu)(x+\mu)}{|[(x+\mu)^2 + y^2]^{3/2}|} - \frac{\mu(x+\mu-1)}{|[(x+\mu-1)^2 + y^2]^{3/2}|} = 0, \\ \Omega_y(x, y) &\equiv y - \frac{(1-\mu)y}{|[(x+\mu)^2 + y^2]^{3/2}|} - \frac{\mu y}{|[(x+\mu-1)^2 + y^2]^{3/2}|} = 0. \end{aligned}$$

Three of the libration points lie on the line joining the masses μ and $1 - \mu$.

† Cf. for instance, T. Levi-Civita, "Sopra alcuni criteri di instabilità," *Annali di Matematico* (3), Vol. 5 (1901), pp. 282-284.

‡ For the cases where $\mu = 0$, and $\mu = 1$ there exists an infinity of points on a circle of radius unity described about the mass as a centre. That in these limit cases these points and only these are libration points follows immediately from equations (4).

These three points are separated by the masses for $0 < \mu < 1$ and their mutual position is, in the notation of E. Strömgren * as follows:



The remaining two libration points denoted by L_4 and L_5 each form an equilateral triangle with the masses μ and $1-\mu$.

In this paper the distances of L_1 and L_2 from the mass μ are designated by ρ_1 and ρ_2 respectively, while that of L_3 from the mass $1-\mu$ is designated by ρ_3. The distances ρ_1, ρ_2 and ρ_3 are functions of μ. In part I of this paper are proved the following theorems on the nature of these functions:

THEOREM I. *The functions* $\rho_1(\mu)$, $\rho_2(\mu)$ *and* $\rho_3(\mu)$ *are monotone in the interval* $0 < \mu < 1$ *and have the boundary values* †

$$\text{(5a)} \qquad \begin{aligned} &\rho_1(+0) = 0, \ \rho_1(1-0) = 1; \\ &\rho_2(+0) = 0, \ \rho_2(1-0) = 1; \\ &\rho_3(+0) = 1, \ \rho_3(1-0) = 0, \end{aligned}$$

so that

$$\text{(5b)} \qquad 0 < \rho_k < 1 \quad \text{for} \quad 0 < \mu < 1; \qquad (k = 1, 2, 3).$$

Moreover it is possible to obtain better inequalities for ρ_k in which the boundary values of the inequalities are functions of μ. In this connection we have

THEOREM II. *The functions* $\rho_1(\mu)$, $\rho_2(\mu)$ *and* $\rho_3(\mu)$ *may be bounded as follows* ‡:

$$\begin{array}{lll} \left\{ \begin{array}{l} (6a_1) \\ (6a_2) \\ (6a_3) \end{array} \right. & \begin{array}{r} \mu < \rho_1 < \tfrac{1}{2} \\ \rho_1 = \tfrac{1}{2} \\ \tfrac{1}{2} < \rho_1 < \mu \end{array} & \begin{array}{l} \text{for } \ 0 < \mu < \tfrac{1}{2}, \\ \text{for } \quad \mu = \tfrac{1}{2}, \\ \text{for } \tfrac{1}{2} < \mu < 1, \end{array} \\ (6b) & \mu < \rho_2 < \mu^{\frac{1}{4}} & \text{for } \ 0 < \mu < 1, \\ (6c) & 1-\mu < \rho_3 < (1-\mu)^{\frac{1}{4}} & \text{for } \ 0 < \mu < 1. \end{array}$$

More definite information on the nature of the functions $\rho_1(\mu)$, $\rho_2(\mu)$ and $\rho_3(\mu)$ is given by the following theorem on the relative values of these functions for the interval $0 < \mu < 1$:

THEOREM III. *There exists in the interval* $0 < \mu < 1$ *one and only one*

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* E. Strömgren, *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, Nr. 39 (1922).

† Cf. footnote ‡ of previous page.

‡ It follows quite readily from the inequalities of Theorem II that inequalities which are functions of μ can be given for the values of $\Omega(x, y)$ at each of the libration points L_1, L_2. and L_3.

value of $\mu = \mu^*$ *for which the following relations hold between the functions* $\rho_1(\mu)$ *and* $\rho_2(\mu)$:

(7a) $\rho_1(\mu) < \rho_2(\mu)$ for $0 < \mu < \mu^*$,

(7b) $\rho_1(\mu^*) = \rho_2(\mu^*)$

(7c) $\rho_1(\mu) > \rho_2(\mu)$ for $\mu^* < \mu < 1$,

and here

(7d) $\mu^* > \frac{1}{2}$.

Part II of the paper is concerned with the nature of the function $\Omega(x, y)$ at the three libration points L_1, L_2 and L_3. In his paper on the restricted problem of three bodies Birkhoff † mentions the fact that the value of $\Omega(x, y)$ at the libration point L_1 is greater than the value at either L_2 or L_3. The proof of this statement for all values of μ is not to be found in the literature. In "Die Mechanik des Himmels" by Charlier, a proof is given employing power series expansions of ρ_k, but recent numerical calculations by E. Strömgren prove that the expansions employed by Charlier are valid only for exceedingly small values of μ. Now by a simple method it is possible to prove this statement, namely (8a) below, for all values of μ in the interval $0 < \mu < 1$. In addition I shall demonstrate the statements (8b) and (8d) below which concern the relative values of $\Omega(x, y)$ at the libration points L_2 and L_3.

THEOREM IV.‡ *Denoting by* $\Omega(L_1)$, $\Omega(L_2)$ *and* $\Omega(L_3)$ *the values of the function* Ω *at the libration points* L_1, L_2 *and* L_3 *respectively,* $\Omega(L_1)$, $\Omega(L_2)$ *and* $\Omega(L_3)$ *satisfy the following relations in the interval* $0 < \mu < 1$:

(8a) $\Omega(L_1) > \Omega(L_2)$ and $\Omega(L_1) > \Omega(L_3)$ for $0 < \mu < 1$,

(8b) $\Omega(L_2) > \Omega(L_3)$ for $0 < \mu < \frac{1}{2}$,

(8c) $\Omega(L_2) = \Omega(L_3)$ for $\mu = \frac{1}{2}$,

(8d) $\Omega(L_2) < \Omega(L_3)$ for $\frac{1}{2} < \mu < 1$.

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† G. D. Birkhoff, "The Restricted Problem of Three Bodies," *Rendiconti del Circolo Matematico di Palermo*, Vol. 39 (1915), pp. 281-283.

‡ If we designate the values of the function $\Omega(x, y)$ at the points L_4 and L_5 by $\Omega(L_4)$ and $\Omega(L_5)$ respectively, we have from (2), since L_4 and L_5 lie equally distant from the x-axis, $\Omega(L_4) = \Omega(L_5)$ for $0 < \mu < 1$. The function $\Omega(x, y)$ becomes infinite at infinity and at both the masses. It has been proved by Plummer (in his paper mentioned below) that $\Omega(x, y)$ possesses a minimax at each of the libration points L_1, L_2 and L_3 and it accordingly follows that $\Omega(x, y)$ must have an absolute minimum at L_4 and L_5.

An apparent paradox arises in the distribution of the libration points L_1, L_2 and L_3 on the x-axis for the masses $\mu = 0$ and $\mu = 1$. For $\mu = 0$ the libration points L_1 and L_2 coincide and for $\mu = 1$ the libration points L_1 and L_3 coincide.* For these values of μ the actual distribution of the mass of the system is symmetrical while the libration points are placed unsymmetrically. The difficulty arises in the fact that the above distributions of the libration points are not for the distributions of mass $\mu = 0$ and $\mu = 1$, but rather for the distributions $\lim_{\mu \to 0} \mu$ and $\lim_{\mu \to 1} \mu$ which are unsymmetrical.

In the appendix the results of part I are used to prove that the function $\Omega(x, y)$ possesses a minimax at each of the libration points L_1, L_2 and L_3. This theorem was first proved by Plummer.†

Part I.

Proof of Theorem I.

For the three collinear libration points L_1, L_2 and L_3, lying on the x-axis, the second of equations (4), namely $\Omega_y(x, 0) = 0$, is identically fulfilled and the first equation will be

$$(9) \qquad \begin{aligned} \Omega_x(x, 0) = x - (1-\mu)(x+\mu)/|x+\mu|^3 \\ - \mu(x+\mu-1)/|x+\mu-1|^3 = 0. \end{aligned}$$

The real roots of this equation are the coördinates of the libration points L_1, L_2 and L_3. Now it follows from (9) that for $0 < \mu < 1$

$$(10) \quad \Omega_x(\pm\infty, 0) = \pm\infty, \ \Omega_x(-\mu \pm 0, 0) = \mp\infty, \ \Omega_x(1-\mu \pm 0, 0) = \mp\infty,$$

$$(11) \qquad \Omega_{xx}(x, 0) = 1 + 2(1-\mu)/|x+\mu|^3 + 2\mu/|x+\mu-1|^3.$$

Therefore

$$(12) \qquad \Omega_{xx}(x, 0) > 0 \qquad \text{for} \quad 0 < \mu < 1.$$

From (10) and (12) we see that the function $\Omega_x(x, 0)$ has for $0 < \mu < 1$ one and only one zero in each of the three intervals

$$-\infty < x < -\mu, \qquad -\mu < x < 1-\mu, \qquad 1-\mu < x < \infty,$$

that is we have exactly three collinear libration points which furthermore are separated by the two finite masses μ and $1 - \mu$. If we denote the x-coördinates

* Cf. Theorem I.

† H. C. Plummer, "Neighbourhood of Centers of Libration," *Monthly Notices of the Royal Astronomical Society*, Vol. 62 (1901), pp. 6-17.

of the three libration points L_1, L_2, and L_3 by x_1, x_2, and x_3 respectively, we may write

$$(13) \qquad x_1 = 1 - \mu - \rho_1, \qquad x_2 = 1 - \mu + \rho_2, \qquad x_3 = -\mu - \rho_3,$$

where ρ_1, ρ_2 and ρ_3 are the positive functions of μ defined in the introduction. Then (9) may be written in the form

$$(14a) \qquad \Omega_1(\rho_1, \mu) \equiv 1 - \rho_1 - \mu - (1-\mu)/(1-\rho_1)^2 + \mu/\rho_1^2 = 0,$$
$$(14b) \qquad \Omega_2(\rho_2, \mu) \equiv 1 + \rho_2 - \mu - (1-\mu)/(1+\rho_2)^2 - \mu/\rho_2^2 = 0,$$
$$(14c) \qquad \Omega_3(\rho_3, \mu) \equiv -\rho_3 - \mu + (1-\mu)/\rho_3^2 + \mu/(1+\rho_3)^2 = 0,$$

where (14a), (14b) and (14c) define $\Omega_1(\rho_1, \mu)$, $\Omega_2(\rho_2, \mu)$ and $\Omega_3(\rho_3, \mu)$ respectively. These equations on simplifying yield

$$(14A) \qquad \rho_1^5 - (3-\mu)\rho_1^4 + (3-2\mu)\rho_1^3 - \mu\rho_1^2 + 2\mu\rho_1 - \mu = 0,$$
$$(14B) \qquad \rho_2^5 + (3-\mu)\rho_2^4 + (3-2\mu)\rho_2^3 - \mu\rho_2^2 - 2\mu\rho_2 - \mu = 0,$$
$$(14C) \qquad \rho_3^5 + (2+\mu)\rho_3^4 + (1+2\mu)\rho_3^3 - (1-\mu)\rho_3^2 - 2(1-\mu)\rho_3 - 1 + \mu = 0.$$

The three positive functions $\rho_k(\mu)$ are defined uniquely by (14A), (14B) and (14C). We will demonstrate by reductio ad absurdum that

$$(15) \qquad \frac{d\rho_k}{d\mu} \neq 0 \qquad \text{for} \quad 0 < \mu < 1; \qquad (k = 1, 2, 3).$$

We demonstrate (15) at first for $k = 1$. If equation (14A) be differentiated with respect to μ and the derivative of ρ_1 assumed to be zero we obtain

$$(16) \qquad \rho_1^4 - 2\rho_1^3 - \rho_1^2 + 2\rho_1 - 1 = 0.$$

Since L_1 lies between the two finite masses for all values of μ it is sufficient if we show that this quartic equation in ρ_1 has no roots in the interval $0 < \rho_1 < 1$. Denoting the left hand member of (16) by $F(\rho)$, we have

$$(17) \qquad \begin{aligned} &F(0) = F(1) = -1, && \left(\frac{dF(\rho_1)}{d\rho_1}\right)_{\rho_1 = 1/2} = F'(1/2) = 0, \\ &F(1/2) = -7/16, && F''(\rho_1) = 12\rho_1^2 - 12\rho_1 - 2. \end{aligned}$$

Since $F''(\rho_1)$ has no roots in the interval $0 < \rho_1 < 1$, we conclude that $F'(\rho_1)$ has only one root in the interval $0 < \rho_1 < \mu$, namely $\rho_1 = 1/2$, and consequently that (16) does not vanish in the interval $0 < \rho_1 < 1$. It follows the original assumption that the derivative of ρ_1 can be zero in the interval $0 < \rho_1 < 1$ is untenable.

In order to prove (15) for $k = 2$, we differentiate (14B) and obtain on assuming the derivative of ρ_2 to be zero

$$(18) \qquad \rho_2^4 + 2\rho_2^3 + \rho_2^2 + 2\rho_2 + 1 = 0.$$

Now (18) obviously has no positive roots and the proof of (15) for $k = 2$

follows immediately. It follows from considerations of symmetry that, if (15) is true for $k=2$, it is also true for $k=3$, and finally it follows from (14a), (14b) and (14c) that (5) are valid.

Proof of Theorem II.

If we place $\rho_2=\mu^{1/4}$ in (14b) we have

$$(19)\quad \Omega_2(\mu^{1/4},\mu)=1+\mu^{1/4}-\mu-(\mu^{1/2}+2\mu^{3/4}+1)/(1+\mu^{1/4})^2,$$

$$(20)\quad \Omega_2(\mu^{1/4},\mu)>1+\mu^{1/4}-\mu-(\mu^{1/2}+2\mu^{1/4}+1)/(1+\mu^{1/4})^2;\ (0<\mu<1)$$

and therefore

$$(21)\qquad \Omega_2(\mu^{1/4},\mu)>0 \qquad \text{for} \qquad 0<\mu<1.$$

If we now place $\rho_2=\mu$ in (14b), we obtain

$$(22)\qquad \Omega_2(\mu,\mu)=(\mu-1)(\mu^2+3\mu+1)/\mu(1+\mu^2).$$

Since $(\mu-1)(\mu^2+3\mu+1)<0$ for $0<\mu<1$, we have

$$(23)\qquad \Omega_2(\mu,\mu)<0 \qquad \text{for} \quad 0<\mu<1.$$

The inequality (6b) now follows immediately from (12), (21) and (23) and the validity of (6c) follows by symmetry from (6b).

We now demonstrate ($6a_1$), ($6a_2$) and ($6a_3$). From (14a) we have

$$(24)\qquad \Omega_1(\mu,\mu)=[1-2\mu]\,[1+\mu(1-\mu)]/\mu(1-\mu);$$

therefore

$$(25)\qquad \Omega_1(\mu,\mu)>0 \qquad \text{for} \quad 0<\mu<\tfrac{1}{2}.$$

The lower bound in ($6a_1$) follows from (12) and (25) and the upper bound in ($6a_1$) is a consequence of Theorem I. Now ($6a_2$) is obvious while ($6a_3$) follows from ($6a_1$) by symmetry.†

Proof of Theorem III.

On eliminating μ between (14A) and (14B), we have

$$(26)\qquad \frac{\rho_1^5-3\rho_1^4+3\rho_1^3}{\rho_1^4-2\rho_1^3-\rho_1^2+2\rho_1-1}+\frac{\rho_2^5+3\rho_2^4+3\rho_2^3}{\rho_2^4+2\rho_2^3+\rho_2^2+2\rho_2+1}=0.$$

We assume $\rho_1=\rho_2=\rho$ in (26) and obtain

$$(27)\qquad \rho^4(\rho^5-6\rho^3-2\rho^2+6)=0,$$

$$(27a)\qquad Q(\rho)\equiv \rho^5-6\rho^3-2\rho^2+6=0.$$

A simple calculation shows that in the interval $0<\mu<1$ the equation (27a)

† While the inequalities (6b) and (6c) are valid throughout the interval $0<\mu<1$ they give a good approximation for ρ_2 and ρ_3 only for $0<\mu<\epsilon$ and $1-\delta<\mu<1$ where ϵ and δ are small positive numbers.

has one and only one root ρ^*. From (27a) we have $Q(3/4) > 0$ and $Q(1) < 0$, that is

$$(28a) \qquad 3/4 < \rho^* < 1,$$

and by (14A)

$$(28b) \qquad 2/3 < \rho_2 < 3/4 \dagger \qquad \text{for} \quad \mu = 1/2.$$

The proof of (7d) follows from (28a) and (28b) by Theorem I.

Inequalities (7a) and (7c) will now be proven together. The proof consists in establishing that

$$(29) \qquad \frac{d\rho_1}{d\mu} \neq \frac{d\rho_2}{d\mu} \qquad \text{for} \quad \mu = \mu^*.$$

The inequalities (7a) and (7c) then follow readily from the uniqueness of μ^* inasmuch as (7a) follows from ($6a_2$), (28a) and (28b). Then (7c) is an immediate consequence of (29). To prove (29) we again appeal to a reductio ad absurdum. We assume (29) is not true, and (26) becomes, for $\mu = \mu^*$, on differentiating with respect to μ

$$(30) \qquad \rho^3(\rho^5 + 3\rho^3 + 14\rho^2 + 24) = 0.$$

But this equation has no positive roots; whence (29) follows as a necessary consequence.

Part II.

Proof of Theorem IV.

We now prove (8a). Denote by $\Omega(\rho)$ and $\Omega(-\rho)$ the values of $\Omega(x, 0)$ at points distant ρ and $-\rho$ from the mass μ respectively. Then, by (9), we have

$$(31) \qquad \begin{aligned} \Omega(-\rho) &= 1/2(1-\mu-\rho)^2 + (1+\mu)/(1-\rho) + \mu/\rho, \\ \Omega(\rho) &= 1/2(1-\mu+\rho)^2 + (1-\mu)/(1+\rho) + \mu/\rho. \end{aligned}$$

Therefore

$$(32) \qquad \Omega(-\rho) - \Omega(\rho) = 2\rho^3(1-\mu)/(1-\rho^2),$$

and accordingly

$$(33) \qquad \Omega(-\rho) - \Omega(\rho) > 0 \qquad \text{for} \quad 0 < \mu < 1; \quad 0 < \rho < 1.$$

The equality (8c) is obvious. We now give a proof of (8b) and (8d). From (31) we obtain

$$(34) \qquad \frac{\partial\Omega(\rho)}{\partial\rho} \equiv \Omega_\mu(\rho) = -(\rho^3 + (2-\mu)\rho^2 + (1-\mu)\rho - 1)/\rho(1+\rho),$$

which can be written

† While better limiting values for ρ_2 have been found, the interval given here is sufficient for the needs of this paper.

(35) $\Omega_\mu(\rho) = -(\rho^5 + (3-\mu)\rho^4 + (3-2\mu)\rho^3 - \mu\rho^2 - \rho)/\rho^2(1+\rho)^2,$

and therefore from (14B), in the notation of the introduction for Theorem IV,

(36) $$\Omega_\mu(L_2) = (\rho_2 - 2\mu\rho_2 - \mu)/\rho_2^2(1+\rho_2)^2.$$

Therefore from (6b)

(37) $$\Omega_\mu(L_2) < 0.$$

The inequalities (8b) and (8d) follow directly from (37).

Appendix.

We now show that the function $\Omega(x, y)$ has a minimax at each of the points L_1, L_2, and L_3 and therefore in L_4 and L_5 certainly an absolute minimum.† It will be sufficient to show that

(38) $$\Omega_{xx}(L_k)\Omega_{yy}(L_k) - \Omega_{xy}(L_k)^2 < 0 \qquad (k = 1, 2, 3).$$

From (2) one obtains $\Omega(x, -y) = \Omega(x, y)$. Consequently $\Omega_{xy}(x, 0) = 0$. Therefore (38) becomes, from (12),

(39) $$\Omega_{yy}(L_k) < 0 \qquad (k = 1, 2, 3).$$

From (2) and (13) we have

(40a) $$\Omega_{yy}(x_1, 0) = 1 - (1-\mu)/(1-\rho_1)^3 - \mu/\rho_1^3$$

and

(40b) $$\Omega_{yy}(x_2, 0) = 1 - (1-\mu)/(1+\rho_2)^3 - \mu/\rho_2^3.$$

As a consequence of Theorem I we have from (40a) and (13)

(41) $$\Omega_{yy}(x_1, 0) < 0 \qquad \text{for} \quad -\mu < x_1 < 1 - \mu.$$

It is clear from (40b) that $\Omega_{yy}(x_2, 0)$ is negative for small values of ρ_2 and positive for sufficiently great values of ρ_2. We now show the values for ρ_2 given as a function of μ by (14B) are small enough so that for x_2 defined by (13) we have

(42) $$\Omega_{yy}(x_2, 0) < 0 \qquad \text{for} \quad 0 < \mu < 1.$$

We have from (40b)

(43) $\Omega_{yy}(x_2, 0) = (\rho_2^6 + 3\rho_2^5 + 3\rho_2^4 - 3\mu\rho_2^2 - 3\mu\rho_2 - \mu)/\rho_2^3(1+\rho_2)^3,$

and this reduces by (14B) to

(44) $$\Omega_{yy}(x_2, 0) = (\mu - 1)(\rho_2^3 + 3\rho_2 + 3)/(1+\rho_2)^3.$$

The inequality (42) follows at once from (44). The proof of (39) for $k = 1$ and 2 follows from (41) and (44) respectively; and the proof for $k = 3$ is a necessary consequence of the validity for $k = 2$.

The Johns Hopkins University.

† Cf. foot-note ‡ to Theorem IV, p. 169.

ON ROTATIONS IN ORDINARY AND NULL SPACES.

By S. A. Schelkunoff.

1. In the following paper I am interested primarily in two problems, one of which deals with determination of invariant lines, planes and angles of rotations and quasi-rotations in a flat space of n dimensions, while the other is the converse problem.

When the paper was being written the author was not aware that either of these problems had been completely solved. He knew only of a paper* in which Professor F. N. Cole proved that every rotation in a 4-flat could be considered as a succession of two simple rotations taking place in two absolutely orthogonal planes. His method was based on direct computation in terms of Cayley's independent parameters of the coefficients of the group of rotations in a 4-flat. The method involves laborious computations even in a case of 4-flat.

At a later date, the author's attention was called to a paper written by Camille Jordan.† Jordan proved that an ordinary rotation leaves relatively invariant certain biplanes (i. e., $(n-2)$-flats immersed in an n-flat). He reached the result by concentrating his attention on infinitesimal rotations. Jordan named Schläfli as the first who had obtained equation (8) of this paper, but he claimed that the latter had not perceived its geometric significance.

The method of this paper seems to be more direct than Jordan's and it is certainly instrumental in the solution of the converse problem which appears never to have been solved in the general case of n-flat. There exists only a well-known solution for 3-flat and one for 4-flat implicitly contained in Jordan's paper.

Among the more important results obtained in this paper, equations (22) and (29) appear to be new.

Further search through literature disclosed that Ludwig Bieberbach was interested in the problem of reduction of the rotation group to a canonical

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* F. N. Cole, "On Rotations in Space of Four Dimensions," *American Journal of Mathematics*, Vol. 12 (1890), pp. 191-210.

† M. Camille Jordan, "Essai sur la geométrie à n dimensions," *Bulletin de la Société Mathématique*, Vol. 3 (1875), pp. 103-174.

form and solved it on the basis of Cayley's representation of such groups in terms of independent parameters.*

Bieberbach made references to Muth,† Schläfli ‡ and Goursat § as those whose papers had touched partially on the subject.

Since our solution of the direct problem is apparently different from any other known to the author, and since we use it as a basis for the solution of the converse problem, we include it in full in the present paper.

2. The group of rotations around the origin in an ordinary n-flat is defined analytically by the following set of equations:

$$(1) \qquad y_k = a_k{}^r x_r, \qquad (k, r = 1, 2, \cdots n), \quad (r \text{ umbral})$$

where the coefficients $a_k{}^r$ are real, subject to the conditions,

$$(2) \qquad a_k{}^r a_k{}^s = 1, \quad \text{if } r = s, \qquad = 0, \quad \text{if } r \neq s,$$

and the determinant $|a_k{}^r|$ is equal to unity. The equivalent conditions are

$$(3) \qquad a_r{}^k a_s{}^k = 1, \quad \text{if } r = s, \qquad = 0, \quad \text{if } r \neq s,$$

with the above assumption regarding the determinant.

The group of quasi-rotations is defined by similar equations:

$$(4) \qquad y_k = A_k{}^r x_r, \qquad (k, r = 1, 2, \cdots n)$$

where the coefficients $A_k{}^r$ are subject to the conditions

$$(5) \qquad A_k{}^r \bar{A}_k{}^s = 1, \quad \text{if } r = s, \qquad = 0, \quad \text{if } r \neq s$$

or their equivalents,

$$(6) \qquad A_r{}^k \bar{A}_s{}^k = 1, \quad \text{if } r = s, \qquad = 0, \quad \text{if } r \neq s.$$

It is easy to prove that the determinant $|A_k{}^r|$ is a unit complex number.

The group of quasi-rotations can be taken as a basis of "metrical" geometry in null spaces.¶

* Ludwig Bieberbach, "Über die Bewegungsgruppen der euklidischen Räume," *Mathematische Annalen*, Vol. 70 (1911), pp. 297-336.

† Muth, *Theorie und Anwendung der Elementarteiler*, Leipzig (1899), s. 176.

‡ Schläfli, *Journal für Mathematik*, Vol. 65 (1866), s. 185.

§ Goursat, *Annales de l'École Normale Superieure* (3), t. 6 (1889).

¶ S. A. Schelkunoff, *On Certain Properties of the Metrical and Generalized Metrical Groups in Linear Spaces of n Dimensions*, Lütcke and Wulff, Hamburg, Germany, (1927).

3. If $x_1, x_2, \cdots x_n$ are the direction components of a straight line thru the origin in an n-flat (F_n) the following set of equations

$$\lambda x_k = a_k^r x_r \tag{7}$$

(where λ is the coefficient of proportionality) determines the lines of F_n invariant under transformation (1), or the axes of rotation as we might appropriately call them.

This system of linear equations has a proper solution if any only if the determinant of its coefficients vanishes, i. e., if λ is a root of the characteristic equation:

$$\begin{vmatrix} a_1^1 - \lambda & a_1^2 & a_1^3 & \cdots & a_1^n \\ a_2^1 & a_2^2 - \lambda & a_2^3 & \cdots & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n^1 & a_n^2 & a_n^3 & \cdots & a_n^n - \lambda \end{vmatrix} = 0. \tag{8}$$

The roots of equation (8), or the *rotation factors* as we might call them, *are unit complex numbers* as we can readily ascertain by multiplying equations (7) by their conjugates and then adding.

Again, if equations (7) are squared and added, we have

$$(\lambda^2 - 1) x_k x_k = 0, \tag{9}$$

i. e. either $\lambda = \pm 1$, or the axis is a null line.

If $\lambda = 1$, we have an absolutely invariant axis, i. e. a line of invariant points.

Since the coefficients of (8) are real, the complex rotation factors are grouped in conjugate pairs. Thus, null axes exist in conjugate pairs. Each such pair determines a real invariant plane that rotates on itself, as we shall prove later.

Equation (8) can be written in the form

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - S_3\lambda^{n-3} + \cdots + (-1)^{n-1}S_{n-1}\lambda + (-1)^n = 0,$$

where S_{n-k} is the sum of the principal minors of $(n-k)$-th order, taken without repetition.

Since the complementary minors of an orthogonant are equal and

$$S_k = S_{n-k},$$

this equation is symmetric if n is even, and antisymmetric if n is odd.

Hence, if n is odd, $+1$ is a root of (8), that is, *in space of odd number of dimensions there is always an absolutely invariant line.**

* According to Jordan this result was first discovered by Schläfli.

If n is even, equation (8) can be transformed into equation of lower degree by the following substitution:

$$\lambda + 1/\lambda = 2z.$$

If n is odd, the above substitution can be used after equation (8) has been divided by $(\lambda - 1)$.

Therefore, $+1$ *is a multiple root of odd or even order if the space has respectively odd or even number of dimensions;* -1 *is always a multiple root of even order.* (Here an absence of a root is regarded as even multiplicity of zero order).

If we multiply equations (7) corresponding to λ_1 and λ_2, and simplify the result we have:

$$(\lambda_1\lambda_2 - 1)x_k{}^1x_k{}^2 = 0, \tag{10}$$

i. e., either

$$x_k{}^1x_k{}^2 = 0,$$

or,

$$\lambda_2 = 1/\lambda_1 = \bar{\lambda}_1.$$

Hence, *any two non-conjugate axes are orthogonal.*

Multiplying equations (7) corresponding to λ_1 and $\bar{\lambda}_2$ we have:

$$(\lambda_1\bar{\lambda}_2 - 1)x_k{}^1\bar{x}_k{}^2 = 0,$$

i. e., either

$$x_k{}^1\bar{x}_k{}^2 = 0, \tag{10_1}$$

or,

$$\lambda_2 = \lambda_1.$$

Hence, *any two axes not having the same rotation factor are quasi-orthogonal.*

Obviously, *if for a given λ equations (7) have exactly "m" linearly independent solutions, this λ must be a multiple root of at least m-th order.*

From the canonical form given by Bieberbach* it follows at once that *for a given λ equations (7) have m linearly independent solutions if and only if λ is a multiple root of m-th order.*

4. *A conjugate pair of axes determines a real invariant plane. This plane rotates on itself.* In fact, if $\| u_r \|$ and $\| \bar{u}_r \|$ are the direction components of axes whose rotation factors are $e^{\pm i\phi}$, we can take $(u_k + \bar{u}_k)/2^{1/2}$ and $(u_k - \bar{u}_k)/i2^{1/2}$ as the direction cosines of two real straight lines in the plane determined by the axes provided $\| u_r \|$ is a quasi-normalized set, i. e.

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* *Ibid.*, p. 302.

$$u_r\bar{u}_r = 1.$$

Now if

(11) $$x_k = (u_k + \bar{u}_k)/2^{1/2}$$

are the direction cosines of a line before the transformation (1), the new direction cosines are given by:

$$y_k = a_r x_r = (\lambda u_k + \lambda \bar{u}_k)/2^{1/2},$$

and the angle thru which this line is rotated is determined from the following equation:

(12) $$\cos\theta = x_k y_k = (\lambda + \bar{\lambda})/2 = \cos\phi,$$

i. e., $$\theta = \pm\phi.$$

Similarly we can show that the line whose direction cosines are $(u_k - \bar{u}_k)/i2^{1/2}$, and, later, that every other line in the plane of rotation rotates thru the same angle ϕ.

The invariant planes corresponding to different conjugate sets of rotation factors are absolutely orthogonal. Indeed, if

$$x_k = au_k + b\bar{u}_k$$

are the direction cosines of a line thru the origin in the plane determined by one conjugate set of axes, and

$$y_k = cv_k + d\bar{v}_k$$

is a similar line in another plane of rotation, we have:

$$\begin{aligned} x_k y_k &= (au_k + b\bar{u}_k)(cv_k + d\bar{v}_k) \\ &= (acu_k v_k + bd\bar{u}_k\bar{v}_k + adu_k\bar{v}_k + bc\bar{u}_k v_k) = 0, \end{aligned}$$

provided the sets of rotation factors are distinct. This proves the above theorem.

Thus, *if the rotation factors are all different a rotation in F_{2n} can be uniquely decomposed into "n" simple rotations taking place in a set of "n" absolutely orthogonal real planes, thru angles determined by the roots of λ-equation.*

Also, *if the rotation factors are all different, a rotation in F_{2n+1} leaves absolutely invariant one real line, and relatively invariant a unique set of absolutely orthogonal (mutually as well as to the invariant line) real planes. The lines of any one invariant plane rotate thru the same angle.**)

* These two theorems are most interesting special cases of the obvious general theorem: *In every flat space F_n there are Subflats (of 1, 2, 3, . . . $n-1$ dimensions)*

Besides real invariant planes there are imaginary. In F_{2n} all of these $[2n(n-1)$ in number$]$ are null planes, as may be readily ascertained; while in F_{2n+1}, besides $2n(n-1)$ null planes, there are also $2n$ aeolotropic invariant planes, namely those determined by a null axis and the absolutely invariant real axis.

5. Interesting exceptions arise when the characteristic equation has multiple roots. As we have already stated equations (7) possess m linearly independent solutions, if the corresponding λ is a multiple root of order m. Thus, to every such root there corresponds an m-fold pencil of invariant lines.

If $\lambda = 1$ *we have an absolutely invariant m-subflat.*

If $\lambda = -1$ *(which corresponds to rotation thru 180°), we have an invariant m-subflat every line of which turns thru 180°.*

If λ is a *bona fide* complex number, $\bar{\lambda}$ is also a multiple root of order m, and hence, there are two conjugate m-fold pencils of invariant lines. Each conjugate set of lines determines a real plane. Thus, instead of the usual m real planes of rotation we have ∞^{m-1} such planes, all of which turn upon themselves and thru the same angle equal to $\cos^{-1}(\bar{\lambda} + \lambda)/2$.

6. Some of the results that we have just obtained can be readily extended to the quasi-orthogonal group defined by equations (4).

As before, the invariant lines (axes of quasi-rotations) are determined by the set of equations,

$$\lambda x_k = A_k{}^r x_r, \tag{13}$$

which possess proper solutions if any only if λ is a root of the characteristic equation:

$$\begin{vmatrix} A_1{}^1 - \lambda & A_1{}^2 & \cdots & A_1{}^n \\ A_2{}^1 & A_2{}^2 - \lambda & \cdots & A_2{}^n \\ \cdot & \cdot & \cdots & \cdot \\ A_n{}^1 & A_n{}^2 & \cdots & A_n{}^n - \lambda \end{vmatrix} = 0. \tag{14}$$

Unless this equation has multiple roots there are n and only n axes of rotation.

Again it is easy to demonstrate that

$$\lambda\bar{\lambda} = 1, \tag{15}$$

i. e., that the rotation factors are unit complex numbers.

If x_r are the direction "quasi-cosines" of an axis, i. e., if

---

determined by the corresponding number of invariant lines (7), *all points of which are permuted among themselves when transformation* (1) *is applied.*

$$x_r\bar{x}_r = 1,$$

the equations

$$y_k = A_k{}^r x_r,$$

determine the new direction quasi-cosines and the "twist" of the axis is given by

$$\cos\Phi = (x_k\bar{y}_k + \bar{x}_k y_k)/2 = (\lambda x_k\bar{x}_k + \bar{\lambda} x_k\bar{x}_k)/2 = (\lambda + \bar{\lambda})/2. \tag{16}$$

Any two axes are, in general, quasi-orthogonal. Indeed if,

$$\lambda_1 x_k{}^1 = A_k{}^r x_r \quad \text{and} \quad \lambda_2 x_k{}^2 = A_k{}^s x_s, \tag{17}$$

determine a pair of axes, we have

$$\lambda_1\bar{\lambda}_2 x_k{}^1\bar{x}_k{}^2 = A_k{}^r\bar{A}_k{}^s x_r{}^1\bar{x}_s{}^2,$$

or,

$$(\lambda_1\bar{\lambda}_2 - 1)x_k{}^1\bar{x}_k{}^2 = 0. \tag{18}$$

Therefore,

$$x_k{}^1\bar{x}_k{}^2 = 0,$$

unless $\lambda_2 = \lambda_1$, i. e., unless the rotation factors of both axes are the same. If the latter is the case, the axes may or may not be quasi-orthogonal, and there are more than minimum number of axes.

7. Suppose we have a set of axes whose direction quasi-cosines form the following quasi-orthogonal matrix:

$$\left\|\begin{matrix} x_1{}^1 & x_2{}^1 & \cdot & \cdot & \cdot & x_n{}^1 \\ x_1{}^2 & x_2{}^2 & \cdot & \cdot & \cdot & x_n{}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1{}^n & x_2{}^n & \cdot & \cdot & \cdot & x_n{}^n \end{matrix}\right\| \tag{19}$$

and let the corresponding quasi-rotation factors be $\lambda_1, \lambda_2, \cdots \lambda_n$. Let us determine the coefficients of the corresponding quasi-orthogonal transformation.

We have the following system of equations at our disposal:

$$\begin{aligned} \lambda_1 x_k{}^1 &= A_k{}^r x_r{}^1, \\ \lambda_2 x_k{}^2 &= A_k{}^r x_r{}^2, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_n x_k{}^n &= A_k{}^r x_r{}^n. \end{aligned} \qquad (k, r = 1, 2, \cdots n). \tag{20}$$

Fixing the value of k we obtain a system of n equations with n unknowns

$A_k{}^1, A_k{}^2, \cdots A_k{}^n$. There are n such systems corresponding to different values of k. Solving these systems, we have:

$$A_s{}^k = \lambda_r x_s{}^r \bar{x}_k{}^r, \qquad (k, s = 1, 2, \cdots n), \quad (r = 1, 2, \cdots n(\text{umbral})). \tag{21}$$

We shall prove now that $A_s{}^k$ given by equations (21) are actually the coefficients of a quasi-orthogonal group. Indeed, take

$$A_s{}^m = \lambda_p x_s{}^p \bar{x}_m{}^p. \tag{22}$$

Multiplying equations (21) by the conjugates of (22) and summing, we have

$$A_s{}^k \bar{A}_s{}^m = \lambda_r \bar{\lambda}_p x_s{}^r \bar{x}_k{}^r \bar{x}_s{}^p x_m{}^p. \tag{23}$$

But since

$$x_s{}^r \bar{x}_s{}^p = 1, \qquad \text{if } p = r,$$
$$= 0, \qquad \text{if } p \neq r,$$

equations (23) take the following form:

$$A_s{}^k \bar{A}_s{}^m = \lambda_r \bar{\lambda}_r x_m{}^r \bar{x}_k{}^r = x_m{}^r \bar{x}_k{}^r,$$

i. e.,

$$A_s{}^k \bar{A}_s{}^m = 1, \qquad \text{if } k = m,$$
$$= 0, \qquad \text{if } k \neq m, \tag{24}$$

which proves that $A_s{}^k$ *given by equations* (21) *are actually the coefficients of a quasi-orthogonal group that leaves invariant a set of lines determined by matrix* (19).

The ordinary orthogonal group is obtained if the invariant axes given by matrix (19) *are conjugate in pairs and the corresponding rotation factors are also conjugate. If any of the axes happens to be real, the corresponding* λ *must be either* $+1$ *or* -1 (-1 *is allowable only if there is even number of real axes*). Under such conditions, we have real $A_s{}^k$, i. e.,

$$A_s{}^k = \bar{A}_s{}^k. \tag{25}$$

In fact, equations (25) are equivalent to

$$\lambda_1 x_s{}^1 \bar{x}_k{}^1 + \lambda_2 x_s{}^2 \bar{x}_k{}^2 + \cdots = \bar{\lambda}_1 \bar{x}_s{}^1 x_k{}^1 + \bar{\lambda}_2 \bar{x}_s{}^2 x_k{}^2 + \cdots,$$

which is an obvious identity if the above conditions hold.

From the results of section 3 we conclude that the above conditions are necessary with the exception of the quasi-orthogonality condition which must be satisfied only for distinct rotation factors. However, in the latter case

there are infinitely many axes, out of which quasi-orthogonal pairs can be chosen.

Thus, equations (21) may serve for determinations of the coefficients of the group of rotations if the axes and the rotation factors are known.

In practice, however, it is more convenient to describe the rotation group in terms of known planes and angles of rotation.

8. Therefore, assume the equations of known planes of rotation in the following form:

$$x_k^m = p_k^m u + q_k^m v, \tag{26}$$

where the superscript refers to the planes while the subscript, as usual, to the coördinates of a point in the plane. Without loss of generality we may assume p_k^m and q_k^m to be the direction cosines of pairs of orthogonal lines in the corresponding planes.

Since the rotation axes are the null lines in planes (26), we can determine them immediately from the condition

$$x_k^m x_k^m = 0, \qquad \text{(only } k \text{ is umbral)}. \tag{27}$$

In fact, we have

$$v = \pm iu,$$

and, hence, the direction quasi-cosines of the axes are given by:

$$x_k^m = (p_k^m \pm iq_k^m)/2^{1/2}. \tag{28}$$

Assume the corresponding rotation factors $e^{\pm i\phi_m}$. Substituting in equations (21) we have

$$a_s^k = \tfrac{1}{2} \sum_m \left[(p_s^m + iq_s^m)(p_k^m - iq_k^m) e^{i\phi_m} + (p_s^m - iq_s^m)(p_k^m + iq_k^m) e^{-i\phi_m} \right],$$

and simplifying,

$$a_s^k = \sum_m \left[(p_s^m p_k^m + q_s^m q_k^m) \cos \phi_m + (p_s^m q_k^m - p_k^m q_s^m) \sin \phi_m \right]. \tag{29}$$

These equations remain true for $s = k$ provided we waive the summation convention with regard to these letters. It is interesting to note that the first group of terms is symmetric in s and k and an even function of each ϕ_m, while the second group is antisymmetric in s and k and an odd function of each ϕ_m. In case of an odd space in which there exists an invariant real line the formula (29) still holds if we merely take the direction cosines of the line as p's and let q's and the corresponding ϕ be zero. We observe that

the coefficients of rotations in absolutely orthogonal planes or higher mainfolds are additive. This fact may be successfully used in building up rotation groups in higher spaces when planes and angles of rotation are known.

9. It is interesting to apply formula (29) to special cases. In three dimensional space it is convenient to begin with the matrix:

$$\begin{Vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{Vmatrix}$$

the first row of which is made up of the direction cosines of the real axis of rotation, and the remaining two rows are the direction cosines of two perpendicular lines in the plane of rotation. Let the angle of rotation be ϕ. Applying equations (29) and eliminating $\beta_1, \beta_2, \cdots \gamma_3$ by means of properties of orthogonal matrices, we obtain the following equations of the group of rotations in F_3:

$$\begin{aligned} y_1 &= (2a_1^2 + 2d^2 - 1)x_1 + 2(a_1a_2 + a_3d)x_2 + 2(a_1a_3 - a_2d)x_3, \\ y_2 &= 2(a_1a_2 - a_3d)x_1 + (2a_2^2 + 2d^2 - 1)x_2 + 2(a_2a_3 + a_1d)x_3, \\ y_3 &= 2(a_1a_3 + a_2d)x_1 + 2(a_2a_3 - a_1d)x_2 + (2a_3^2 + 2d^2 - 1)x_3, \end{aligned} \tag{30}$$

where,

$$d = \cos(\phi/2), \qquad a_k = \cos\alpha_k \sin(\phi/2).$$

These formulae are identical with those given by L. E. Dickson,* except for the convention concerning the direction of rotation.

In four-dimensional space we may start with a plane

$$x_3 = ax_1 + bx_2, \qquad x_4 = cx_1 + dx_2, \tag{31}$$

that rotates on itself thru angle ϕ. Let the plane absolutely orthogonal to (31) rotate thru angle ψ.

The equation of the plane absolutely orthogonal to the plane (31) may be written

$$x_1 = -ax_3 - cx_4, \qquad x_2 = -bx_3 - dx_4.$$

The plane (31) contains the points $(1, 0, a, c)$ and $(0, 1, b, d)$. The corresponding set of p's and q's are the direction cosines of the bisectors of the angles between the lines determined by the origin and those points. Similarly for the other plane.

If we use the following abbreviations

* L. E. Dickson, *Modern Algebraic Theories*, p. 100.

$$
\begin{array}{ll}
D = ad - bc, & R^2 = 1 + a^2 + b^2 + c^2 + d^2 + D^2, \\
A^2 = 1 + a^2 + c^2, & A^{1^2} = R^2 - A^2, \\
B^2 = 1 + b^2 + d^2, & B^{1^2} = R^2 - B^2, \\
P^2 = 1 + a^2 + b^2, & P^{1^2} = R^2 - P^2, \\
Q^2 = 1 + c^2 + d^2, & Q^{1^2} = R^2 - Q^2,
\end{array}
$$

we can write the coefficients of the group of rotations as follows:

$$
\begin{aligned}
R^2a_1{}^1 &= B^2 \cos\phi + B^{1^2} \cos\psi \\
R^2a_2{}^2 &= A^2 \cos\phi + A^{1^2} \cos\psi \\
R^2a_3{}^3 &= Q^{1^2} \cos\phi + Q^2 \cos\psi \\
R^2a_4{}^4 &= P^{1^2} \cos\phi + P^2 \cos\psi,
\end{aligned}
$$

$$
\begin{aligned}
R^2a_1{}^2 &= (ab + cd)(-\cos\phi + \cos\psi) - R(\sin\phi + D\sin\psi), \\
R^2a_1{}^3 &= (a + dD)(\cos\phi - \cos\psi) - R(b\sin\phi + c\sin\psi), \\
R^2a_1{}^4 &= (c - bD)(\cos\phi - \cos\psi) - R(d\sin\phi - a\sin\psi), \\
R^2a_2{}^3 &= (b - cD)(\cos\phi - \cos\psi) - R(-a\sin\phi + d\sin\psi), \\
R^2a_2{}^4 &= (d + aD)(\cos\phi - \cos\psi) + R(c\sin\phi + b\sin\psi), \\
R^2a_3{}^4 &= (ac + bd)(\cos\phi - \cos\psi) - R(D\sin\phi + \sin\psi),
\end{aligned} \tag{32}
$$

and $a_r{}^s$ is obtained from $a_s{}^r$ by changing simultaneously the signs of ϕ and ψ.

If $\psi = \phi$, equations (32) degenerate into

$$
\begin{array}{ll}
a_1{}^1 = a_2{}^2 = a_3{}^3 = a_4{}^4 = \cos\phi, & Ra_1{}^2 = Ra_3{}^4 = -(1 + D)\sin\phi, \\
Ra_1{}^3 = -Ra_2{}^4 = -(b + c)\sin\phi, & Ra_1{}^4 = Ra_2{}^3 = (a - d)\sin\phi.
\end{array} \tag{33}
$$

Since,

$$
R^2 = (a - d)^2 + (b + c)^2 + (1 + D)^2,
$$

coefficients $a_s{}^r$ depend only on two parameters besides the angle of rotation. Hence one parameter in (31) is arbitrary, and there exist ∞^1 *real* planes thru the origin that rotate on themselves, which is in keeping with the previously stated general theorem (end of section 5).

BELL TELEPHONE LABORATORIES,
NEW YORK.

GENERATING INVOLUTIONS OF INFINITE DISCONTINUOUS CREMONA GROUPS OF S_4 WHICH LEAVE A GENERAL CUBIC VARIETY INVARIANT.

By Virgil Snyder and Marguerite Lehr.

The present paper derives the equations and obtains the table of characteristics of three types A, B, C of involutorial birational transformations of S_4; they are obtained by means of the general cubic variety which remains invariant.

Of these three types the first and second each contain six parameters, and the third more. The product of any two transformations of the set, of the same or of different types, is a non-periodic transformation.

These types and two others discussed previously * generate all known transformations which leave the variety V_3 invariant.

A. Given a general cubic variety V_3 of four way space S_4 and a line l not lying on it. A birational transformation of the S_4 in which V_3 is immersed may be constructed as follows. Let M be one of the points in which l meets V_3. A point P determines the plane P, l which meets V_3 in a plane cubic curve. The tangent to it at M meets it in the tangential M'. The line $M'P$ meets the polar quadric variety of M' as to V_3 in M''. The harmonic conjugate of P as to $M'M''$ is P'. The transformation $P \sim P'$ is birational and involutorial, and under it V_3 remains invariant. Call it I.

The locus of M' is the cubic surface γ_3, intersection of V_3 with the tangent space at M; it has a node at M, hence every line of the bundle M in the tangent space τ meets it in just one point M'. The plane M', l cuts from the first polar of M' as to V_3 a conic c_2; the tangent space at M' meets l in one point, and the line joining this point to M' is the tangent line t to c_2 at M'.

In the plane l, M' the transformation I is a perspective quadratic involution determined by c_2 and M'. The only fundamental point is M', and its associated principal locus is the tangent t which meets l. The conjugates of the lines of the plane under I are conics having t for common tangent at M' and having three point contact with each other.

As the plane l, M' takes all positions about l, the locus of t is a fundamental ruled variety Γ conjugate of γ_3. The locus of c_2 is the variety H of invariant points. It touches Γ at every point of γ_3.

* *Rendiconti Circolo Matematico di Palermo*, Vol. 38 (1914), pp. 344-352.

Every plane through l is transformed into itself. In each position, l has an image conic having three point contact with the image conics of the other lines of the plane at M'. The locus of this conic, as M' describes γ_3, is the complete conjugate L of l.

2. When the section of the polar quadric at M' with the plane l, M' is composite, and the cubic section is not, one component is the tangent t and the other, the harmonic polar of the point of inflexion M' of the plane section V_3, l, M', is the axis of a harmonic homology having M' for center. Thus in each such plane the transformation is linear; the conjugate of any line in the plane is made quadratic by adjoining the inflexional tangent t. Hence every inflexional tangent of V_3 at points of γ_3, which meets l is a fundamental line of the second kind; the image of any point on one of these lines is the whole line t passing through it. These lines are therefore on H, on Γ, on L, and on the conjugate of every S_3 of S_4.

Since the conjugates of the $| S_3 |$ are varieties having three point contact on γ_3, its conjugate Γ appears three times as component of the Jacobian of these conjugate varieties. The other component is L, the conjugate of l.

3. A general S_3 meets its own conjugate in the section made by it on H and in the elliptic cubic cone with vertex on l and containing the plane cubic curve S_3, γ_3. Every line meeting l and γ_3 contains an infinite number of pairs of conjugate points, hence the involution I belongs to the generalized monoidal type, analogous to those thus defined in S_3 by Montesano.

4. Analytic procedure. Let $x_1=0,\ x_2=0,\ x_3=0$ be l and let M be $(0,0,0,0,1)$. Let $x_4=0$ be the tangent space to V at M. Then the equation of V has the form

$$V \equiv x_4x_5^2 + x_5 f + \phi = 0,$$

in which f is a quadratic and ϕ a cubic quaternary form in $x_1,\ x_2,\ x_3,\ x_4$. The equations of γ_3 are then

$$\gamma_3 : x_4 = 0, \qquad x_5 f_0 + \phi_0 = 0,$$

f_0, ϕ_0 being f, ϕ with x_4 replaced by zero.

A point (y) determines the plane

$$x_1/y_1 = x_2/y_2 = x_3/y_3,$$

through l. This plane meets γ_3 in M' defined by

$$M' : f_0y_1,\ f_0y_2,\ f_0y_3,\ 0,\ -\phi_0.$$

The polar quadric of M' as to V is

$$f_0y_1(x_5f_1+\phi_1)+f_0y_2(x_5f_2+\phi_2)+f_0y_3(x_5f_3+\phi_3)-\phi_0(2x_4x_5+f)=0,$$

in which f_i, ϕ_i are partial derivatives of f, ϕ as to x_i. By eliminating y_1, y_2, y_3 between this equation and those of the plane l, (y) we obtain the equation of H, the locus of invariant points,

$$H: f_0x_1(x_5f_1 + \phi_1) + f_0x_2(x_5f_2 + \phi_2) + f_0x_3(x_5f_3 + \phi_3) - \phi_0(2x_4x_5 + f) = 0.$$

It is of order 5, contains l to multiplicity 3, and contains γ_3.

The tangent S_3 to V_3 at M' is

$$x_1(-\phi_0f_0f_{1,0} + f_0^2\phi_{1,0}) + x_2(-\phi_0f_0f_{2,0} + f_0^2\phi_{2,0})$$
$$+ x_3(-\phi_0f_0f_{3,0} + f_0^2\phi_{3,0}) + x_4(\phi_0^2 - \phi_0f_{4,0}f_0 + f_0^2\phi_{4,0}) + x_5 \cdot f_0^3 = 0.$$

By eliminating y as before, this equation represents Γ. It is of order 7, contains l to multiplicity 6, and contains γ_3. Any plane through l contains one and only one generator of Γ. Any point on the line $M'y$ has coördinates of the form

$$x_1 = \sigma f_0 y_1 + \tau y_1, \quad \text{etc.}$$

The point M'' in which this line meets the polar quadric of M' again is given by $\tau^2 H(y) + 2\sigma\tau H(y, M') = 0$ or

$$\tau = 2H(y, M'), \qquad \sigma = -H(y),$$

where $H(y, M') = 0$ is the polar of M' as to $H = 0$. The harmonic conjugate of y as to M', M'' is then the point

$$\rho y_1' = (\Gamma - Hf_0)y_1, \quad \rho y_2' = (\Gamma - Hf_0)y_2, \quad \rho y_3' = (\Gamma - Hf_0)y_3,$$
$$\rho y_4' = \Gamma y_4, \quad \rho y_5' = \Gamma y_5 + H\phi_0.$$

Thus, the transformation of order 8 has

$$L: \Gamma - Hf_0 = 0$$

for the image of l. The Jacobian is of order $5(n-1) = 35$. It consists of L to multiplicity two and Γ to multiplicity three.

$$J: \Gamma^3(\Gamma - Hf_0)^2 = 0.$$

A table of characteristics of I_8 may now be constructed. Everything is determined except the loci of parasitic lines, or fundamental lines of the second kind.

5. The six planes π, defined by $f_0 = 0$, $\phi_0 = 0$ are double on each V_8, conjugates of the $|S_3|$. These planes are fundamental of the second kind, that is, the whole plane is the conjugate of any point on it. From the forms of the equations it follows that these planes are double on L_7, Γ_7 and simple on H. The line l is six fold on each V_8; every point P of l has for conjugate

a surface of order 6, π_6 lying on the three dimensional cone P, γ_3. The tangents to V_8 at points of γ_8 which meet l form the ruled variety Γ; an arbitrary S_3 meets it in a surface of order 7, having a five fold point at the point S_3, l. No plane section c_3 of γ_3 can lie in a plane meeting l except at M, as l does not lie in the tangent space τ containing γ_3. An arbitrary S_3 meets γ_3 in a plane section c_3, and meets l in a point. The lines of Γ from points of c_3 do not lie in this S_3. They form a ruled surface of S_4, of order 9.

The other fundamental lines of the involution are the inflexional tangents to V_8 at points γ_3 which meet l. These lines form a ruled surface Γ_{17} of S_4, of order 17, having l for six fold line, as can be seen by passing an S_3 through l. It meets γ_3 in a plane cubic curve g_3 having a node at M. The ruled surface of inflexional tangents to $G \equiv V, S_3(l)$ has g_3 for double curve and its plane has three inflexional and two nodal tangents at M. Hence the surface is of order 11. The same result can be obtained by expressing the condition that the plane l, M' shall touch the polar quadric of M' as to G. Thus in any S_3 through l are 11 inflexional tangents to V_8 at points of γ_3 which meet l. The order of the surface in S_4 is therefore 11 plus the multiplicity of l upon it. From any point of l can be drawn six such tangents, hence the order of the surface is 17. The conjugate of any point on every generator is the entire generator passing through it. The table of characteristics can now be constructed as follows:

$$
\begin{aligned}
&S_3 \sim V_8 : 6\pi^2 l^6 \gamma_3^{(3)} F_{17}\\
&S_2 \sim M_{14}{}^2;\\
&S_1 \sim C_8;\ [C_8; S_1] = 5\\
&\;\; l \sim L_7 : 6\pi^2 l^5 \gamma_3^{(3)} F_{17}\\
&\gamma_3 \sim \Gamma_7 : 6\pi^2 l^6 \gamma_3^{(2)} F_{17}\\
&H_5 \sim H_5 : 6\pi l^3 \gamma_3^{(2)} F_{17}.
\end{aligned}
$$

The symbol $\gamma_3^{(3)}$ means that all the V_8 of the system have three point contact with each other at every point of γ_3. The intersection of V_8 and L_7, a composite surface of order 56, consists of $6 \times 2 \times 2 = 24$ for the double planes, γ_3 counted three times, F_{17} and the surface of order 6, conjugate of the point in which the conjugate S_3 of the given V_8 meets l. Similarly, the intersection of V_8 and Γ_7 consists of $6\pi^2 = 24$, F_{17}, γ_3 taken twice, and the ruled surface of tangents to V_8 at points of S, γ_3 which meet l. This surface is of order 9.

The conjugate of an arbitrary plane is a surface of order 14 containing the quintic curve in which the given plane meets H. When the given plane meets l, its conjugate consists of a surface of order 8 and the sextic surface, image of the point on l.

The conjugate of a line is a curve of order 8 meeting it in 5 points. If the line meets l, its proper image is a conic having three point contact with γ_3.

6. Let l meet V in a second point N. If $N \equiv (0, 0, 0, 1, 0)$ and the tangent space to V at N be taken as $x_5 = 0$, the equation of V has the form

$$V: x_4^2 x_5 + x_4 x_5^2 + a x_4 x_5 + b x_4 + b' x_5 + c = 0$$

wherein a, b, b', c are ternary, of orders 1, 2, 2, 3, respectively, or

$$V: x_4^2 x_5 + x_4 g + \psi = 0.$$

If ψ^0, g^0 denote the values of ψ, g when $x_5 = 0$ then $\psi^0 = \phi_0$. The equations of the transformation S associated with N are of the same form as those of T, and can be obtained from them by making a few obvious changes.

In TS, L_7^6 appears as a factor in the second members of the equations, hence the transformation is of order 22, and is not periodic. The 12 parasitic planes all lie on the cubic cone $\phi_0 = 0$ which has the line l for vertex.

The line l meets V in a third point $P: (0, 0, 0, 1, -1)$, with $a + x_4 + x_5 = 0$ for tangent space. Let the transformation associated with this point be denoted by U. The three involutions T, S, U generate a noncyclic infinite discontinuous group.

7. Let l_1, l_2 be two lines through M, and T_1, T_2 the corresponding transformations. Their product is not periodic. The plane of $l_1 l_2$ is composed of invariant points under $T_1 T_2$.

The general transformation of this type may be thought of as follows: Let F be any rational surface of order n lying on V_3, and l any line not on V_3, meeting F in $n-1$ points. Then any plane through l meets F in one residual point, which takes the place of M'. But $n \ngtr 4$, since l is not on V_3. Since V_3 contains no surfaces other than complete intersections, it follows that the surfaces γ_3 in tangent S_3 are the only possible surfaces satisfying the conditions.

8. B. Given two skew lines l_1, l_2 on V_3 and a plane π not on V_3. Given any point (y) on S_4. The $S_3 \equiv \pi$, (y) meets l_i in P_i. The line P_1P_2 meets V_3 in K. The line $(y), K$ meets the polar quadric of K as to V_3 in K'. The harmonic conjugate (y') of (y) as to K, K' determines an involutorial birational transformation I under which V_3 remains invariant.

The two lines l_1, l_2 determine an S_3 meeting V_3 in a cubic surface F on which K lies. Since P_1, P_2 are projective, the line P_1P_2 describes a quadric surface meeting F in a residual rational C_4, having three points on each line l_i. This C_4 is the locus of K. Every line P_1P_2 lies in some S_3 through π,

hence the line meets π. The S_3 of l_1, l_2 therefore meets π in a directrix of the quadric, that is, π meets C_4 in 3 collinear points.

Let π be defined by $x_1 = 0$, $x_2 = 0$ and C_4 by

$$x_1 = \lambda_1\phi(\lambda_1, \lambda_2), \quad x_2 = \lambda_2\phi, \quad x_3 = f_3(\lambda_1, \lambda_2), \quad x_4 = f_4(\lambda_1, \lambda_2), \quad x_5 = 0$$

wherein ϕ is a cubic and each f_i a quartic form in (λ). The polar quadric of a point (λ) on C_4 as to V is

$$H(x) : \lambda_1\phi(\lambda_1, \lambda_2)\partial V/\partial x_1 + \lambda_2\phi\partial V/\partial x_2 + f_3\partial V/\partial x_3 + f_4\partial V/\partial x_4 = 0.$$

The space $x_1y_2 - x_2y_1 = 0$ or π, (y) meets C_4 in $\lambda_1y_2 - \lambda_2y_1 = 0$, $\lambda_1 = y_1$, $\lambda_2 = y_2$. The line $(y)(\lambda)$ meets $H(x)$ in (λ) and in the point $2H(y, \lambda)(y) = H(y)(\lambda)$. The point (y') is given by

$$I_{10}: \begin{array}{ll} y_1' = [H(y, \lambda) - H(y)\phi(\lambda)]\, y_1, & y_2' = [H(y, \lambda) - H(y)\phi(\lambda)]\, y_2, \\ y_3' = H(y, \lambda)y_3 - H(y)f_3(\lambda), & y_4' = H(y, \lambda)y_4 - H(y)f_4(\lambda), \\ \multicolumn{2}{c}{y_5' = H(y, \lambda)y_5,} \end{array}$$

in which $H(y, \lambda) = 0$ is the polar of the point (λ) on C_4 as to $H = 0$. The transformation is of order 10. The plane π is of multiplicity 8 and C_4 simple on the $| V_{10} |$, conjugates of $| S_8 |$.

9. The tangent S_3 to V_3 at a point (λ) on C_4 meets π in a line. The pencil of tangents at (λ) meeting this line are all generators of $H(y, \lambda)$. The same tangent S_3 meets the polar quadric variety of (λ) in a quadric cone. It meets π in a line having two points on the cone. Lines thru these points meet V_3 in three points coincident at (λ), hence these lines are parasitic. They generate a ruled surface containing C_4 doubly. The section of the space π, (y) with the polar quadric variety is a quadric surface lying on $H(y)$. When this quadric surface is a cone, it lies on $H(y)$ and $H(y, \lambda)$; each generator is a parasitic line. It is part of the base of $| V_{10} |$, and lies on Π_9, the conjugate of π in I_{10}. The cone is the residual intersection of $H(y)$ and the tangent space π, (y) to V_3.

The plane π meets V_3 in a general cubic curve γ_3. One line in π meets γ_3 in three points on C_4; another, not in π, meets it in points of l_1, l_2, C_4. The image of any point in π is a curve of order 8, having the point to order 4, and lying on the cone having the given point as vertex, with C_4 as directrix curve. Any line meeting π in one point has for image a conic meeting it in two points. Any line lying in π has for image a surface generated by a conic in the plane through the line and a variable point of C_4. Any S_3 through π is transformed into itself. Apart from parasitic lines, the only fundamental elements are π and C_4. The former is a proper two

dimensional basis element, whereas C_4 is of dimensionality 1, hence does not appear as part of the base of two V_{10} of the system $|V_{10}|$.

Any S_2 meets π in one point or lies in an S_3 containing π. The order of the surface of parasitic planes (pencils of lines) is found from the intersection of V_{10} with H_6 to be 22. The order of the surface conjugate to an arbitrary plane is then found from $[V_{10}, V_{10}]$ to be 14. From $[V, H(y,\lambda)]$ the 4 remaining units $90 = 64 + 22 + 4$ are accounted for by the planes conjugate to the four points in which S_3 meets C_4. The space $y_5 = 0$ which contains C_4 goes into itself, as its conjugate contains the variety conjugate to C_4. $[H_9(y,\lambda), \Pi_9] = 81 = 56 + 22 + 3$, the 3 being planes conjugate to the three points of C_4 on π. Finally, $[\Pi_9, H_6]$ from $\Pi_9 = H(y,\lambda) - H_6\phi(\lambda)$ has the same value as H_6 with $H(y,\lambda)$. This completes the table. We may now write

$$\begin{aligned}
&S_3 \sim V_{10}{}^3 : \pi^8 C_4{}^{(3)} \cdot F_{22} \\
&S_2 \sim M_{14} \\
&S_1 \sim C_{10} : [C_{10}, \pi] = 9,\ [C_{10}, C_4] = 9,\ [C_{10}, S_1] = 6 \\
&C_4 \sim H_9(y,\lambda) :\ \pi^8 C_9{}^{(2)} F_{22} \\
&\pi \sim \Pi_9 : \pi^7 C_4{}^{(3)} F_{22} \\
&H_6 \sim H_6 : \pi^4 C^{(2)} F_{22} \\
&J_{45} \equiv H^4(y,\lambda) \cdot \Pi_9.
\end{aligned}$$

This case can be generalized to include the following one. Define π as before, and let C_n be a rational curve of order n, lying on V_3 and having $n-1$ points on π. With obvious changes in ϕ, f_i the equations of the transformation have the same form as before. The general table of characteristics has the form

$$\begin{aligned}
&S_3 \sim V_{2n+2} :\ \pi^{2n} C_n{}^{(3)} F_{5n+2}, \\
&S_2 \sim M_{3n+2}, \\
&S_1 \sim C_{2n+2},\ [S_1, C_{2n+2}] = n + 2 :\ [\pi, C_{2n+2}] = 2n + 1 = [C_n, C_{2n+2}], \\
&\pi \sim H(y,\lambda) - H(y)\phi(\lambda) \equiv \Pi_{2n+1} :\ \pi^{2n-1} C_n{}^{(3)} F_{5n+2}, \\
&C_n \sim H_{2n+1}(y,\lambda) :\ \pi^{2n} C_n{}^{(2)} F_{5n+2}, \\
&H_{n+2} \sim H_{n+2} :\ \pi^n C_n{}^{(2)} F_{5n+2}, \\
&J_{10n+5} \equiv H^4(y,\lambda)\, \Pi_{2n+1}.
\end{aligned}$$

10. C. Let $l : x_1 = 0,\ x_2 = 0,\ x_3 = 0$ be a line on a general V_3 and R_n be a rational surface of order n, not lying on V_3 but having $n-1$ points on l. A point (y) of S_4 determines a plane (y, l) which meets R_n in one point P. This plane meets V_3 in a residual conic C_2. Let p be the polar line of P as to C_2. The conjugate (y') of the given point (y) in the

harmonic homology P, p generates a birational involutorial transformation of S_4, under which each conic of V_3 in a plane through l is transformed into itself; hence V_3 remains invariant.

The parametric representation of R_n has the form

$$x_i = f_i(r_1, r_2, r_3), \ (i = 1, 2, 3), \qquad x_4 = g_4(r), \qquad x_5 = g_5(r),$$

in which the net $| f(r) |$ is Cremonian, of order n' and $g_4(r) = 0$, $g_5(r) = 0$ contain base points of $| f |$ to multiplicity $n'^2 - n$.

The equation of V_3 has the form

$$V_3 \colon \ ux_1 + vx_2 + wx_3 = 0.$$

The plane

$$l,(y) \colon \ x_i = y_i k \qquad\qquad (i = 1, 2, 3),$$

cuts from V_3 the conic

$$k^2 t + k(px_4 + sx_5) + q(x_4, x_5) = 0,$$

in which t is cubic, p and s each quadratic and q linear in y_1, y_2, y_3; q is quadratic in x_4, x_5. If k be replaced by x_1/y_1, the equation of the conic in x_1, x_4, x_5 has coefficients cubic in y_1, y_2, y_3.

The coördinates of P have the form

$$P \colon \ y_1\theta(y), \quad y_2\theta(y), \quad y_3\theta(y), \quad \phi_4(y), \quad \phi_5(y)$$

wherein θ, ϕ_i are obtained as follows: the equations $f_i(r) = y_i$ can be solved for $r_i = f_i^{-1}(y)$, and substituted in $g_i(r)$ and $f_i(r) = y_i F(y)$. The forms $g_k(f^{-1})$ and F contain fundamental curves of orders $n'^2 - n$, so that θ is of order $n - 1$ and each ϕ_k of order n in y_1, y_2, y_3. The polar p of $(y_1\theta, \phi_4, \phi_5)$ as to the conic C_2 is of the form

$$a_1 x_1 + a_4 x_4 + a_5 x_5 = 0,$$

in which a_i is of order $n + 2$ in y_1, y_2, y_3. The factor y_1 has been removed from the equation. The equations of the harmonic homology P, p have the form

$$\begin{aligned} I\colon \quad & y_1' = (\Gamma - 2\theta K)y_1, \quad y_2' = (\Gamma - 2\theta K)y_2, \quad y_3' = (\Gamma - 2\theta K)y_3, \\ & y_4' = \Gamma y_4 - 2K\phi_4, \quad y_5' = \Gamma y_5 - 2K\phi_5. \end{aligned}$$

Here K is the result of substituting y_1, y_4, y_5 for x in the equation of the polar line p of P as to C_2. The factor y_1 can be removed again. $K = 0$ is the variety of invariant points in the involutorial transformation of order $2n + 2$. It is of order $n + 2$ and contains l to multiplicity $n + 1$.

The variety $\Gamma = 0$ is a cone of order $2n + 1$ having l for vertex. Its

equation is obtained by replacing x_1, x_4, x_5 by $y_1\theta$, ϕ_4, ϕ_5 in the equation of the polar p, and removing the factor y_1. It is the projection of the curve γ_{3n}, intersection of R_n, V_3 from l.

The conjugate of the line l is the variety $L=0$ defined by $L \equiv \Gamma - 2\theta K = 0$. It is of order $2n+1$, contains l to multiplicity $2n$, and contains the curve γ_{3n}. Every S_2 through l is transformed into itself.

The base surface M of the system $|V_{2n+2}|$ of varieties conjugate to the $|S_3|$ of S_4 is the complete intersection $\Gamma = 0$, $K=0$. It is of order $(2n+1)(n+2)$.

The table of characteristics of I_{2n+2} may be written in the form

$$\begin{aligned}
&S_3 \sim V_{2n+2}: M_{(2n+1).(n+2)}\gamma_{3n}{}^{(2)} \cdot l^{2n+1},\\
&S_2 \sim M_{2n^2+3n+2},\\
&S_1 \sim C_{2n+2},\ [C_{2n+2}, l] = 2n+1:\ [C_{2n+2}, S_1] = 2n+1,\\
&l \sim L_{2n+1}: M_{(2n+1).(n+2)}\gamma_{3n} \cdot l^{2n},\\
&\gamma_{3n} \sim \Gamma_{2n+1}: M_{(2n+1)(n+2)} l^{2n+1}\gamma_{3n},\\
&K_{n+2} \sim K_{n+2}: M_{(2n+1)(n+2)} l^{n+1}\gamma_{3n}.
\end{aligned}$$

Any two V_{2n+2}, varieties conjugate to two S_3 touch each other at every point of γ_{3n}. The conjugate C_{2n+2} of any S_1 lies in the S_3 determined by the given S_1 and l. Any S_3 not containing l meets it in a point. The conjugate V_{2n+2} meets L_{2n+1} in the cubic cone connecting this point with γ_{3n}. The S_3 meets R_n in n lines, each containing 3 points of γ_{3n}. The conjugate V_{2n+2} meets Γ in $M_{(2n+1)(n+2)}$ and in the $3n$ planes connecting these points on γ_{3n} with l. Finally, S_3 meets K_{n+2} in a surface F_{n+2} of invariant points, which is also on its conjugate V_{2n+2}.

The jacobian of the system $|V_{2n+2}|$ is

$$J \equiv L^2{}_{2n+1}\Gamma^3{}_{2n+1}.$$

These results can be generalized immediately to apply to a three dimensional variety V_m of S_4, having l to multiplicity $m-2$. The simplest form of R_n is a plane $x_4=0$, $x_5=0$. An extensive category is that of the monoids in S_3:

$$g_{n-1}(x_1, x_2, x_3)x_4 + h_n(x_1, x_2, x_3) = 0; \quad x_5 = 0.$$

11. A set A', B', C', of generating involutions of another infinite discontinuous Cremona group under which V_3 remains invariant, this time point by point, can be obtained by the central perspective Jonquieres involutions in each plane through l, with P as center and the residual section of the plane with V_3 as curve of invariant points.

THE DERIVATION OF TENSORS FROM TENSOR FUNCTIONS.*

By Alfred K. Mitchell.

Introduction. E. Schroedinger † has given a rule for deriving a tensor from an invariant tensor function. In the first part of the present paper a proof is given of the theorem that if Φ is any invariant function of a tensor its derivative with respect to a component of this tensor is itself a component of a tensor. It is also proved that the derivative of a tensor function of a tensor produces a tensor of higher rank. The first of these theorems is then applied to the invariants of a mixed tensor of rank 2, and the invariants of the tensor so derived are considered. In this way it is seen that if $F_s{}^r$ is any polynomial function $F = a_0 + a_1E + a_2E^2 + \cdots$ of the matrix E, then its invariants can be expressed in terms of the invariants of E and hence the derivatives of the invariants of F are functions of the derivatives of the invariants of E.

1. By a space of n dimensions we mean a continuous arrangement of points; a point being a set of n ordered real numbers $(x^1, x^2, x^3, \cdots, x^n)$.

A set of n equations

$$\bar{x}^r = \bar{x}^r(x^1, x^2, \cdots, x^n) \qquad (r = 1, 2, \cdots, n)$$

in which the functions $\bar{x}^r$ are single valued for all points and which can be solved ‡ so as to yield a set of n equations

$$x^r = x^r(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$$

in which the functions x^r are singled valued, define a transformation of coördinates in the n dimensional space.

A set of $n^{(p+q)}$ functions $X^{r_1r_2\cdots r_p}_{s_1s_2\cdots s_q}$, defined with respect to a coördinate system (x), and from which we obtain, in any other coördinate system $(\bar{x})$, the corresponding $n^{(p+q)}$ functions $\bar{X}^{r_1r_2\cdots r_p}_{s_1s_2\cdots s_q}$, by means of the $n^{(p+q)}$ equations of transformation §

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† See *Annalen der Physik*, Vol. 82 (1927), p. 265.

‡ This implies that the Jacobian determinant $\partial(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)/\partial(x^1, x^2 \cdots x^n)$ is not identically zero. See Goursat-Hedrick, *Mathematical Analysis*, Vol. I, Chap. 2.

§ In these equations and throughout this paper Greek letters occurring as indices in pairs will be used as summation or umbral labels, the summation from 1 to n. See O. Veblen, *Cambridge Tracts*, No. 24, Chaps. 1 and 2.

$$\bar{X}^{r_1 r_2 \cdots r_p}_{s_1 s_2 \cdots s_q} = X^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q} \frac{\partial \bar{x}^{r_1}}{\partial x^{\alpha_1}} \frac{\partial \bar{x}^{r_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial \bar{x}^{r_p}}{\partial x^{\alpha_q}} \frac{\partial x^{\beta_1}}{\partial \bar{x}^{s_1}} \frac{\partial x^{\beta_2}}{\partial \bar{x}^{s_2}} \cdots \frac{\partial x^{\beta_q}}{\partial \bar{x}^{s_q}}$$

is said to define a mixed tensor of rank $p + q$ which is contravariant of rank p and covariant of rank q.

Any function X of the coördinates whose value is unchanged when we change the coördinate system is called an invariant or scalar function. The equation of transformation is

$$\bar{X} = X.$$

According to the rule of composition of tensors* we can form an invariant from a contravariant tensor of rank p, $X^{r_1 r_2 \cdots r_p}$ and a covariant tensor of rank p, $X_{r_1 r_2 \cdots r_p}$ by forming their scalar product

$$X^{\alpha_1\alpha_2\cdots\alpha_p} X_{\alpha_1\alpha_2\cdots\alpha_p} \; (\alpha_1, \alpha_2 \cdots \alpha_p \text{ summation labels}).$$

The converse of the rule of composition of tensors may be stated for the general case as follows. If the functions $X^{a_1 \cdots a_p r_1 \cdots r_l}_{b_1 \cdots b_q s_1 \cdots s_m}$ have such a law of transformation that the summation

$$X^{\alpha_1 \cdots \alpha_p r_1 \cdots r_l}_{\beta_1 \cdots \beta_q s_1 \cdots s_m} \cdot Y^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}$$

is a mixed tensor, covariant of rank m and contravariant of rank l, where $Y^{t_1 \cdots t_q}_{u_1 \cdots u_p}$ is an arbitrary mixed tensor contravariant of rank q and covariant of rank p, then the $n^{(p+l+q+m)}$ functions $X^{a_1 \cdots a_p r_1 \cdots r_l}_{b_1 \cdots b_q s_1 \cdots s_m}$ form a mixed tensor, contravariant of rank $(p + l)$ and covariant of rank $(q + m)$.

THEOREM 1.† *If Φ is any invariant function of a tensor $E^{r_1 \cdots r_p}_{s_1 \cdots s_q}$, contravariant of rank p, covariant of rank q and of any other tensors which are independent of $E^{r_1 \cdots r_p}_{s_1 \cdots s_q}$, then $\partial\Phi/\partial E^{r_1 \cdots r_p}_{s_1 \cdots s_q}$ is a mixed tensor which is contravariant of rank q and covariant of rank p.*

Proof: Let

$$\partial\Phi/\partial E^{r_1 \cdots r_p}_{s_1 \cdots s_q} = X^{s_1 \cdots s_q}_{r_1 \cdots r_p}.$$

Then

$$\bar{X}^{s_1 \cdots s_q}_{r_1 \cdots r_p} = \partial\Phi/\partial \bar{E}^{r_1 \cdots r_p}_{s_1 \cdots s_q} = \partial\Phi/\partial E^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} \times \partial E^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}/\partial \bar{E}^{r_1 \cdots r_p}_{s_1 \cdots s_q}.$$

But, by hypothesis,

$$E^{\alpha_1\alpha_2 \cdots \alpha_p}_{\beta_1\beta_2 \cdots \beta_q} = \bar{E}^{\rho_1 \cdots \rho_p}_{\sigma_1 \cdots \sigma_q} \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\rho_1}} \cdots \frac{\partial x^{\alpha_p}}{\partial \bar{x}^{\rho_p}} \frac{\partial \bar{x}^{\sigma_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \bar{x}^{\sigma_q}}{\partial x^{\beta_q}}.$$

* See F. D. Murnaghan, *Vector Analysis and Theory of Relativity*, p. 25.

† The author finds that this theorem is not essentially different from the "Aronhold Process." See Weitzenböck, *Invarianten Theorie*, p. 18.

Hence

$$\partial E^{a_1 \dots a_p}_{\beta_1 \dots \beta_q} / \partial \bar{E}^{r_1 \dots r_p}_{s_1 \dots s_q} = \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \cdots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Therefore

$$\bar{X}^{s_1 \dots s_q}_{r_1 \dots r_p} = X^{\beta_1 \dots \beta_q}_{a_1 \dots a_p} \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \cdots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Q. E. D.

Now suppose we have a mixed tensor $X^{r_1 \dots r_p}_{s_1 \dots s_q}$ which is contravariant of rank p and covariant of rank q and is a function of a mixed tensor $E^{a_1 \dots a_l}_{b_1 \dots b_m}$ and possibly other tensors. Let us form an invariant Φ from the tensor $X^{r_1 \dots r_p}_{s_1 \dots s_q}$ and the arbitrary tensor $Y^{m_1 \dots m_q}_{n_1 \dots n_p}$, which is covariant of rank p and contravariant of rank q and which is independent of the tensor $E^{a_1 \dots a_l}_{b_1 \dots b_m}$. By the rule of composition, we have

$$\Phi = X^{a_1 \dots a_p}_{\beta_1 \dots \beta_q} \cdot Y^{\beta_1 \dots \beta_q}_{a_1 \dots a_p},$$

Now by Theorem 1

$$\partial \Phi / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m} = Z^{b_1 \dots b_m}_{a_1 \dots a_l},$$

is a mixed tensor contravariant of rank m and covariant of rank l. But

$$\partial \Phi / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m} = (\partial X^{a_1 \dots a_p}_{\beta_1 \dots \beta_q} / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m}) Y^{\beta_1 \dots \beta_q}_{a_1 \dots a_p} = Z^{b_1 \dots b_m}_{a_1 \dots a_l}.$$

Hence, denoting

$$\partial X^{a_1 \dots a_p}_{\beta_1 \dots \beta_q} / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m} \quad \text{by} \quad F^{a_1 \dots a_p b_1 \dots b_m}_{\beta_1 \dots \beta_q a_1 \dots a_l},$$

we have

$$F^{a_1 \dots a_p b_1 \dots b_m}_{\beta_1 \dots \beta_q a_1 \dots a_l} Y^{\beta_1 \dots \beta_q}_{a_1 \dots a_p} = Z^{b_1 \dots b_m}_{a_1 \dots a_l},$$

and, therefore, by the converse of the rule of composition of tensors,

$$F^{r_1 \dots r_p b_1 \dots b_m}_{s_1 \dots s_q a_1 \dots a_l} = \partial X^{r_1 \dots r_p}_{s_1 \dots s_q} / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m}$$

is a mixed tensor which is contravariant of rank $(p + m)$ and covariant of rank $(q + l)$. We have therefore proved

THEOREM 2. *If $X^{r_1 \dots r_p}_{s_1 \dots s_q}$ is a mixed tensor, contravariant of rank p and covariant of rank q which is a function of a mixed tensor $E^{a_1 \dots a_l}_{b_1 \dots b_m}$, and possibly other tensors, then $\partial X^{r_1 \dots r_p}_{s_1 \dots s_q} / \partial E^{a_1 \dots a_l}_{b_1 \dots b_m}$ is a mixed tensor which is covariant of rank $(q + l)$ and contravariant of rank $(p + m)$.*

2. We shall now apply the first of the above theorems to derive tensors from the invariants of the mixed tensor of rank 2, E_s^r, and shall consider the problem of expressing the invariants of these derived tensors in terms

of the invariants of $E_s{}^r$. Note: The invariants of a covariant tensor of rank 2, E_{rs}, are the invariants of the n-ary quadratic form $f = E_{\alpha\beta}x^\alpha x^\beta$ and there is only one invariant of this form, namely its discriminant.* But $E_s{}^r = g^{r\alpha}E_{\alpha s}$ where $g^{rs} =$ cofactor of $g_{sr} \div |g|$ and g_{rs} are the coefficients of the differential form $ds^2 = g_{\alpha\beta} \cdot dx^\alpha dx^\beta$, thus the invariants of $E_s{}^r$ may be considered as the simultaneous invariants of the two n-ary quadratic forms $f = E_{\alpha\beta}x^\alpha x^\beta$ and $g = g_{\alpha\beta}x^\alpha x^\beta$. Or, if one prefers, the invariants of $E_s{}^r$ may be regarded as the simultaneous invariants of $E_s{}^r$ and the unit tensor $\delta_s{}^r$ whose components are 1 or zero according as r is equal to s or not.

Introducing the generalized Kronecker delta † $\delta^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ (which, if the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts, has the value $+1$ or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; and which in all other cases has the value zero) we can form the following invariants of the tensor $E_s{}^r$:

$$I_1 = \delta_\beta{}^\alpha E_\alpha{}^\beta; \qquad I_2 = (1/2\,!)\,\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2},$$
$$I_3 = (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}\,E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$I_n = (1/n\,!)\,\delta^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}\cdots E_{\alpha_n}{}^{\beta_n}.$$

Applying Theorem 1 to these invariants, we derive from each of them a mixed tensor. Denote the tensors derived from $I_1, I_2, \cdots I_n$ by $T_{1r}{}^s$, $T_{2r}{}^s, \cdots T_{nr}{}^s$ respectively. Then

$$T_{1r}{}^s = \partial I_1/\partial E_s{}^r = \delta_r{}^s,$$
$$T_{2r}{}^s = \partial I_2/\partial E_s{}^r = \delta^{s\alpha}_{r\beta}E_\alpha{}^\beta = \{\delta_r{}^s\delta_\beta{}^\alpha - \delta_\beta{}^s\delta_r{}^\alpha\}E_\alpha{}^\beta = \delta_r{}^s I_1 - E_r{}^s,$$
$$T_{3r}{}^s = \partial I_3/\partial E_s{}^r = (1/2\,!)\,\delta^{s\alpha_1\alpha_2}_{r\beta_1\beta_2}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}$$
$$= (1/2\,!)\,\{\delta_r{}^s\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} - \delta_{\beta_1}{}^s\delta^{\alpha_1\alpha_2}_{r\beta_2} - \delta_{\beta_2}{}^s\delta^{\alpha_1\alpha_2}_{\beta_1 r}\}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}$$
$$= \delta_r{}^s I_2 - (1/2\,!)\,\{T_{2r}{}^{\alpha_1}E_{\alpha_1}{}^s + T_{2r}{}^{\alpha_2}E_{\alpha_2}{}^s\} = \delta_r{}^s I_2 - T_{2r}{}^\alpha E_\alpha{}^s$$
$$= \delta_r{}^s I_2 - (\delta_r{}^\alpha I_1 - E_r{}^\alpha)E_\alpha{}^s = \delta_r{}^s I_2 - I_1 E_r{}^s + E_r{}^\alpha E_\alpha{}^s,$$
$$T_{4r}{}^s = (1/3\,!)\,\delta^{s\alpha_1\alpha_2\alpha_3}_{r\beta_1\beta_2\beta_3}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3}$$
$$= (1/3\,!)\,\{\delta_r{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} - (\delta_{\beta_1}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{r\beta_2\beta_3} + \delta_{\beta_2}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1 r\beta_3} + \delta_{\beta_3}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2 r})\}\,E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3}$$
$$= \delta_r{}^s I_3 - (1/3)\,\{T_{2r}{}^{\alpha_1}E_{\alpha_1}{}^s + T_{2r}{}^{\alpha_2}E_{\alpha_2}{}^s + T_{2r}{}^{\alpha_3}E_{\alpha_3}{}^s\} = \delta_r{}^s I_3 - T_{2r}{}^\alpha E_\alpha{}^s$$
$$= \delta_r{}^s I_3 - I_2 E_r{}^s + I_1 E_r{}^\alpha E_\alpha{}^s - E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}E_{\alpha_2}{}^s,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T_{pr}{}^s = \delta_r{}^s I_{p-1} - I_{p-2}E_r{}^s + I_{p-3}E_r{}^\alpha E_\alpha{}^s - I_{p-4}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}E_{\alpha_2}{}^s$$
$$+ \cdots (-1)^{t+1}I_{p-t}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}\cdots E^s_{\alpha_{t-2}} + \cdots (-1)^{p+1}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}\cdots E^s_{\alpha_{p-2}}.$$

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* See Dickson, *Algebraic Invariants*, p. 48.

† See F. D. Murnaghan, *American Mathematical Monthly*, Vol. 32, p. 233.

From each of these mixed tensors we can form invariants analogous to the invariants $I_1, I_2, \cdots I_n$ of the tensor $E_s{}^r$. Denote the invariants formed from

$$\begin{array}{lll} T_{1r}{}^s & \text{by} & I_{11}, I_{21}, \cdots I_{n1}, \\ T_{2r}{}^s & \text{by} & I_{12}, I_{22}, \cdots I_{n2}, \\ T_{3r}{}^s & \text{by} & I_{13}, I_{23}, \cdots I_{n3}, \\ T_{ir}{}^s & \text{by} & I_{1i}, I_{2i}, \cdots I_{ni}. \end{array}$$

For the invariants of $T_{1r}{}^s$, we have

$$I_{11} = \delta_\beta{}^\alpha \delta_\alpha{}^\beta = \delta_\alpha{}^\alpha = n,$$
$$I_{21} = (1/2\,!)\,\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = (1/2\,!)\,\delta_{\alpha_1\alpha_2} = [n(n-1)/2\,!],$$
$$I_{31} = (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2}\delta_{\alpha_3}{}^{\beta_3} = (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}_{\alpha_1\alpha_2\alpha_3} = [n(n-1)(n-2)/3\,!],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$I_{m1} = (1/m\,!)\,\delta^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_m}\delta_{\alpha_1}{}^{\beta_1}\cdots\delta_{\alpha_m}{}^{\beta_m} = (1/m\,!)\,\delta^{\alpha_1\cdots\alpha_m}_{\alpha_1\cdots\alpha_m} = [n\,!/m\,!(n-m)\,!],$$
$$I_{n1} = (n\,!/n\,!) = 1.$$

Now denote by $f(\lambda)$ the polynomial

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} + \cdots (-1)^n I_n = f(\lambda),$$

where $I_1, I_2, \cdots I_n$ are the invariants of the tensor $E_r{}^s$. It is evident, from the definition of these invariants (see page 198) and the definition of the generalized Kronecker delta, that I_1 is the sum of the diagonal elements of the matrix $\| E_r{}^s \|$; I_2 is the sum of the two rowed principle minors; I_3 the sum of the three rowed principle minors, etc. and I_n is the determinant of the matrix $\| E_r{}^s \|$. Hence we see that $f(\lambda) = 0$ is the characteristic equation of the matrix $\| E_r{}^s \|$ i. e. the polynomial $f(\lambda)$ is the determinant of the matrix $\| \delta_r{}^s\lambda - E_r{}^s \|$.

Thus the invariants of a mixed tensor $E_r{}^s$ are the coefficients of the characteristic equation of the matrix $\| E_r{}^s \|$.

With this fact and with the theorem from algebra which says that if you have a matrix $\| E_r{}^s \|$, with characteristic equation $f(\lambda) = 0$ and latent roots $\lambda_1, \lambda_2, \cdots \lambda_n$, the latent roots of any polynomial function of $\| E_r{}^s \|$ are the corresponding polynomial functions of the latent roots $\lambda_1, \lambda_2, \cdots \lambda_n$, we shall be able to investigate the nature of the invariants of the tensors $T_{2r}{}^s, T_{3r}{}^s, \cdots T_{pr}{}^s$.

Consider the tensor $T_{2r}{}^s$. Its characteristic polynomial will be the determinant of the matrix $\| \delta_r{}^s\lambda - T_{2r}{}^s \|$. But $T_{2r}{}^s = \delta_r{}^s I_1 - E_r{}^s$, so that the determinant

$$| \delta_r{}^s\lambda - T_{2r}{}^s | = | \delta_r{}^s(\lambda - I_1) + E_r{}^s |$$
$$= (-1)^n \, | \delta_r{}^s(I_1 - \lambda) - E_r{}^s | = (-1)^n f(I_1 - \lambda),$$

where

$$f(\lambda)=\lambda^n-I_1\lambda^{n-1}+I_2\lambda^{n-2}+\cdots(-1)^nI_n=|\,\delta_r{}^s\lambda-E_r{}^s\,|\,.$$

By Taylor's expansion for a polynomial of degree n,

$$f(I_1-\lambda)=f(I_1)-\lambda f'(I_1)+(\lambda^2/2\,!)f''(I_1)+\cdots(-1)^n\lambda^n;$$

so that the characteristic polynomial for $T_{2r}{}^s$ is

$$\lambda^n-[f^{(n-1)}(I_1)/(n-1)\,!]\lambda^{n-1}+[f^{(n-2)}(I_1)/(n-2)\,!]\lambda^{n-2}$$
$$+\cdots(-1)^r[f^{(n-r)}(I_1)/(n-r)\,!\,]\lambda^{n-r}+\cdots(-1)^nf(I_1),$$

from which we obtain the invariants of $T_{2r}{}^s$. They are

$$I_{12}=[f^{(n-1)}(I_1)/(n-1)\,!]=1/(n-1)\,!\;\{n\,!\,I_1-(n-1)\,!\,I_1\}=(n-1)I_1$$
$$I_{22}=[f^{(n-2)}(I_1)/(n-2)\,!]$$
$$=1/(n-2)\,!\;\{(n\,!/2\,!)I_1{}^2-(n-1)\,!\,I_1I_1+(n-2)\,!\,I_2\}$$
$$=[n(n-1)/2\,!\,]I_1{}^2-(n-1)I_1I_1+I_2,$$
$$I_{32}=[f^{(n-3)}(I_1)/(n-3)\,!\,]$$
$$=1/(n-3)\,!\;\{(n\,!/3\,!)I_1{}^3-[(n-1)\,!/2\,!\,]I_1I_1{}^2$$
$$+[(n-2)\,!/1]I_2I_1-(n-3)\,!\,I_3\}=[n(n-1)(n-2)/3\,!\,]I_1{}^3$$
$$-[(n-1)(n-2)/2\,!\,]I_1I_1{}^2-(n-2)I_2I_1-I_3,$$
$$I_{s2}=[f^{(n-s)}(I_1)/(n-s)\,!\,]=1/(n-s)\,!\;\{(n\,!/s\,!)I_1{}^s$$
$$-[(n-1)\,!/(s-1)\,!\,]I_1I_1{}^{s-1}+[(n-2)\,!/(s-2)\,!\,]I_2I_1{}^{s-2}$$
$$+\cdots(-1)^r[(n-r)\,!/(s-r)\,!\,]I_rI_1{}^{s-r}+\cdots(-1)^s(n-s)\,!\,I_s\}$$
$$=[n\,!/(n-s)\,!\,s\,!\,]I_1{}^s-[(n-1)\,!/(n-s)\,!\;(s-1)\,!\,]I_1I_1{}^{s-1}$$
$$+[(n-2)\,!/(n-s)\,!\;(s-2)\,!\,]I_2I_1{}^{s-2}$$
$$+\cdots(-1)^r[(n-r)\,!/(n-s)\,!\;(s-r)\,!\,]I_rI_1{}^{s-r}+\cdots(-1)^sI_s,$$
$$I_{n2}=f(I_1)=I_1{}^n-I_1I_1{}^{n-1}+I_2I_1{}^{n-2}+\cdots(-1)^rI_rI_1{}^{n-r}+\cdots(-1)^nI_n.$$

Likewise the invariants $I_{13}, I_{23},\cdots I_{n3}$ of the tensor $T_{3r}{}^s$ will be the coefficients of the characteristic equation of the matrix $\|\,T_{3r}{}^s\,\|$ and since (from the expression for $T_{3r}{}^s$ on page 198)

$$\|\,T_{3r}{}^s\,\|=\|\,\delta_r{}^sI_2\,\|-I_1\|\,E_r{}^s\,\|+\|\,E_r{}^s\,\|^2$$

by the theorem stated above, $I_{13}, I_{23},\cdots I_{n3}$ will be the coefficients of the equation whose roots are $(I_2-I_1\lambda_1+\lambda_1{}^2)$, $(I_2-I_1\lambda_2+\lambda_2{}^2),\cdots$ $(I_2-I_1\lambda_n+\lambda_n{}^2)$ where $\lambda_1, \lambda_2\cdots\lambda_n$ are the roots of

$$f(\lambda)=\lambda^n-I_1\lambda^{n-1}+I_2\lambda^{n-2}+\cdots(-1)^nI_n=0.$$

From the elementary theory of equations we see that for this last equation

$$I_1 = \Sigma\lambda_1$$
$$I_2 = \Sigma\lambda_1\lambda_2$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$I_n = \Sigma\lambda_1\lambda_2\cdots\lambda_n.$$

Similarly, since $I_{13}, I_{23}, \cdots I_{n3}$ play a corresponding rôle in the characteristic equation of the matrix $\| T_{3r}{}^s \|$, we shall have

$$\begin{aligned} I_{13} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2), \\ I_{23} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2), \\ I_{33} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2)(I_2 - I_1\lambda_3 + \lambda_3^2), \quad \text{etc.} \end{aligned}$$

By multiplying out the factors of a term in the above summations we find

$$\begin{aligned} I_{13} &= nI_2 - I_1\Sigma\lambda_1 + \Sigma\lambda_1^2, \\ I_{23} &= [n(n-1)/2!]I_2^2 - (n-1)I_2I_1\Sigma\lambda_1 \\ &\quad + (n-1)I_2\Sigma\lambda_1^2 + I_1^2\Sigma\lambda_1\lambda_2 - I_1\Sigma\lambda_1\lambda_2^2 + \Sigma\lambda_1^2\lambda_2^2 \\ &= [n(n-1)/2!]I_2^2 - (n-1)I_2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} + \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\}, \\ I_{33} &= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2I_1\Sigma\lambda_1 \\ &\quad + [(n-1)(n-2)/2!]I_2^2\Sigma\lambda_1^2 + (n-2)I_2I_1^2\Sigma\lambda_1\lambda_2 \\ &\quad - (n-2)I_2I_1\Sigma\lambda_1\lambda_2^2 + (n-2)I_2\Sigma\lambda_1^2\lambda_2^2 \\ &\quad + \Sigma\lambda_1^2\lambda_2^2\lambda_3^2 - I_1\Sigma\lambda_1\lambda_2^2\lambda_3^2 + I_1^2\Sigma\lambda_1\lambda_2\lambda_3^2 - I_1^3\Sigma\lambda_1\lambda_2\lambda_3 \\ &= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} \\ &\quad + (n-2)I_2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\} \\ &\quad - \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)(I_1\lambda_3 - \lambda_3^2)\}. \end{aligned}$$

From which we see that

$$\begin{aligned} I_{43} &= [n(n-1)(n-2)(n-3)/4!]I_2^4 \\ &\quad - [(n-1)(n-2)(n-3)/3!]I_2^3\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} \\ &\quad + [(n-2)(n-3)/2!]I_2^2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\} \\ &\quad - (n-3)I_2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)(I_1\lambda_3 - \lambda_3^2)\} \\ &\quad + \{\Sigma(I_1\lambda_1 - \lambda_1^2)(\qquad)\cdots(I_1\lambda_4 - \lambda_4^2)\}, \end{aligned}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\begin{aligned} I_{r3} &= [n!/(n-r)!\,r!]I_2^r - [(n-1)!/(n-r)!\,(r-1)!]I_2^{r-1}\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} \\ &\quad + \cdots (-1)^{s-1}[(n-s+1)!/(n-r)!\,(r-s+1)!]I_2^{r-s+1} \\ &\quad \times \{\Sigma(I_1\lambda_1 - \lambda_1^2)\cdots(I_1\lambda_{s-1} - \lambda^2_{s-1})\} \\ &\quad + \cdots (-1)^r\{\Sigma(I_1\lambda_1 - \lambda_1^2)\cdots(I_1\lambda_r - \lambda_r^2)\}. \end{aligned}$$

Substituting $I_1 = \Sigma\lambda_1$ and $\Sigma\lambda_1^2 = I_1^2 - 2I_2$,

$$I_{13} = nI_2 - I_1^2 + I_1^2 - 2I_2 = (n-2)I_2.$$

Furthermore it is easily seen that

$$\Sigma\lambda_1\lambda_2^2 = I_1I_2 - 3I_3, \quad \Sigma\lambda_1^2\lambda_2^2 = I_2^2 - 2I_1I_3 + 2I_4, \quad \Sigma\lambda_1\lambda_2 = I_2.$$

Substituting these results in the expression (page 201) for I_{23} we obtain

$$I_{23} = [n(n-1)/2!]I_2^2 - 2(n-1)I_2^2 + I_2^2 + I_1I_3 + 2I_4.$$

Since $\Sigma\lambda_1^2\lambda_2^2\lambda_3^2$ is of degree 6 in the λ's, it is of weight six in the coefficients I, and we write

$$\Sigma\lambda_1^2\lambda_2^2\lambda_3^2 = AI_2I_5 + BI_2I_4 + CI_3^2 + DI_6.$$

Let

$$\lambda_1 = \lambda_2 = \lambda_3 = 1; \qquad \lambda_4 = \lambda_5 = \cdots = \lambda_n = 0.$$

Then

$$I_1 = 3, \quad I_2 = 3, \quad I = 1, \quad I_4 = I_5 = 0.$$

Substituting we obtain $C = 1$.

Next let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$; $\lambda_5 = \cdots = \lambda_n = 0$. Then

$$I_1 = 4, \quad I_2 = 6, \quad I_3 = 4, \quad I_4 = 1, \quad I_5 = I_6 = 0.$$

We get $B = -2$.

Continuing in this way we find that

$$\Sigma\lambda_1^2\lambda_2^2\lambda_3^2 = 2I_1I_5 - 2I_2I_4 + I_3^2 - 2I_6.$$

And by a similar process we obtain

$$\Sigma\lambda_1\lambda_2^2\lambda_3^2 = I_2I_3 - 3I_1I_4 + 5I_5, \quad \Sigma\lambda_1\lambda_2\lambda_3^2 = I_1I_3 - 4I_4, \quad \Sigma\lambda_1\lambda_2\lambda_3 = I_3.$$

Substituting these and the foregoing results in the expression (page 13) for I_{33}, we find

$$\begin{aligned} I_{33} = {} & [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2(2I_2) \\ & + (n-2)I_2(I_2^2 + I_1I_3 + 2I_4) \\ & - (2I_6 + I_1I_2I_3 + I_1^2I_4 + 3I_1I_5 + 2I_2I_4 + I_3^2). \end{aligned}$$

We observe from the expression for I_{r3} on page 201 that this invariant is a polynomial of degree r in I_2 and that the coefficients of this polynomial are symmetric functions of the roots of $f(\lambda) = 0$. These symmetric functions being of the form $\Sigma\lambda_1^p\lambda_2^q\lambda_3^r \cdots \lambda_s^t$ where the exponents are at most 2, can be calculated by means of known* formulae in terms of the coefficients of $f(\lambda) = 0$, i.e. in terms of the invariants $I_1, I_2, \cdots I_n$. It is evident from the foregoing expressions for the I_{13}, I_{23}, I_{33}, however, that the expression for I_{r3} is not conveniently put explicitly in terms of I_1, I_2, etc.

Finally we observe from the expressions (page 198) for the tensors T_r^s that each of the matrices $\| T_r^s \|$ is a polynomial function of the matrix $\| E_r^s \|$.

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* See Burnside and Panton, *Theory of Equations*, p. 269.

Thus

$$\| T_{2r}{}^s \| = \| \delta_r{}^s I_1 \| - \| E_r{}^s \|,$$
$$\| T_{3r}{}^s \| = \| \delta_r{}^s I_2 \| - I_1 \| E_r{}^s \| + \| E_r{}^s \|^2,$$
$$\| T_{4r}{}^s \| = \| \delta_r{}^s I_3 \| - I_2 \| E_r{}^s \| + I_1 \| E_r{}^s \|^2 - \| E_r{}^s \|^3, \quad \text{etc.}$$

It follows from the theorem stated on page 199 that the latent roots of each of these matrices are the corresponding polynomial functions of the roots of $f(\lambda) = 0$. Now given an equation of degree n, $f(\lambda) = 0$ whose roots are $\lambda_1, \lambda_2 \cdots \lambda_n$ we can, at least theoretically, by means of Tschirnhausen's transformation,* form the equation whose roots are any polynomial function of degree $(n-1)$ or less of the roots of $f(\lambda) = 0$, and the coefficients of the transformed equation will be given in terms of the coefficients of $f(\lambda) = 0$. Thus the invariants (coefficients of the characteristic equation of the matrix $\| T_{pr}{}^s \|$) of any one of the tensors $T_{pr}{}^s$ are functions of the invariants (coefficients of the equation $f(\lambda) = 0$) of the tensor $E_r{}^s$.

In general we can say, as is evident from the foregoing considerations, if $F_s{}^r$ is any polynomial function

$$F = a_0 + a_1 E + a_2 E^2 + \cdots,$$

of the matrix E then its invariants can be expressed in terms of the invariants of E, and hence the derivatives, with respect to $E_s{}^r$, of its invariants can be expressed in terms of the tensors $T_{1r}{}^s, T_{2r}{}^s \cdots T_{nr}{}^s$.

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* See Burnside and Panton, *op. cit.*, p. 318.

THE BEHAVIOR OF MEAN-SQUARE OSCILLATION AND CONVERGENCE UNDER REGULAR TRANSFORMATIONS.*

By Ralph Palmer Agnew.

1. *Introduction.* Recently the writer † has considered the behavior of continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of complex and real sequences of functions under complex and real regular transformations with triangular matrices. It is the object of this paper to extend that investigation to consider the behavior of mean square oscillation and convergence in the mean of sequences of complex and of real measurable functions under complex and real regular transformations with triangular matrices.

2. *Transformations.* We recall that a transformation with a triangular matrix is a sequence-to-sequence transformation of the form

$$\sigma_n = a_{n1}s_1 + a_{n2}s_2 + \cdots + a_{nn}s_n, \qquad (n = 1, 2, 3, \cdots),$$

where the a_{nk} are constants, and that a transformation is said to be *regular* when it carries every convergent sequence $\{s_n\}$ into a sequence $\{\sigma_n\}$ which converges to the same value. Such a transformation is said to be *real* when a_{nk} is real for all n and k; otherwise it is complex. The transformations and sequences considered in this paper may, except in cases where a specific statement to the contrary is made, be complex.

The following six conditions which are to be used in this paper, are listed together for convenience:

$$C_1: \quad \sum_{k=1}^{n} | a_{nk} | \text{ is bounded for all } n;$$

$$C_2: \quad \text{for each } k, \lim_{n\to\infty} a_{nk} = 0;$$

$$C_3: \quad \lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1;$$

$$C_4: \quad \lim_{n\to\infty} \sum_{k=1}^{n} | a_{nk} | = 1;$$

$$C_5: \quad \text{for each } k, a_{nk} = 0 \text{ for almost ‡ all } n;$$

$$C_6: \quad \sum_{k=1}^{n} a_{nk} = 1 \text{ for almost all } n.$$

* Presented to the American Mathematical Society, April 18, 1930, together with the paper referred to in § 1.

† "The Behavior of Bounds and Oscillations of Sequences of Functions under Regular Transformations," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 669-708.

‡ i. e. $a_{nk} = 0$ except for at most a finite number, depending on k, of values of n.

Let the symbol (T) represent a regular transformation with a triangular matrix. By the Silverman-Toeplitz theorem, C_1, C_2, and C_3 are necessary and sufficient for the regularity of a complex transformation when applied to complex sequences, and of a real transformation when applied to real sequences. Hence (T), complex or real, satisfies C_1, C_2, and C_3.

A property* of regular transformations which we shall have occasion to use is included in the following

LEMMA 2.1. *If* (T) *fails to satisfy* C_4, *then there is a bounded sequence* $\{s_n\}$ *of constants such that*

$$\lim_{m\to\infty,\, n\to\infty} \sup |\sigma_m - \sigma_n| > \lim_{m\to\infty,\, n\to\infty} \sup |s_m - s_n|;$$

if (T) *is real,* $\{s_n\}$ *may be taken real.*

3. *Mean Square Oscillation and Convergence in the Mean.* Let a sequence $\{f_n(x)\}$ of measurable functions† be defined over a set A in a Euclidean space, and let the Lebesgue integral

$$\int_A |f_m(x) - f_n(x)|^2 \, dx$$

exist for all sufficiently great values of m and n; then

$$\mathfrak{M}(\{f_n\}, A) = \lim_{m\to\infty,\, n\to\infty} \sup \int_A |f_m(x) - f_n(x)|^2 \, dx$$

may be called the *mean square oscillation* of $\{f_n(x)\}$ over A.

For the purpose of relating mean square oscillation to the well known concept of convergence in the mean, we shall give some lemmas.

LEMMA 3.1. *In order that a measurable function* $f(x)$ *may be summable,‡ it is necessary and sufficient that* $|f(x)|$ *be summable.§*

The following lemma is easily established.

LEMMA 3.2. *In order that a measurable function* $f(x) = u(x) + iv(x)$, u *and* v *being real, may be of summable square, it is necessary and sufficient that* u *and* v *be of summable square.*

Using this lemma, it can readily be shown that the sums and differences of measurable complex functions of summable square are measurable func-

* W. A. Hurwitz, *American Journal of Mathematics*, Vol. 52 (1930), pp. 611-616.

† A complex function $f(x) = u(x) + iv(x)$, u and v being real, is said to be measurable when u and v are measurable.

‡ A complex function $f(x) = u(x) + iv(x)$ is said to be summable (to the value $\int u dx + i \int v dx$) when u and v are summable.

§ For real functions, this is a standard result. Caratheodory, *Vorlesungen über Reellen Funktionen* (1927), p. 434.

tions of summable square, and that if two measurable complex functions are of summable square, then their product is summable.

By the Riesz-Fischer theorem* if $f_n(x)$ is real, measurable, and of summable square for all (or almost all) values of n and $\mathfrak{M}(\{f_n\}, A) = 0$, there is a measurable function $f(x)$ of summable square, uniquely determined except over a set of measure 0, such that

$$\lim_{n\to\infty} \int_A |f(x) - f_n(x)|^2 \, dx = 0;$$

then $\{f_n(x)\}$ is said to converge in the mean to $f(x)$. We may show that sequences of complex functions have the same property by proving

LEMMA 3.3. *If $f_n(x)$ is measurable and of summable square for almost all values of n and $\mathfrak{M}(\{f_n\}, A) = 0$, then there is a measurable function $f(x)$ of summable square, uniquely determined except over a set of measure 0, such that*

$$\lim_{n\to\infty} \int_A |f(x) - f_n(x)|^2 \, dx = 0.$$

Obviously the condition $f_n(x)$ is measurable and of summable square for almost all n is a sufficient but not a necessary condition for the existence, finite or infinite, of $\mathfrak{M}(\{f_n\}, A)$; it follows at once from the definition of $\mathfrak{M}(\{f_n\}, A)$ and from lemma 3.1 that, for sequences of measurable functions, a necessary and sufficient condition is that $f_m(x) - f_n(x)$ shall be of summable square for all sufficiently great m and n. We shall show that a sequence $\{f_n(x)\}$ of measurable functions converges in the mean to a measurable function whenever its mean square oscillation is zero by proving

LEMMA 3.4. *If $\{f_n(x)\}$ is a sequence of measurable functions and $\mathfrak{M}(\{f_n\}, A) = 0$, then there is a measurable function $f(x)$, uniquely determined except over a set of measure 0, such that $f(x) - f_n(x)$ is of summable square for almost all n and*

$$\lim_{n\to\infty} \int_A |f(x) - f_n(x)|^2 \, dx = 0.$$

Choose an index p such that $f_m(x) - f_n(x)$ is of summable square for $m \geqq p$ and $n \geqq p$, and let $\phi_n(x) = f_n(x) - f_p(x)$. Then $\phi_n(x)$ is measurable and of summable square for $n \geqq p$ and $\phi_m - \phi_n = f_m - f_n$ so that $\mathfrak{M}(\{\phi_n\}, A) = 0$; hence, applying the preceding lemma, there is a measurable function $\phi(x)$ of summable square, uniquely determined except over a set of measure 0, such that

* H. Weyl, *Mathematische Annalen*, Vol. 68 (1910), p. 242.

$$\lim_{n\to\infty} \int_A |\, \phi(x) - \phi_n(x) \,|^2\, dx = 0,$$

or

$$\lim_{n\to\infty} \int_A |\, \phi(x) + f_p(x) - f_n(x) \,|^2\, dx = 0.$$

It is easily seen that $f(x) = \phi(x) + f_p(x)$ is the required function. It may be noted that, since $\phi(x)$ is measurable and of summable square and $f_p(x)$ is measurable, $f(x)$ is or is not of summable square according as f_p is or is not of summable square.

4. *Oscillation when $s_n(x)$ is of Summable Square for All n.* The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of which each element is measurable and of summable square.

THEOREM 4.1. *In order that (T) may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A) \tag{4.11}$$

for every sequence $\{s_n(x)\}$, defined over a set A of measure $m(A) > 0$, such that $s_n(x)$ is measurable and of summable square over A for all n, C_4 *is necessary and sufficient.*

C_4 is necessary; for when C_4 is denied the bounded sequence of constants of lemma 2.1 contradicts (4.11). If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required for (4.11) is automatically satisfied. If $\mathfrak{M}(\{s_n\}. A)$ is finite, let q be any greater number; then there is an index p such that

$$\int_A |\, s_\mu(x) - s_\nu(x) \,|^2\, dx < q \quad \text{for} \quad \mu \geqq p,\ \nu \geqq p. \tag{4.12}$$

It follows from (4.12) and Schwarz's inequality that

$$\left[\left(\int_A |\, s_n(x) \,|^2\, dx\right)^{\frac{1}{2}} - \left(\int_A |\, s_p(x) \,|^2\, dx\right)^{\frac{1}{2}}\right]^2 < q \quad \text{for} \quad n \geqq p;$$

hence there is a constant, say Q, such that

$$\int_A |\, s_n(x) \,|^2\, dx \leqq Q \quad \text{for} \quad n \geqq p. \tag{4.121}$$

Since $s_n(x)$ is of summable square for $n = 1, 2, 3, \cdots, p$ there is a constant, say R, such that

$$\int_A |\, s_n(x) \,|^2\, dx \leqq R \quad \text{for all } n;$$

thus

$$\int_A |\, s_m(x) \,|\, |\, s_n(x) \,|\, dx \leqq R \quad \text{for all } m \text{ and } n. \tag{4.122}$$

Again it follows from (4.12) and Schwarz's inequality that

$$(4.13)\qquad \int_A |s_\mu(x)-s_\nu(x)|\,|s_\xi(x)-s_\eta(x)|\,dx < q \quad \text{for} \quad \mu \geqq p,\ \nu \geqq p,\ \xi \geqq p,\ \eta \geqq p.$$

We may write for $m > p$, $n > p$,

$$\begin{aligned}\sigma_m(x)-\sigma_n(x) &= \sum_{k=1}^{m} a_{mk}s_k(x) - \sum_{k=1}^{n} a_{nk}s_k(x)\\ &= \sum_{k=1}^{p} a_{mk}s_k - \sum_{k=1}^{p} a_{nk}s_k + \sum_{\mu=p+1}^{m} a_{m\mu}s_\mu - \sum_{\nu=p+1}^{n} a_{n\nu}s_\nu,\end{aligned}$$

and obtain the identity

$$\begin{aligned}(4.14)\qquad \sigma_m(x)-\sigma_n(x) &= \sum_{k=1}^{p} a_{mk}s_k - \sum_{k=1}^{p} a_{nk}s_k\\ &+ \left(\sum_{\mu=p+1}^{m} a_{m\mu}s_\mu\right)\left(1-\sum_{\nu=p+1}^{n} a_{n\nu}\right)\\ &- \left(\sum_{\nu=p+1}^{n} a_{n\nu}s_\nu\right)\left(1-\sum_{\mu=p+1}^{m} a_{m\mu}\right) + \sum_{\mu=p+1}^{m}\sum_{\nu=p+1}^{n} a_{m\mu}a_{n\nu}(s_\mu - s_\nu).\end{aligned}$$

Thus

$$\begin{aligned}\mathfrak{M}(\sigma_n, A) &= \limsup_{m\to\infty,\ n\to\infty} \int_A |\sigma_m(x)-\sigma_n(x)|^2\,dx\\ (4.15)\qquad &= \limsup_{m\to\infty,\ n\to\infty} \int_A |F_1+F_2+F_3+F_4+F_5|^2\,dx\\ &\leqq \limsup_{m\to\infty,\ n\to\infty} \int_A (|F_1|+|F_2|+|F_3|+|F_4|+|F_5|)^2\,dx\end{aligned}$$

where F_1, F_2, F_3, F_4, and F_5 are in order the terms of the right member of (4.14). For $m > p$, $n > p$ we may write

$$\begin{aligned}\int_A |F_5|^2\,dx &\leqq \int_A \left[\sum_{\mu=p+1}^{m}\sum_{\nu=p+1}^{n} |a_{m\mu}|\,|a_{n\nu}|\,|s_\mu(x)-s_\nu(x)|\right]^2 dx\\ &\leqq \int_A \sum_{\mu=p+1}^{m}\sum_{\nu=p+1}^{n}\sum_{\xi=p+1}^{m}\sum_{\eta=p+1}^{n} |a_{m\mu}|\,|a_{n\nu}|\,|a_{m\xi}|\,|a_{n\eta}|\,|s_\mu - s_\nu|\,|s_\xi - s\ \ |\,dx;\end{aligned}$$

and using (4.13) we obtain for $m > p$, $n > p$

$$\int_A |F_5|^2\,dx \leqq \sum_{\mu=p+1}^{m}\sum_{\nu=p+1}^{n}\sum_{\xi=p+1}^{m}\sum_{\eta=p+1}^{n} |a_{m\mu}|\,|a_{n\nu}|\,|a_m\ |\,|a_n\eta|\,q \leqq qB_m^2B_n^2$$

where

$$B_n = \sum_{k=1}^{n} |a_{nk}|$$

so that

$$(4.16)\qquad \limsup_{m\to\infty,\ n\to\infty} \int_A |F_5|^2\,dx \leqq \limsup_{m\to\infty,\ n\to\infty} (qB_m^2B_n^2).$$

Employing (4.122) and the regularity of (T), methods similar to that by which (4.16) was obtained suffice to prove that

$$\lim_{m\to\infty,\, n\to\infty} \int_A | F_r |^2 \, dx = 0, \qquad (r = 1, 2, 3, 4). \tag{4.17}$$

The right member of (4.16) being finite by C_1, it follows from (4.16), (4.17), and Schwarz's inequality that

$$\lim_{m\to\infty,\, n\to\infty} \int_A | F_r | \; | F_\rho | \, dx = 0, \tag{4.18}$$

$$(r = 1, 2, \cdots, 5;\ \rho = 1, 2, \cdots, 5, \text{ except when } r = \rho = 5).$$

Using (4.16) and (4.18), we find from (4.15) that

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2). \tag{4.19}$$

Using, for the first time,* C_4 we obtain $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$; and since q is any number greater than $\mathfrak{M}(\{s_n\}, A)$, (4.11) follows and the theorem is proved.

The same proof establishes the three theorems 4.2, 4.3, and 4.4 outlined in the summary of § 8.

THEOREM 4.5. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A, *such that* $s_n(x)$ *is measurable and of summable square over* A *for all* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, *no further conditions need be imposed on* a_{nk}.

To prove this theorem, we may choose an arbitrarily small positive number q and proceed exactly as in the sufficiency proof of theorem 4.1 to obtain 4.19. Thus $\mathfrak{M}(\{\sigma_n\}, A) \leqq qB^4$ where $B = \limsup B_n$; and since B is finite by C_1, qB^4 is arbitrarily small, $\mathfrak{M}(\{\sigma_n\}, A) = 0$ and the theorem is proved.

Using the properties of regular transformations and properties of measurable functions of summable square, we obtain from the preceding theorem the

THEOREM 4.6. *Any regular transformation with a triangular matrix carries any sequence of measurable functions of summable square, which converges in the mean over a given set, into a sequence of measurable functions of summable square which converges in the mean over the set to the same function.*

5. *Oscillation when* $s_n(x)$ *is of Summable Square for Almost All* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of which all sufficiently advanced elements are measurable and of summable square.

* The use of C_4 has been delayed at a slight inconvenience so that the preceding work may furnish the basis of the proofs of later theorems.

THEOREM 5.1. *In order that* (T) *may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A) \tag{5.11}$$

for every sequence $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable and of summable square over* A *for almost all* n, C_4 *and* C_5 *are necessary and sufficient.*

The necessity of C_4 follows from lemma 2.1. To show that C_5 is necessary, we shall suppose that (T) fails to satisfy C_5 and construct an admissible real sequence $\{s_n(x)\}$, bounded above (below) over A for all n, such that $\mathfrak{M}(\{s_n\}, A) = 0$ and such that the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails. From a denial of C_5 it follows that there is a value of k, say λ, and a sequence $\{n_\alpha\}$ of indices such that $\lim n_\alpha = +\infty$ and $a_{n_\alpha,\lambda} \neq 0$ for $\alpha = 1, 2, 3, \cdots$; furthermore owing to C_2 the sequence $\{n_\alpha\}$ may be so chosen that $|a_{n_i,\lambda}| \neq |a_{n_j,\lambda}|$ when $i \neq j$. Let $\{A_\alpha\}$ be a sequence of subsets of A such that no two subsets have points in common and such that $m(A_\alpha) = m(A)/2^{2\alpha}$, $\alpha = 1, 2, 3, \cdots$.* Define the sequence $\{s_n(x)\}$ as follows: $s_n(x) = 0$ over A for $n \neq \lambda$; $s_\lambda(x) = (-1)^h 2^\alpha / |a_{n_\alpha,\lambda}|$ over A_α, $\alpha = 1, 2, 3, \cdots$, and $s_\lambda(x) = 0$ for all remaining points x of A. Evidently $s_n(x)$ is bounded above or below over A for all n according as $h = 1$ or 2 and is measurable and of summable square for almost all n, and $\mathfrak{M}(\{s_n\}, A) = 0$. But for $n_i > \lambda$, $n_j > \lambda$, and x in A_α,

$$|\sigma_{n_i}(x)| = 2^\alpha |a_{n_i,\lambda}| / |a_{n_\alpha,\lambda}| \quad \text{and} \quad |\sigma_{n_j}(x)| = 2^\alpha |a_{n_j,\lambda}| / |a_{n_\alpha,\lambda}|$$

so that

$$|\sigma_{n_i}(x) - \sigma_{n_j}(x)| \geqq 2^\alpha \big|\, |a_{n_i,\lambda}| - |a_{n_j,\lambda}| \,\big| / |a_{n_\alpha,\lambda}| \quad \text{over } A_\alpha$$

and

$$\begin{aligned}\int_{A_\alpha} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2\, dx &\geqq 2^{2\alpha}[\,|a_{n_i,\lambda}| - |a_{n_j,\lambda}|\,]^2 m(A_\alpha)/|a_{n_\alpha,\lambda}|^2 \\ &\geqq m(A)[\,|a_{n_i,\lambda}| - |a_{n_j,\lambda}|\,]^2/|a_{n_\alpha,\lambda}|^2.\end{aligned}$$

From the preceding inequality and C_2 we obtain for $n_i > \lambda$, $n_j > \lambda$ and $i \neq j$

$$\lim_{p\to\infty} \sum_{\alpha=1}^{p} \int_{A_\alpha} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2\, dx = +\infty;$$

hence $\int_A |\sigma_m(x) - \sigma_n(x)|^2\, dx$ does not exist for all sufficiently large values of m and n, $\mathfrak{M}(\{\sigma_n\}, A)$ does not exist,† the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails, and necessity of C_5 is established.

* The possibility of subdividing A in this manner is assured by the fact that the measure of the set of points of A, which lie within a "sphere" with a fixed center and radius r, is a continuous monotonically increasing function of r.

† If one cares to admit $+\infty$ as a value of a Lebesgue integral, and to consider sequences of which elements are $+\infty$, then $\mathfrak{M}(\{\sigma_n\}, A)$ is $+\infty$.

If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number. Choose an index p so great that $s_n(x)$ is measurable and of summable square for $n \geqq p$ and also so great that (4.12) holds and obtain (4.13). We may obtain (4.121) and, using Schwarz's inequality,

$$(5.12) \qquad \int_A |s_m(x)| \; |s_n(x)| \, dx \leqq Q \quad \text{for} \quad m \geqq p, \; n \geqq p.$$

Considering (4.15) we can, owing to C_5, choose an index $N \geqq p$ so great that $F_1 = F_2 = 0$ for $m \geqq N$, $n \geqq N$; hence

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} \int_A (|F_3| + |F_4| + |F_5|)^2 \, dx.$$

Using (4.13), (5.12) and the methods of the proof of theorem 4.1, we find that

$$\limsup_{m\to\infty,\, n\to\infty} \int_A |F_5|^2 \, dx \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

and

$$\lim_{m\to\infty,\, n\to\infty} \int_A |F_r| \; |F_\rho| \, dx = 0 \text{ for } r = 3, 4, 5; \; \rho = 3, 4, 5, \text{ except when } r = \rho = 5$$

and hence that

$$(5.13) \qquad \mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Finally, using C_4, we have $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and the theorem follows. The same proof establishes the two theorems 5.2 and 5.3 of the summary.

THEOREM 5.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_s(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable and of summable square over* A *for almost all* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, C_5 *is necessary and sufficient.*

Necessity is established exactly as in theorem 5.1. The sufficiency proof is a modification of that of theorem 5.1 in the same sense that the proof of theorem 4.5 is a modification of the sufficiency proof of theorem 4.1. The same proof establishes theorems 5.5 and 5.6 of the summary.

6. *Oscillation when* $s_m(x) - s_n(x)$ *is of Summable Square for All* m *and* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of measurable functions such that the difference of any two functions of the sequence is of summable square.

THEOREM 6.1. *In order that* (T) *may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, defined over a set A of measure $m(A) > 0$, such that $s_n(x)$ is measurable for all n and $s_m(x) - s_n(x)$ is of summable square for all m and n, C_4 *and* C_6 *are necessary and sufficient.*

The necessity of C_4 follows from lemma 2.1. To show that C_6 is necessary, we shall suppose that (T) fails to satisfy C_6 and construct an admissible real sequence $\{s_n(x)\}$, bounded above (below) over A for all n, such that $\mathfrak{M}(\{s_n\}, A) = 0$ and the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails. From a denial of C_6 it follows that there is a sequence $\{n_a\}$ of indices such that

$$\lim_{a\to\infty} n_a = +\infty \text{ and } \left| \sum_{k=1}^{n_a} a_{n_a,k} - 1 \right| \neq 0 \qquad \text{for} \quad \alpha = 1, 2, 3, \cdots;$$

furthermore owing to C_3 the sequence $\{n_a\}$ may be so chosen that

$$\left| \sum_{k=1}^{n_i} a_{n_ik} - 1 \right| \neq \left| \sum_{k=1}^{n_j} a_{n_jk} - 1 \right| \qquad \text{when} \quad i \neq j.$$

Let $\{A_a\}$ be a sequence of subsets of A such that no two subsets have points in common and such that $m(A_a) = m(A)/2^{2a}$ for $\alpha = 1, 2, 3, \cdots$. Let a function $s(x)$ be defined as follows:

$$s(x) = (-1)^h 2^a / \left| 1 - \sum_{k=1}^{n_a} a_{n_ak} \right| \text{ over } A_a, \qquad \alpha = 1, 2, 3, \cdots,$$

and $s(x) = 0$ for all other x in A; and let $s_n(x) = s(x)$ for all n. Evidently $s_n(x)$ is measurable over A for all n and is bounded above or below over A for all n according as $h = 1$ or 2, $s_m(x) - s_n(x)$ is of summable square for all m and n, and $\mathfrak{M}(\{s_n\}, A) = 0$. But for x in A

$$|\sigma_{n_i}(x) - \sigma_{n_j}(x)| \geqq |s(x)| \left| \left| 1 - \sum_{k=1}^{n_i} a_{n_ik} \right| - \left| 1 - \sum_{k=1}^{n_j} a_{n_jk} \right| \right|$$

and hence for x in A_a

$$|\sigma_{n_i}(x) - \sigma_{n_j}(x)| \geqq 2^a \left| \left| 1 - \sum_{k=1}^{n_i} a_{n_ik} \right| - \left| 1 - \sum_{k=1}^{n_j} a_{n_jk} \right| \right| / \left| 1 - \sum_{k=1}^{n_a} a_{n_ak} \right|$$

and

$$\int_{A_a} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2 \, dx \geqq m(A) \left[\left|1 - \sum_{k=1}^{n_i} a_{n_ik}\right| - \left|1 - \sum_{k=1}^{n_j} a_{n_jk}\right| \right]^2 / \left|1 - \sum_{k=1}^{n_a} a_{n_ak}\right|^2$$

From the preceding inequality and C_3 we obtain for $i \neq j$

$$\lim_{p\to\infty} \sum_{a=1}^{p} \int_{A_a} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2 \, dx = +\infty,$$

and it follows as in the proof of theorem 5.1 that the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails and necessity is established.

If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number and choose an index p so great that (4.12) and hence (4.13) hold and also so great that

$$\sum_{k=1}^{n} a_{nk} = 1 \quad \text{for} \quad n \geqq p. \tag{6.11}$$

Choose a constant $R > q$ such that

$$\int_A |s_\mu(x) - s_\nu(x)|^2 < R \quad \text{for} \quad \mu \leqq p,\ \nu \leqq p;$$

then

$$\int_A |s_\mu(x) - s_\nu(x)|\ |s_\xi(x) - s_\eta(x)|\, dx < R \qquad \text{for} \quad \mu \leqq p,\ \nu \leqq p,\ \xi \leqq p,\ \eta \leqq p. \tag{6.12}$$

If $\mu \geqq p$ and $\nu \leqq p$, then

$$|s_\mu(x) - s_\nu(x)|^2 \leqq 2\, |s_\mu(x) - s_p(x)|^2 + 2\, |s_p(x) - s_\nu(x)|^2$$

so that

$$\int_A |s_\mu(x) - s_\nu(x)|^2\, dx < 2q + 2R < 4R;$$

therefore

$$\int_A |s_\mu(x) - s_\nu(x)|\ |s_\xi(x) - s_\eta(x)|\, dx < 4R \qquad \text{for.} \quad \mu \geqq p,\ \xi \geqq p,\ \nu \leqq p,\ \eta \leqq p. \tag{6.13}$$

Using (6.11) we obtain for $m > p$, $n > p$

$$\begin{aligned}
\sigma_m(x) - \sigma_n(x) &= \left(\sum_{\mu=1}^{m} a_{m\mu} s_\mu(x)\right) \left(\sum_{\nu=1}^{n} a_{n\nu}\right) - \left(\sum_{\nu=1}^{n} a_{n\nu} s_\nu(x)\right) \left(\sum_{\mu=1}^{m} a_{m\mu}\right) \\
&= \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) \\
&= \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) + \sum_{\mu=p+1}^{m} \sum_{\nu=1}^{p} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) \\
&\quad + \sum_{\mu=1}^{p} \sum_{\nu=p+1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) + \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu)
\end{aligned} \tag{6.14}$$

so that

$$\int_A |\sigma_m(x) - \sigma_n(x)|^2\, dx \leqq \int_A (|\Phi_1| + |\Phi_2| + |\Phi_3| + |\Phi_4|)^2\, dx \tag{6.15}$$

where the Φ's are in order the four terms of the right member of (6.14). We may note that $\Phi_4 \equiv F_5$ and hence, owing to (4.13) and (4.16), write

$$\limsup_{m\to\infty,\ n\to\infty} \int_A |\Phi_4|^2\, dx \leqq \limsup_{m\to\infty,\ n\to\infty} (q B_m^2 B_n^2). \tag{6.16}$$

Employing (6.12), (6.13) and the regularity of (T), the method by which (4.16) was obtained suffices to prove that

$$\limsup_{m\to\infty,\ n\to\infty} \int_A |\Phi_r|^2\, dx = 0 \qquad (r = 1, 2, 3). \tag{6.17}$$

Using (6.16), (6.17), and Schwarz's inequality, we find from (6.15) that

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Making use, for the first time, of C_4 we see that $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and sufficiency follows. The same proof establishes theorems 6.2 and 6.3.

THEOREM 6.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable for all* n *and* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, C_6 *is necessary and sufficient.*

Necessity is established as in theorem 6.1; the sufficiency proof is a modification of that of theorem 6.1. The same proof establishes theorems 6.5 and 6.6.

7. *Oscillation when* $s_m(x) - s_n(x)$ *is of Summable Square for All Sufficiently Great* m *and* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of the most general sequences of measurable functions for which mean square oscillation is defined.

THEOREM 7.1. *In order that* (T) *may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable for all* n *and* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n, C_4, C_5, *and* C_6 *are necessary and sufficient.*

The necessity of each condition follows at once from a preceding theorem. If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number. Choose an index p such that $s_m(x) - s_n(x)$ is of summable square for $m \geqq p$, $n \geqq p$, and such that (4.12) and hence (4.13) hold. Due to C_5 and C_6 we can choose an index $N \geqq p$ such that $a_{nk} = 0$, $k = 1, 2, 3, \cdots, p$ for $n > N$ and $\sum_{k=p+1}^{n} a_{nk} = 1$ for $n > N$. Then, referring to (6.15), we see that $\Phi_1 = \Phi_2 = \Phi_3 = 0$ over A, and obtain

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} \int_A |F_5|^2\, dx \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Using C_4, we find that $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and the theorem follows. The same proof establishes theorems 7.2 and 7.3.

THEOREM 7.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such*

that $s_n(x)$ *is measurable for all* n *and* $\mathfrak{M}(\{s_n\}, A) = 0$, C_5 *and* C_6 *are necessary and sufficient.*

The necessity of each condition follows at once from a preceding theorem; the sufficiency proof is a modification of that of theorem 7. 1. The same proof establishes theorems 7. 5 and 7. 6.

8. *Summary of Results.* Each of the following is an outline of a theorem giving necessary and sufficient conditions that a complex (or real) regular transformation may be such that $\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$ for every complex (or real) sequence $\{s_n(x)\}$ of a stated character defined over a set A of measure $m(A) > 0$.

THEOREM 4. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for all* n; C_4.

THEOREM 4. 2. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for all* n; C_4.

THEOREM 4. 3. *Complex* (T); *Complex* $s_n(x)$, *bounded and measurable for all* n; C_4.

THEOREM 4. 4. *Real* (T); *Real* $s_n(x)$, *bounded and measurable for all* n; C_4.

THEOREM 5. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 5. 2. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 5. 3. *Real* (T); *Real* $s_n(x)$, *bounded above (below) for all* n *and measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 6. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 6. 2. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$, *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 6. 3. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 7. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

THEOREM 7. 2. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

THEOREM 7.3. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

Each of the following is an outline of a theorem giving necessary and sufficient conditions that a complex (or real) regular transformation may be such that $\mathfrak{M}(\{\sigma_n\}, A) = 0$ for every complex (or real) sequence $s_n(x)$ of a stated character, defined over a set A of measure $m(A) > 0$, such that $\mathfrak{M}(\{s_n\}, A) = 0$.

THEOREM 4.5. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for all* n; *none.*

THEOREM 5.4. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for almost all* n; C_5.

THEOREM 5.5. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for almost all* n; C_5.

THEOREM 5.6. *Real* (T); *Real* $s_n(x)$, *bounded above (below) for all* n *and measurable and of summable square for almost all* n; C_5.

THEOREM 6.4. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 6.5. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 6.6. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 7.4. *Complex* (T); *Complex* $s_n(x)$ *measurable for all* n; C_5 *and* C_6.

THEOREM 7.5. *Real* (T); *Real* $s_n(x)$, *measurable for all* n; C_5 *and* C_6.

THEOREM 7.6. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n; C_5 *and* C_6.

9. *Conclusion.* In this paper we have considered in turn the four conditions of which any one might naturally be selected as sufficient for the existence (finite or infinite) of mean square oscillations of sequences. In each case we have found necessary and sufficient conditions that (T) shall not increase mean square oscillations, and that (T) shall preserve convergence in the mean. Thus eight groups of theorems have been obtained. It is a curious fact that these eight groups of theorems should involve as necessary and sufficient conditions the eight possible combinations of none, some or all of the conditions C_4, C_5, and C_6.

CORNELL UNIVERSITY,
ITHACA, NEW YORK.

TACTICAL CONFIGURATIONS OF RANK TWO.

By R. D. Carmichael.

1. *Definition and Immediate Examples.* In accordance with the terminology of E. H. Moore * I use the term *tactical configuration of rank two* to denote a combination of l elements into m sets, each set consisting of λ distinct elements and each element occurring in μ distinct sets: it is to be understood that order of sets and order within a set are both immaterial. For such a configuration I use the symbol $\Delta^{\lambda,\mu}_{l,m}$ employed by Coble.† It is obvious that $l\mu = \lambda m$.

In the configurations which we shall usually consider there is a certain symmetry in the rôle played by the l elements and the m sets. We may consider the m sets themselves as elements. The μ sets which contain a given symbol may be thought of as a combination of sets which define that symbol, provided that there is no additional symbol common to these μ sets. From this point of view the configuration gives rise to a new configuration having the symbol $\Delta^{\mu,\lambda}_{m,l}$. The two configurations $\Delta^{\lambda,\mu}_{l,m}$ and $\Delta^{\mu,\lambda}_{m,l}$ are therefore essentially the same when they satisfy the restrictive condition just named. They may be called *associated* configurations.

With each of the m sets of λ elements each we may associate the complementary set of $l-\lambda$ elements, thus forming m sets each of which contains $l-\lambda$ elements; in these sets each element occurs $m-\mu$ times. Thus we have what Coble (*l. c.*, p. 2) calls the configuration *complementary* to $\Delta^{\lambda,\mu}_{l,m}$. Its symbol is $\Delta^{l-\lambda,m-\mu}_{l,m}$.

In the course of the memoir it will become apparent that tactical configurations of the sort defined are important in the theory of permutation groups. Coble (*l. c.*) has found them of essential use in constructing poristic forms in connection with the study of geometrical configurations similar to and including those associated with the Poncelet polygons which arise in the theory of conic sections. He points out (*l. c.*, p. 6) that the same tactical problem appears in the formation of irrational (i. e., non-symmetric) invariants of a set of l points in an $S_{\lambda-1}$ of weight m and degree μ. Hence we see that this tactical problem plays an important role in three widely separated fields each of great interest in itself. To construct such tactical configurations is sometimes a difficult problem. Concerning the difficulty

* *Mathematische Annalen*, Vol. 50 (1898), pp. 226-227 (ftn.).

† *American Journal of Mathematics*, Vol. 43 (1921), pp. 1-19.

Coble says (*l. c.*, p. 6): "Indeed the complications of this tactical problem are the most serious bar to any general discussion of the porisms. Certain series of cases may be treated as a class. . . . As a rule however each case presents its own peculiarities."

Emch * has also recently employed these tactical configurations in connection with several geometric problems.

In this section we shall exhibit a few classes of tactical configurations which are immediately available. In several of the following sections we shall show how to use the finite geometries for the systematic construction of several infinite classes of these configurations. We shall also give a brief account of quadruple systems similar to the usual theory of triple systems. In addition we shall treat briefly certain other remarkable special cases and in particular configurations associated with the Mathieu groups and affording a ready means of constructing these groups.

The finite geometries $PG(k, p^n)$, $k > 1$, furnish at once a certain infinite class of these tactical configurations. The points of the geometry constitute the l elements and the lines of the geometry constitute the m classes. In a similar way from the Euclidean geometry $EG(k, p^n)$, $k > 1$, we may obtain at once certain other configurations of rank two. Since the configurations thus defined are of great importance in the theory of finite groups it is apparent that the general theory of these configurations must have a wide use in this chapter of algebra.

The configuration arising from $EG(k, p^n)$ when $k = 2$ and $p^n = 2$ has the symbol $\Delta^{2,3}_{4,6}$. This particular configuration belongs not only to the infinite class from which it has been taken but also to another; it consists of four things taken in pairs; since the number of pairs is six it follows that all possible pairs appear. In general, one can form $\frac{1}{2}n(n-1)$ pairs from n things, each element occurring in $n-1$ pairs. Thus we have a configuration

$$\Delta^{2,n-1}_{n,n(n-1)/2}.$$

More generally, let us form from n given elements all the sets consisting each of a combination of k distinct elements, k being less than n. Thus we have a configuration $\Delta^{\lambda,\mu}_{l,m}$ where

$$l = n,\ \lambda = k,\ m = \frac{n(n-1)\cdots(n-k+1)}{k!},\ \mu = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!}.$$

* See *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 25-42 and *Journal für Mathematik*, Vol. 162 (1930), pp. 238-255.

We may also form configurations by grouping the points of $PG(k, p^n)$ into the sets formed by the subspaces of a given number s of dimensions where $0 < s < k$. In a similar manner configurations may be formed from $EG(k, p^n)$.

In the case when $p^n = 2$ and $k > 1$ we have in $EG(k, 2)$ the number 2^k of points. Any three points of $EG(k, 2)$ determine a plane, and this plane contains just one additional point of $EG(k, 2)$. Moreover, any three of the points in such a quadruple uniquely determines the quadruple itself. Hence the 2^k points in $EG(k, 2)$ may be taken in 4's, in the way indicated, so that any given triple of these 2^k points occurs in one and just one of the named quadruples. This tactical configuration is therefore a quadruple system. If it is denoted by $\Delta_{l,m}^{\lambda,\mu}$ then we have

$$l = 2^k, \quad \lambda = 4, \quad \mu = (2^k - 1)(2^{k-1} - 1)/3, \quad m = 2^{k-2}(2^k - 1)(2^{k-1} - 1)/3.$$

Thus for $k = 2$ we have just one quadruple—a trivial case. For $k = 3$ we have 14 quadruples of 8 things, each triple occurring just once. For $k = 4$ we have 140 quadruples of 16 things. In the general case, the named quadruple system is left invariant by the collineation group in $EG(k, 2)$ and by no larger permutation group on its elements, as one may show without difficulty.

2. *Dual Configurations.* Owing to the principle of duality in the $PG(k, p^n)$ the principal configurations already formed from it have a certain dual character. It is of interest to construct certain other dual configurations from the special case of $PG(2, p^n)$. From $PG(2, p^n)$ omit a line and all the points on that line, also omit one additional point and all the lines on that point. Then we have left $p^{2n} - 1$ points and $p^{2n} - 1$ lines; moreover, there are p^n retained points on a retained line and also p^n retained lines on a retained point. Considering the points as elements and the lines as sets of elements we are thus led to a configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = m = p^{2n} - 1, \qquad \lambda = \mu = p^n.$$

For $p^n = 2, 3$ these configurations are $\Delta_{3,3}^{2,2}$, $\Delta_{8,8}^{3,3}$. The latter configuration may be represented explicitly by the sets of symbols

136, 147, 158, 238, 245, 267, 357, 468,

where the eight digits are the elements and the triples are those indicated.

This particular configuration was communicated to me orally by A. B. Coble, having been obtained by him in connection with a geometrical investiagtion: it was from an analysis of this particular configuration that I was led to the infinite class of configurations just described.

Let us next consider the configuration obtained from $PG(2, p^n)$ by omitting all the points on a line and all the lines on one point of this line. There remain p^{2n} points and p^{2n} lines; each retained line contains p^n of the retained points, and each retained point is on p^n of the retained lines. Thus we have a $\Delta_{l,m}^{\lambda,\mu}$ with $l = m = p^{2n}$, $\lambda = \mu = p^n$. For $p^n = 3$ we have a $\Delta_{9,9}^{3,3}$ of considerable interest.

Let us now consider the dual configuration formed from $PG(2, p^n)$ in the following manner. We omit all the points on two lines, leaving $p^{2n} - p^n$ points. We also omit all the lines on the common point of these first two lines and also all the lines on one other point of one of these lines. We have thus omitted two lines of points, these two lines having a common point, and also two bundles of lines, these two bundles having a common line. The omitted configuration is dual in character. Hence the points which remain form a set that is dual in character. Grouping these remaining points in collinear sets on the retained lines we have a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = m = p^{2n} - p^n, \qquad \lambda = \mu = p^n - 1.$$

For $p^n = 4$ we have an interesting $\Delta_{12,12}^{3,3}$ which may be represented by the following twelve triples of twelve things:

$$\begin{array}{cccccc} CLO, & CQT, & CMS, & DKT, & DMP, & DOU, \\ EPS, & ELU, & EKQ, & MQU, & KQS, & LPT. \end{array}$$

Let us next omit from $PG(2, p^n)$ three non-collinear points and all the points on the three lines determined by pairs of them and also all the lines on each of these three points. There remain of the $PG(2, p^n)$ the same number of lines and of points, namely, $(p^n - 1)^2$; they fall into sets of $p^n - 2$ each on $p^n - 2$ lines thus giving a configuration $\Delta_{\lambda,\mu}^{l,m}$ with

$$l = m = (p^n - 1)^2, \qquad \lambda = \mu = p^n - 2.$$

The foregoing general configuration may be readily generalized. Let P_r denote a polygon in $PG(2, p^n)$ whose vertices are $A_1, A_2, \cdots, A_r$ and whose sides are $A_1A_2, A_2A_3, A_3A_4, \cdots, A_{r-1}A_r, A_rA_1$. Omit all the points on these r lines and also all the lines on these r vertices. The number of omitted points [omitted lines] is rp^n. Each of the retained lines holds $p^n - r + 1$ of the retained points while each of the retained points is on

the same number of retained lines. We suppose that r is such that $p^n - r + 1$ is an integer s greater than unity and less than $p^n - 1$ (in order to avoid trivial cases). Then we have a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ with

$$\lambda = \mu = s = p^n - r + 1, \qquad l = m = sp^n + 1.$$

Let us now consider the $PG(2, 2^n)$, $n > 2$. Let Q be any complete quadrangle in this plane. Since its diagonal points are collinear it consists of seven points and seven lines. Omitting all the lines on these seven points and all the points on these seven lines, we have from the retained points and lines a $\Delta_{l,m}^{\lambda,\mu}$ for which one readily shows that

$$l = m = 2^{2n} - 6 \cdot 2^n + 8, \qquad \lambda = \mu = 2^n - 6.$$

Several of the configurations which we have obtained from $PG(2, p^n)$ are readily extended to $PG(k, p^n)$ for $k > 1$. Two of these generalizations will now be given.

Let us omit from $PG(k, p^n)$, $k > 1$, one particular $(k-1)$-dimensional subspace $PG(k-1, p^n)$ together with all its points. There remains an $EG(k, p^n)$ containing p^{kn} points. Omit one of these points and each of the $(k-1)$-dimensional subspaces $PG(k-1, p^n)$ which contain this omitted point. The number of $(k-1)$-dimensional subspaces retained is then $p^{kn} - 1$; in each of these we take only those points which are in the named $EG(k, p^n)$. By means of these subspaces we have thus grouped the $p^{kn} - 1$ retained points into $p^{kn} - 1$ sets, each set containing $p^{(k-1)n}$ points and each point appearing in $p^{(k-1)n}$ sets. Thus we are led to a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = m = p^{kn} - 1, \qquad \lambda = \mu = p^{(k-1)n}.$$

In constructing another configuration Δ, let us omit from the $PG(k, p^n)$, $k > 1$, one particular $(k-1)$ space $PG(k-1, p^n)$ together with its points, thus forming an $EG(k, p^n)$ of p^{kn} points. Omit also all $(k-1)$-spaces on a particular one of the points already omitted, retaining the remaining p^{kn} $(k-1)$-spaces. Each of these remaining $(k-1)$-spaces has $p^{(k-1)n}$ points of the $EG(k, p^n)$ on it, while each of these points is on $p^{(k-1)n}$ such spaces. Thus we are led to a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where $l = m = p^{kn}$, $\lambda = \mu = p^{(k-1)n}$. Of particular interest are the cases $p^n = 2$, $k = 3$; $p^n = 2$, $k = 4$; $p^n = 3$; $k = 3$: these lead to configurations with the respective symbols

$$\Delta_{8,8}^{4,4}, \qquad \Delta_{16,16}^{8,8}, \qquad \Delta_{27,27}^{9,9}.$$

It is possible to construct various other dual configurations generalizing several of those here given. In particular, configurations may be constructed in which the elements are lines or other subspaces. But these seem to be of less interest than those already given.

By means of the collineation groups in the finite geometries one may readily determine the permutation groups which are characterized as being the largest permutation groups leaving invariant the configurations described in this section. But the work will not be carried out here.

3. *Other Immediate Examples.* From a cycle $a_1a_2 \cdots a_n$ of n elements we may select cyclically the set $a_1a_2 \cdots a_k$, $a_2a_3, \cdots a_{k+1}, \cdots, a_na_1a_2 \cdots a_{k-1}$ thus obtaining a configuration $\Delta_{n,n}^{k,k}$.

Let us next take two sets of n things each, say, $a_1, a_2, \cdots, a_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$. Each element in one set may be paired with each element in the other set, giving rise to n^2 pairs of the $2n$ elements. Thus we have a $\Delta_{2n,n^2}^{2,n}$. Again, each element in one of the two sets may be paired with every other element in the same set: thus we have a $\Delta_{2n,n(n-1)}^{2,n-1}$. Again, we may make the pairs from each one of the sets run in cyclical order, thus obtaining a configuration with the symbol $\Delta_{2n,2n}^{2,2}$.

The three configurations of the foregoing paragraph are capable of a ready generalization. Generalizing the first of them we have a configuration with the symbol $\Delta_{ln,n^l}^{l,n^{l-1}}$ obtained from l sets of n elements each by forming all the possible combinations of l elements each gotten by taking one element from each of the l sets. One may similarly generalize the other two configurations in the preceding paragraph. Moreover, various other similar configurations are readily formed.

Now let us take l sets of n things each, l being greater than 2. Let these l sets be arranged in cyclical order. Form pairs by taking each element in one set with each element in the set which follows it in cyclical order. Thus we form ln^2 pairs from the ln elements, using each element $2n$ times. This gives rise to a configuration $\Delta_{ln,ln^2}^{2,2n}$, $l > 2$, due to Coble. This configuration is capable of generalization in the following manner. Let us consider l sets of n things each where $l > \lambda$, these l sets being arranged in cyclical order; and let us form combinations of λ elements each, such combinations being formed from λ consecutive sets from the l sets in their fixed cyclical order by taking one element from each of the λ sets in all possible ways. Thus we have ln elements formed into sets of λ elements each, the number of sets being ln^λ and each element appearing in $\lambda n^{\lambda-1}$ sets. This gives rise to a configuration

$$\Delta^{\lambda,\lambda n\lambda-1}_{ln,ln\lambda}, \qquad l > \lambda.$$

For the case in which $l = n + 1$ and $\lambda = n$ we have $\Delta^{n,\quad n^n}_{n(n+1),n^n(n+1)}$. For $n = 2$ this becomes $\Delta^{2,4}_{6,12}$; and this configuration consists of all the pairs of the six elements involved except the three pairs from which the configuration was formed.

Other configurations are readily formed by various modifications of the methods employed in this section.

4. *Configurations Associated with Coble's Box Porism.* In the 3-space $PG(3, p^n)$ there are $p^n + 1$ points on a line l and $p^n + 1$ planes on the same line. Let us take p^n of these planes and a point P not on any of these p^n planes (and hence on the remaining plane through l). Let Q be any point on l. In addition to the p^n planes already retained, keep also the p^{2n} planes which are not on the line PQ. We thus retain $p^{2n} + p^n$ planes. Retain the p^{3n} points which are not on the plane through P and l; these points form an $EG(3, p^n)$. The points retained in a given one of the p^n planes first selected and the lines in which that plane is cut by the retained planes on P form a configuration

$$\Delta^{p^n,p^n}_{p^{2n},p^{2n}}.$$

Hence the p^{3n} retained points appear in sets of p^n each on the $p^{2n} + p^n$ retained planes; moreover, each of the retained points appears on one of the retained planes through l and on just p^n of the retained planes on P. We are thus led to a configuration having the symbol

$$\Delta^{p^{2n},p^n+1}_{p^{3n},p^{2n}+p^n}.$$

When $p^n = 2$ we have here a configuration with the symbol $\Delta^{4,3}_{8,6}$. It is based on the $PG(3, 2)$. It may be shown that this leads to a configuration equivalent to that defined by the following scheme:

DEFG, LMNO, DELM, FGNO, DGLO, EFMN.

If these six quadruples in the order written are numbered $1, 2, \cdots, 6$, then the eight letters named in them are determined by triples of digits according to the following correspondence:

135	136	145	146	235	236	245	246
D	*E*	*G*	*F*	*L*	*M*	*O*	*N*

These eight triples of six elements form the configuration $\Delta_{6,8}^{8,4}$ belonging to the box porism of Coble (*l. c.*, p. 15). The latter is therefore exhibited as belonging to an infinite class of configurations: the class was suggested by this example.

Another infinite class of configurations having the same symbols as the foregoing may be constructed in the following manner. From $PG(3, p^n)$ form the corresponding $EG(3, p^n)$ by omitting a plane with its points. Let P be a point on this plane; then there are $p^{2n} + p^n$ additional planes on P; these are to be retained. The p^{3n} points of the $EG(3, p^n)$ fall on these planes, p^{2n} points on each plane thus considered. Moreover a given one of these points is on each of the planes containing the line joining this point to P, and hence it is on $p^n + 1$ of the retained planes. We are thus led to another general configuration with the same symbol as that which appears in the preceding paragraph. These configurations, however, have a certain degenerate character so that they do not give rise to associated configurations by means of the method just employed in the preceding paragraph.

5. *Certain Additional Configurations.* From the $PG(k, p^n)$, $k > 2$, let us omit a line of points and also all the lines on each of these points. The number of points remaining is l where

$$l = p^{2n} + p^{3n} + \cdots + p^{kn}.$$

By computing the number of omitted lines it is readily shown that the number m of retained lines is

$$m = \frac{(p^{(k+1)n} - 1)(p^{kn} - 1)}{(p^{2n} - 1)(p^n - 1)} - (1 + p^n)(p^n + p^{2n} + \cdots + p^{(k-1)n}) - 1.$$

The retained points fall λ at a time on the retained lines, where $\lambda = 1 + p^n$, each point appearing on μ of the lines, where

$$\mu = p^{2n} + p^{3n} + \cdots + p^{(k-1)n}.$$

This gives rise to a $\Delta_{l,m}^{\lambda,\mu}$ where l, m, λ, μ have the values given.

To form another configuration let us omit from $PG(3, p^n)$ the points on two non-intersecting lines and all the lines through these points. This leaves $p^n(p^{2n} - 1)(p^n - 1)$ lines of the $PG(3, p^n)$. Each of these contains $p^n + 1$ points of the $PG(3, p^n)$; and each of these retained points is on $p^{2n} - p^n$ lines. Hence the retained points and lines yield a configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l=(p^n+1)(p^{2n}-1), \qquad m=p^n(p^n-1)(p^{2n}-1),$$
$$\lambda=p^n+1, \qquad \mu=p^n(p^n-1).$$

Again, from the $PG(2k+1, p^n)$ let us omit the points of a k-dimensional subspace S_k and also all the $(2k)$-spaces containing S_k. We thus retain l points and l $(2k)$-spaces where

$$l=p^{(2k+1)n}+p^{2kn}+\cdots+p^{(k+1)n}.$$

Each of the retained $(2k)$-spaces has λ retained points where

$$\lambda=p^{2kn}+p^{(2k-1)n}+\cdots+p^{kn},$$

and each of the retained points lies on λ of the retained $(2k)$-spaces. Thus we are led to a configuration $\Delta^{\lambda,\lambda}_{l,l}$ where λ and l have the values just given.

6. *Subgeometries and the Complementary Sets.* Let ν be any proper factor of n. Then in $PG(k, p^n)$ there is included the geometry $PG(k, p^\nu)$, namely, those points of $PG(k, p^n)$ whose coördinates may be taken as marks of the $GF[p^\nu]$ included in the $GF[p^n]$. We shall denote by $C(k, p^n, p^\nu)$ the complementary set of points, namely, the points of $PG(k, p^n)$ which are not contained in the included $PG(k, p^\nu)$. The number l of points in $C(k, p^n, p^\nu)$ is

$$l=(p^{kn}-p^{k\nu})+(p^{(k-1)n}-p^{(k-1)\nu})+\cdots+(p^n-p^\nu).$$

If a line in $PG(k, p^n)$ contains two points of $PG(k, p^\nu)$ it contains all the points of a line in $PG(k, p^\nu)$. Hence the lines of $PG(k, p^n)$ may be separated into three classes: the first class consists of those lines each of which contains a whole line of the $PG(k, p^\nu)$; the second class consists of those lines each of which contains just one point of the $PG(k, p^\nu)$; the third class consists of those lines containing no point of the $PG(k, p^\nu)$. The numbers of lines in these three classes are readily shown to be respectively

$$\frac{(p^{(k+1)\nu}-1)(p^{k\nu}-1)}{(p^{2\nu}-1)(p^\nu-1)}, \quad \left(\frac{p^{kn}-1}{p^n-1}-\frac{p^{k\nu}-1}{p^\nu-1}\right)\frac{p^{(k+1)\nu}-1}{p^\nu-1},$$
$$\frac{(p^{(k+1)n}-1)(p^{kn}-1)}{(p^{2n}-1)(p^n-1)}-\frac{p^{(k+1)\nu}-1}{p^\nu-1}\left(\frac{p^{kn}-1}{p^n-1}-\frac{p^\nu(p^{k\nu}-1)}{p^{2\nu}-1}\right).$$

It is not difficult to show that the third class is the null class when and only when $k=2$ and $n=2\nu$.

With these classes we readily construct tactical configurations as follows.

Let us consider the second class of lines in the case when $k=2$ and $n=2\nu$. Each of the $p^{2\nu}+p^{\nu}+1$ lines of the $PG(2,p^{\nu})$, when extended to a line of $PG(2,p^{2\nu})$, contains just $p^{2\nu}-p^{\nu}$ points of $C(2,p^{2\nu},p^{\nu})$; and no point P of $C(2,p^{2\nu},p^{\nu})$ occurs on two such extended lines. Hence each of the $(p^{2\nu}-p^{\nu})(p^{2\nu}+p^{\nu}+1)$ points of $C(2,p^{2\nu},p^{\nu})$ occurs on one and just one line which contains a line of $PG(2,p^{\nu})$. Hence each point P of $C(2,p^{2\nu},p^{\nu})$ lies on just $p^{2\nu}$ lines of the second class, this being the number of lines joining P to points of $PG(2,p^{\nu})$ other than the line of $PG(2,p^{\nu})$ on the extension of which P lies. Moreover, each line of the second class contains just $p^{2\nu}$ points of $C(2,p^{2\nu},p^{\nu})$. Hence the $(p^{2\nu}-p^{\nu})(p^{2\nu}+p^{\nu}+1)$ points of $C(2,p^{2\nu},p^{\nu})$ lie $p^{2\nu}$ at a time on the $(p^{2\nu}-p^{\nu})(p^{2\nu}+p^{\nu}+1)$ lines of the second class and each point is on just $p^{2\nu}$ of these lines. This gives rise to a tactical configuration $\Delta^{\lambda,\mu}_{l,m}$, where

$$\lambda=\mu=p^{2\nu},\qquad l=m=(p^{2\nu}-p^{\nu})(p^{2\nu}+p^{\nu}+1).$$

In the case when $p^{\nu}=2$ this gives a $\Delta^{4,4}_{14,14}$; this can be represented explicitly in the following form in which the fourteen columns denote the fourteen sets of four points each (each point occurring in four sets):

E	D	I	H	E	D	R	D	I	E	N	E	D	H
M	L	M	L	I	H	S	M	L	H	O	L	I	M
O	Q	Q	P	P	O	T	P	O	Q	P	N	N	N
T	S	U	T	S	U	U	R	R	R	Q	U	T	S

From the last foregoing general configuration a certain reduced configuration is readily obtained. Let us omit from $PG(2,p^{\nu})$ one of its lines and at the same time omit from $PG(2,p^{2\nu})$ the line L which has $p^{\nu}+1$ points in common with the omitted line $PG(2,p^{\nu})$. This line contains $p^{2\nu}-p^{\nu}$ points of $C(2,p^{2\nu},p^{\nu})$. The remaining points of $C(2,p^{2\nu},p^{\nu})$ are $(p^{2\nu}-p^{\nu})(p^{2\nu}+p^{\nu})$ in number. These points fall $p^{2\nu}$ at a time on those lines of the second class other than the lines containing each a point of L which is in the set $C(2,p^{2\nu},p^{\nu})$. These latter lines are $(p^{2\nu}-p^{\nu})p^{2\nu}$ in number, since each of the excluded $p^{2\nu}-p^{\nu}$ points is on just $p^{2\nu}$ lines of the second class and no two of them are on the same line of the second class. Excluding these lines and retaining the others of the second class we have $p^{\nu}(p^{2\nu}-1)$ retained lines. Each of the retained points is on just p^{ν} of the retained lines. Hence we have a tactical configuration $\Delta^{\lambda,\mu}_{l,m}$ where

$$l=p^{2\nu}(p^{2\nu}-1),\quad \lambda=p^{2\nu},\quad \mu=p^{\nu},\quad m=p^{\nu}(p^{2\nu}-1).$$

For $p^\nu = 2$ this is a $\Delta^{4,2}_{12,6}$. The associated configuration $\Delta^{2,4}_{6,12}$ has an obvious generalization to a configuration

$$\Delta^{2,\ 2(n-1)}_{2n,2n(n-1)}$$

consisting of all the pairs of the $2n$ symbols $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ except the pairs α_1, α_2; α_3, α_4; $\cdots$; $\alpha_{2n-1}, \alpha_{2n}$, each α occurring in $2(n-1)$ pairs. And this in turn is capable of an immediate generalization to the case of kn things taken k at a time except for the omission of n sets of k each, the latter sets involving each symbol once and just once.

Let us consider the second class of lines in the case when $k = 2$ and $n = \rho\nu$, $\rho > 2$. The number l of points in $C(2, p^{\rho\nu}, p^\nu)$ and the number m of lines in the second class and the number N of lines in the third class are now respectively:

$$l = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} + p^\nu + 1), \qquad m = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1),$$
$$N = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}).$$

Moreover, each line of the second class contains just $p^{\rho\nu}$ points of $C(2, p^{\rho\nu}, p^\nu)$. We may separate the points of $C(2, p^{\rho\nu}, p^\nu)$ into two subclasses $C_1(2, p^{\rho\nu}, p^\nu)$ and $C_2(2, p^{\rho\nu}, p^\nu)$, those of the subclass C_1 being each on a line of $PG(2, p^{\rho\nu})$ which contains $p^\nu + 1$ points of the $PG(2, p^\nu)$ while the subclass C_2 consists of the remaining points of C. Now these subclasses C_1 and C_2 contain l_1 and l_2 points respectively where

$$l_1 = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1), \qquad l_2 = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}),$$

a result which may be proved as follows. The $PG(2, p^\nu)$ contains $p^{2\nu} + p^\nu + 1$ lines and each of these lines has $p^{\rho\nu} - p^\nu$ points of C_1 while no point of C_1 is on two of these lines, since two such lines have a point of $PG(2, p^\nu)$ in common. Hence l_1 has the value just given; then l_2 is obtained from the formula $l_2 = l - l_1$.

Each point of C_1 is on just $p^{2\nu}$ lines of the second class, since it is on just one line of the first class and this line contains just $p^\nu + 1$ of the $p^{2\nu} + p^\nu + 1$ points of the $PG(2, p^\nu)$; and each point of C_2 is on just $p^{2\nu} + p^\nu + 1$ lines of the second class. But just $p^{\rho\nu} + 1$ lines of $PG(2, p^{\rho\nu})$ pass through any given point of this geometry. Hence each point of C_1 is on just $p^{2\nu} - p^\nu$ lines of the third class, and each point of C_2 is on just $p^{\rho\nu} - p^{2\nu} - p^\nu$ lines of the third class. Every line of the second class contains just as many points of C_1 as there are lines in $PG(2, p^\nu)$ not containing the point which this line of the second class has in common with

$PG(2, p^\nu)$, and this number is $p^{2\nu}$; therefore every line of the second class contains just $p^{\rho\nu} - p^{2\nu}$ points of C_2.

Now we have seen that the l_1 points of C_1 fall, in sets of $p^{2\nu}$ each, on the m lines of the second class, each point of C_1 belonging to just $p^{2\nu}$ lines of the second class. Thus we have a $\Delta_{l,m}^{\lambda,\mu}$ with

$$l = m = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1), \qquad \lambda = \mu = p^{2\nu}.$$

For $\rho = 3$ and $p^\nu = 2$ we have thus a $\Delta_{42,42}^{4,4}$.

Again, the l_2 points of C_2 fall, in sets of $p^{\rho\nu} - p^{2\nu}$ each, on the m lines of the second class, each point of C_2 belonging to just $p^{2\nu} + p^\nu + 1$ lines of the second class. Thus we have a $\Delta_{l,m}^{\lambda,\mu}$ where

$$\lambda = p^{\rho\nu} - p^{2\nu}, \qquad \mu = p^{2\nu} + p^\nu + 1, \qquad l = l_2,$$

and where m and l_2 have the values already given.

Each line of the third class contains just $p^{2\nu} + p^\nu + 1$ points of C_1 since it contains no point of $PG(2, p^\nu)$ and has one and just one point in common with each of the $p^{2\nu} + p^\nu + 1$ lines each of which contains $p^\nu + 1$ points of $PG(2, p^\nu)$. Hence each line of the third class contains also $p^{\rho\nu} - p^{2\nu} - p^\nu$ points of C_2.

From the foregoing results we see that the l_1 points of C_1 fall, in sets of $p^{2\nu} + p^\nu + 1$ each, on the N lines of the third class. This defines a tactical configuration of rank two.

Similarly, we see that the l_2 points of C_2 fall, in sets of $p^{\rho\nu} - p^{2\nu} - p^\nu$ each, on the N lines of the third class, each point of C_2 belonging to just $p^{\rho\nu} - p^{2\nu} - p^\nu$ lines of the third class. Thus we have a $\Delta_{l,l}^{\lambda,\lambda}$, where

$$\lambda = p^{\rho\nu} - p^{2\nu} - p^\nu, \qquad l = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}).$$

It is evident that other configurations may readily be constructed by means of finite geometries of more than two dimensions and the subgeometries contained within them.

7. *Quadruple Systems.* If n elements $x_1, x_2, \cdots, x_n$ can be arranged in quadruples so that each triple $x_\alpha x_\beta x_\gamma$ of distinct elements occurs in one and just one quadruple, then the arrangement so made is called a *quadruple system.* The number n of elements in a quadruple system must be of one of the forms $6m + 2$ and $6m + 4$, as one may readily prove by showing that each of the numbers

$$n(n-1)(n-2)/4 \cdot 3 \cdot 2, \quad (n-1)(n-2)/3 \cdot 2, \quad (n-2)/2,$$

must be an integer: the first of these numbers is the number of quadruples in the system; the second is the number of quadruples containing a given element; while the third is the number of quadruples containing a given pair of elements. The quadruples containing a given element evidently lead to a triple system on the remaining elements.

From a given quadruple system on the n elements $x_1, x_2, \cdots, x_n$ one may form a quadruple system on the $2n$ elements $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n'$ in the following manner. For each quadruple $x_\alpha x_\beta x_\gamma x_\delta$ of the given set form also the quadruple $x_\alpha' x_\beta' x_\gamma' x_\delta'$ and retain $x_\alpha x_\beta x_\gamma x_\delta$. For each quadruple containing the given pair $x_\alpha x_\beta$, as for instance $x_\alpha x_\beta x_\lambda x_\mu$, form also the quadruple $x_\alpha' x_\beta' x_\lambda x_\mu$. Form also the quadruples $x_\alpha x_\beta x_\alpha' x_\beta'$ for every pair (α, β) of the set $1, 2, \cdots, n$. The total number of quadruples thus formed is

$$2(n)(n-1)(n-2)/24 + n(n-1)(n-2)/4 + n(n-1)/2,$$

or

$$2n(2n-1)(2n-2)/24.$$

This is just the required number of quadruples for a quadruple system of $2n$ elements. Therefore the named quadruples form a quadruple system provided that no triple occurs in two quadruples. That this condition is met is readily shown by considering the triples of each of the forms $x_\rho x_\sigma x_\tau$, $x_\rho' x_\sigma' x_\tau'$, $x_\rho x_\sigma x_\tau'$, $x_\rho' x_\sigma' x_\tau$. Hence from a given quadruple system on n elements one may construct (in the manner indicated) a quadruple system on $2n$ elements.

Now $x_1 x_2 x_3 x_4$ forms a (trivial) quadruple system. Applying to it the method of the previous paragraph one obtains a quadruple system on eight elements; and it is easy to show that this is the only quadruple system on eight elements. From the quadruple system on eight elements one may form one on 16 elements; from this, one on 32 elements; and so on. Thus one has quadruple systems on 2^k elements for $k = 3, 4, 5 \cdots$. These are the same as the quadruple systems formed at the end of § 1 by means of the finite geometries.

Now consider the collineation group $C(1, 3^k)$ of the $PG(1, 3^k)$. It has a subgroup consisting of those transformations

$$x' = (\alpha x + \beta)/(\gamma x + \delta),$$

for which $\alpha, \beta, \gamma, \delta$ belong to the Galois field $GF[3]$; this subgroup is of order $4 \cdot 3 \cdot 2$; it permutes among themselves the elements $\infty, 0, 1, 2$; these

elements are left individually fixed by the transformation $x' = x^3$; this transformation and the group of order 24 just mentioned generate a group of order $24k$, each element of which leaves fixed the set ∞, 0, 1, 2. Hence the group $C(1, 3^k)$ transforms this quadruple into $(3^k + 1)3^k(3^k - 1)/24$ quadruples [as does also the projective group $P(1, 3^k)$ of $PG(1, 3^k)$]. Since the group is triply transitive it follows that every triple of the $3^k + 1$ points of $PG(1, 3^k)$ occurs among these quadruples. The quadruples therefore constitute a quadruple system. When $k = 1$ we have a trivial case. When $k = 2$ we have a quadruple system on 10 elements.

From the three preceding paragraphs it follows that quadruple systems of n elements certainly exist for every number n of the form

$$n = (3^k + 1)2^l, \qquad (k = 1, 2, 3, \cdots, \ l = 0, 1, 2, \cdots).$$

The general problem of the existence of quadruple systems of n elements when n is of the form $6m + 2$ or $6m + 4$ appears not to have been solved.

Let us return to the quadruple system on $3^k + 1$ elements already constructed. Those quadruples which contain the element ∞ lead to a triple system on the 3^k elements exclusive of ∞. It may be shown that this is the same as the triple system afforded by the lines of $EG(k, 3)$. Its group is therefore the projective group $EP(k, 3)$, a doubly transitive group of degree 3^k and order

$$3^k(3^k - 1)(3^k - 3)(3^k - 3^2) \cdots (3^k - 3^{k-1}).$$

This triple system may also be constructed (in a manner now obvious) by means of the transformation group $x' = \alpha x + \beta$ in the $GF[3^k]$; and when so constructed it leads at once to the larger doubly transitive group just named—a good example of the way in which configurations often lead from a given multiply transitive group to a larger one containing it.

8. *Configurations Associated with the Mathieu Groups.* The Mathieu groups of degrees 11, 12, 22, 23, 24 (one of each degree) are remarkable for two things: (a) they seem to be the only known simple groups which do not appear among the known infinite classes of simple groups; (b) among them are found the only known four-fold and five-fold transitive groups other than the alternating and symmetric groups. Examples which stand apart in such a way possess a peculiar interest on account of their isolation. It therefore seems worth while to present (without any details) a very direct method for constructing these groups by means of configurations and to indicate some

of their properties which are made manifest by means of these configurations.

The linear fractional group modulo 11 of order $12 \cdot 11 \cdot 5$ is often represented as a doubly transitive group of degree 12 on the symbols ∞, 0, 1, 2, $\cdots$, 10. From the twelve symbols which this transitive group permutes one may select a set of six, namely, ∞, 1, 3, 4, 5, 9, such that the set is transformed into itself by just five elements of this group. The whole group therefore permutes the set of six symbols into 132 such sets. If any five symbols are selected from the twelve they appear in one and just one of these sextuples. The 132 sextuples therefore afford an interesting configuration on 12 symbols which may well be called a sextuple system, in analogy with the terminology employed in the preceding section. The symbol ∞ appears in just 66 of these sextuples, whence it follows readily that these 66 sextuples afford a configuration of 66 quintuples on the set 0, 1, 2, $\cdots$, 10. These may be said to form a quintuple system since each set of four of the symbols appears in one and just one of the quintuples. Any one of the 11 elements occurs in just 30 quintuples from which a quadruple system on 10 elements may be formed by omitting that element. From this in turn the triple system on nine elements may be constructed.

If one seeks the largest permutation group G on the twelve symbols, each element of which leaves invariant the named sextuple system, it is found [illegible] a five-fold transitive group of degree 12 and order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$. Th[illegible]e Mathieu group of degree 12. Its largest subgroup, each element of which leaves one given symbol fixed, is the Mathieu group of degree 11, a fourfold transitive group of order $11 \cdot 10 \cdot 9 \cdot 8$. Moreover it is the group belonging to the quintuple system already named.

From the foregoing considerations it follows also that the Mathieu group of degree 12 contains a subgroup of order $10 \cdot 9 \cdot 8$ each element of which leaves fixed a given one of the 132 sextuples. This subgroup is intransitive, having two transitive constituents each of degree 6. It thus sets up a simple isomorphism of the symmetric group of degree 6 with itself; and the isomorphism so established is an outer isomorphism. This outer isomorphism is therefore an essential element in the structure of the Mathieu group of degree 12.

The linear fractional group modulo 23 of order $24 \cdot 23 \cdot 11$ is often represented as a doubly transitive group of degree 24 on the symbols ∞, 0, 1, 2, $\cdots$, 22. This transitive group contains a subgroup of order 8 each element of which transforms into itself the set ∞, 0, 1, 3, 12, 15, 21, 22 of eight elements, while the whole group transforms this set into $3 \cdot 23 \cdot 11$ sets of eight each. This configuration of octuples has the remarkable property

that any given set of five of the 24 symbols occurs in one and just one of these octuples. The largest permutation group Γ on the 24 symbols, each element of which leaves this configuration invariant, is a five-fold transitive group of degree 24 and order $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$. This is the Mathieu group of degree 24. Its four-fold and three-fold transitive subgroups of degrees 23 and 22 are the Mathieu groups of these degrees. With these two subgroups respectively we may associate (in a manner now obvious) configurations on 23 and 22 letters respectively. The former consists of septuples such that any set of four of the 23 elements occurs in one and just one septuple; the latter consists of sextuples such that any set of three of the 22 elements in it occurs in one and just one sextuple.

The latter set of sextuples on 22 symbols leads readily to 21 quintuples on 21 symbols; it may be shown that these quintuples constitute the lines of the geometry $PG(2, 2^2)$ of 21 points.

The Mathieu group of degree 24 contains a subgroup of index $3 \cdot 23 \cdot 11$ each element of which leaves invariant a given octuple of the previously named configuration of octuples. This subgroup permutes the eight symbols in this octuple according to the alternating group of degree 8; it permutes the remaining 16 symbols according to a triply transitive group of degree 16 and order $16 \cdot 15 \cdot 14 \cdot 12 \cdot 8$; the latter of these two groups is $(16, 1)$ isomorphic with the former. This isomorphism is essential in the structure of the Mathieu group of degree 24. By means of this isomorphism and the known lists of groups of degree not exceeding 8 it is easy to find all the primitive groups of degree 16 contained in the named triply transitive group of degree 16: it turns out that they are 20 in number: these are all the primitive groups of degree 16 except the alternating and symmetric groups of this degree (Miller, *American Journal of Mathematics,* Vol. 20 (1899), pp. 229-241). By means of the named $(16, 1)$ isomorphism it may also be shown without much difficulty that for every transitive group of degree 5 there exists a doubly transitive group of degree 16 which is $(48, 1)$ isomorphic with the group of degree 5.

9. *Configurations of Marks in $GF[p^n]$.* Let ω be a primitive mark of the Galois field $GF[p^n]$ and let μ be any (positive) factor of $p^n - 1$. Write $p^n - 1 = \mu k$. Let G_k, $G_k \equiv G_k[p^n]$, be the group, of order μp^n, consisting of the transformations $x' = \alpha x + \beta$, where β runs over all the marks of $GF[p^n]$ and α over the k-th power marks of this field. The set of marks $1, \omega^k, \omega^{2k}, \cdots, \omega^{(\mu-1)k}$ is left invariant as a set by the transformations $x' = \alpha x$ of G_k and by no other transformations of G_k, since μ is prime to p, while the transformations $x' = x + \beta$ of G_k permute this set into the p^n (distinct)

sets in the following columns:

$$\begin{array}{llllll}
1 & 1+\omega & 1+\omega^2 & \cdots & 1+\omega^{p^n-1} \\
\omega^k & \omega^k+\omega & \omega^k+\omega^2 & \cdots & \omega^k+\omega^{p^n-1} \\
\omega^{2k} & \omega^{2k}+\omega & \omega^{2k}+\omega^2 & \cdots & \omega^{2k}+\omega^{p^n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\omega^{(\mu-1)k} & \omega^{(\mu-1)k}+\omega & \omega^{(\mu-1)k}+\omega^2 & \cdots & \omega^{(\mu-1)k}+\omega^{p^n-1}.
\end{array}$$

Hence the configuration A_k, $A_k \equiv A_k[p^n]$, on the marks of $GF[p^n]$, defined by this array is invariant under G_k.

Let Γ_k, $\Gamma_k \equiv \Gamma_k[p^n]$, be the largest transformation group on the marks of $GF[p^n]$ each element of which leaves invariant the configuration A_k. Then Γ_k contains G_k as a subgroup. It is easy to show that G_k is the largest subgroup of Γ_k that is contained in the group of all linear transformations in $GF[p^n]$. The groups G_k and Γ_k induce permutation groups G_k and Γ_k respectively on the p^n marks of $GF[p^n]$; it is sometimes more convenient to deal with these permutation groups than with the transformation groups by means of which they are defined.

Let us consider the case when each pair of marks occurs in ν and just ν of the sets in $A_k[p^n]$. Then by a count of pairs it is seen to be necessary that $\mu = k\nu + 1$ and hence that $p^n = 1 + k + \nu k^2$. Since 0 and 1 must occur together in just ν of the sets it follows that the $GF[p^n]$ must have the property that there are just ν pairs of k-th powers in $GF[p^n]$ such that the elements of each pair differ by 1. The configurations thus arising would doubtless reward further investigation, especially the case when $\nu = 2$ and $p^n = 1 + k + 2k^2$.

We consider further the general case when $k = 2$. Then $p^n = 4\nu + 3$ and G_2 contains a transformation replacing the pair $0, 1$ by any preassigned pair. Hence any preassigned pair of marks occurs in the same number of sets of $A_2[p^n]$ as any other pair. Thus we have the theorem:

If $p^n = 4\nu + 3$ then each pair of marks occurs in ν and just ν sets belonging to the configuration $A_2[p^n]$; there are just ν pairs of squares in $GF[p^n]$ such that the elements of each pair differ by 1; any two sets in $A_2[p^n]$ have just ν elements in common.

It is easy to show that $A_2[7]$ is the same as $PG(2, 2)$ and hence that the group belonging to it is the doubly transitive group of degree 7 and order $7 \cdot 6 \cdot 4$.

The $A_2[11]$ is an interesting configuration consisting of the 11 quintuples into which the set 1, 4, 5, 9, 3 is changed by the cyclic permutation $(0, 1, 2, \cdots, 10)$. The group belonging to it is the doubly transitive group

of degree 11 and order $11 \cdot 10 \cdot 6$. A given pair of symbols occurs in just two quintuples of $A_2[11]$. Furthermore, any two of these quintuples have just two symbols in common. With each of the 55 pairs of these quintuples we may associate the quintuple formed by taking the two symbols common to the pair and the three symbols absent from both pairs; thus we have 55 additional quintuples. If we adjoin them to the 11 quintuples in $A_2[11]$ we have 66 quintuples forming the quintuple system described in § 8. The group belonging to it is of order $11 \cdot 10 \cdot 9 \cdot 8$, as we have seen, and hence is much larger than the group of order $11 \cdot 10 \cdot 6$ which belongs to each of the parts from which we have formed the quintuple system.

In general, when $p^n = 4\nu + 3$, any two sets in $A_2[p^n]$ have just ν elements in common; these and the $\nu + 1$ elements absent from both pairs form a set of $2\nu + 1$ elements; thus we have a configuration $B_2[p^n]$ consisting of $\frac{1}{2}p^n(p^n - 1)$ sets of $2\nu + 1$ elements each; combining $A_2[p^n]$ and $B_2[p^n]$ we have a configuration $C_2[p^n]$ containing the quintuple system of the preceding paragraph as a special case. These configurations $C_2[p^n]$ would doubtless reward further investigation.

Let us now consider the group G of transformations of the form

$$t' = (\alpha t + \beta)/(\gamma t + \delta), \qquad \alpha\delta - \beta\gamma = \text{square},$$

in the Galois field $GF[p^n]$ where p is an odd prime. Let S denote the set of $\frac{1}{2}(p^n + 1)$ elements consisting of ∞ and the square marks of $GF[p^n]$ and denote by $D[p^n]$ the configuration consisting of the sets into which S is transformed by G. I have not developed a theory of the configurations $D[p^n]$ though they seem to be of considerable interest. The configuration $D[7]$ is the quadruple system on eight elements; the group belonging to it is the triply transitive group of degree 8 and order $8 \cdot 7 \cdot 6 \cdot 4$. The $D[11]$ is the sextuple system on 12 elements which (§ 8) characterizes the Mathieu five-fold transitive group of degree 12.

10. *Configurations Associated with Multiply Transitive Groups.* Let G be a multiply transitive group of degree n whose degree of transitivity is k; and let G have the property that a set of m elements $(k < m < n)$ exists in G such that, when k of these m elements are changed by a permutation of G into k of these m elements, then all the m elements are permuted among themselves. Then the largest subgroup of G which permutes these m elements among themselves permutes them according to a transitive group which is at least k-fold transitive. Moreover, of $k > \frac{1}{3}m + 1$ (and also under certain other conditions) it follows (from the theory of multiply transitive groups)

that these m elements are then permuted by the named subgroup according to the alternating or the symmetric group of degree m; and it is certainly the symmetric group when $m = k + 1$.

Denote the order of G by $n(n-1)\cdots(n-k+1)\lambda$ and let G_1 be the subgroup of G of order λ leaving fixed each of a given set of k elements. Let H be the largest subgroup of G which permutes the named m elements among themselves and denote by $m(m-1)\cdots(m-k+1)\mu$ the order of the group Γ by which these m elements are permuted by H. Then μ is the order of the largest subgroup Γ_1 of Γ which leaves fixed each of k given elements; hence Γ_1 is a subgroup of the group induced by G_1 on the m elements on which Γ operates: therefore μ is a factor of λ.

Let ρ be the order of the largest subgroup K of G which leaves fixed each of the given m elements and let σ be the order of the largest subgroup L of G which leaves fixed each symbol of G not in the set of m given elements. Then G_1 is (ρ, σ) isomorphic with Γ_1. Hence $\lambda\sigma = \rho\mu$ and the order of H is $m(m-1)\cdots(m-k+1)\lambda\sigma$.

Thence it follows that G permutes the given m elements into

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1)\sigma$$

sets of m elements each, thus forming a configuration which we denote by E. Since G is k-fold transitive it follows that a given set of k elements occurs in just as many of these sets of m each as any other set of k elements. Since each set of k elements must appear at least once it follows that the number of sets must be at least as large as

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1).$$

Therefore $\sigma = 1$ and each set of k elements appears in one and just one set of m elements in the configuration E.

Thus we have the following theorem:

Let G be a multiply transitive group of degree n whose degree of transitivity is k; and let G have the property that a set S of m elements exists in G $(k < m < n)$ such that when k of these elements S are changed by a permutation of G into k of these elements then all these m elements are permuted among themselves. Then the identity is the only element in G which leaves fixed each of the $n-m$ elements not in S; G permutes the m elements of S into

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1)$$

sets of m elements each, thus forming a configuration E having the property that any (whatever) set of k elements appears in one and just one of the sets which constitute E.

It is clear that a necessary condition for meeting the hypotheses of the theorem is that k, m, n shall be such that each of the numbers

$$\frac{n-k+1}{m-k+1},\ \frac{(n-k+2)(n-k+1)}{(m-k+2)(m-k+1)},\cdots,\ \frac{n(n-1)\cdots(n-k+1)}{m(m-1)\cdots(m-k+1)}$$

shall be an integer. If $m=6$ and $k=5$ it may then be readily shown that n is of one of the forms 6ρ or $6\rho+10$ where ρ is not divisible by 5. In the case $n=12$ we have already had an illustration of the theorem afforded by the Mathieu five-fold transitive group of degree 12. If $m=8$ and $k=5$ it may be shown that n must be of one of the forms $20\mu+4$ and $20\mu+8$ and that it must also be congruent modulo 7 to 1, 2, 3 or 4. The smallest values of n $(n>8)$ meeting these conditions are 24, 44, 88, 108. The Mathieu five-fold transitive group of degree 24 affords an illustration of the theorem for the case $n=24$, $m=8$, $k=5$, as may be seen from § 8.

The finite geometries $PG(\lambda, p^\nu)$ afford examples of the configurations defined in the foregoing theorem, k being 2 and m being the number of points on a line. Similar examples arise also from the $EG(\lambda, p^\nu)$. In these cases one first constructs the configuration and then determines the group.

Again, to take up a different problem, let G be a multiply transitive group of degree n whose degree of transitivity is k and let G_1 be the largest subgroup of G which leaves fixed each of a given set of k symbols on which G operates. Let us suppose that G_1 has ν sets of transitivity of degrees $t_1, t_2, \cdots, t_\nu$ where the t's are all different when $\nu>1$ and that it has at least one additional set of transitivity of still another degree. Let s_i be the number of transitive constituents in G_1 each of which is of degree t_i. Then with each set of k symbols from the n on which G operates associate all the remaining symbols in the transitive constituents of degrees $t_1, t_2, \cdots, t_\nu$. We thus have l symbols in the set where $l=k+s_1t_1+s_2t_2+\cdots+s_\nu t_\nu<n$. With each set of k symbols in G form in this way a set of l symbols and let λ denote the number of sets so formed. They constitute a configuration F. Then λ is not greater than the number of combinations of n things taken k at a time.

It is clear that F is invariant under G and that its sets are permuted transitively by G. Moreover, one set of k symbols will appear in just as many of the λ sets of F as any other set of k symbols. Let μ denote this number,

so that each set of k symbols will appear in just μ of the λ sets of F. Then it is easy to show that

$$\lambda = \frac{n(n-1)\cdots(n-k+1)\mu}{l(l-1)\cdots(l-k+1)}, \qquad \mu \leqq \frac{l(l-1)\cdots(l-k+1)}{k!}.$$

Thus, for any given multiply transitive group G whose subgroup G_1 has transitive constituents of different degrees (always realized when $n-k$ is a prime) we have in this way a configuration F left invariant by G, while its sets are permuted transitively by G. (There is nothing to indicate that F may not characterize a larger group containing G as a proper subgroup).

Particular interest attaches to the case when $\mu = 1$, that is, the case in which each set of k symbols appears in only one of the sets which constitute F. When $k \geqq 4$ the conditions thus arising greatly restrict the possible values of n, the restrictions increasing rapidly with increasing k.

Returning to the general case, suppose that a given set M of l elements in F is obtained from each of just ρ sets of k elements, by the method employed in the construction of F. Consider a permutation P of G which transforms M into a set N of F; then the ρ sets each of which leads to M are transformed into ρ sets each of which leads to N; by transforming N to M by P^{-1} we then see that just ρ sets of k each lead to N. Since G permutes the λ sets of F transitively it follows that each of the λ sets in F is obtained from just ρ sets of k elements each. Since each set of k elements occurs in just μ sets of F we then have

$$\lambda = n(n-1)\cdots(n-k+1)/k!\,\rho,$$

while from the previous value of λ we have

$$\rho\mu = l(l-1)\cdots(l-k+1)/k!.$$

11. *Some Generalizations.* By a complete λ-μ-ν-configuration of n elements we shall mean a configuration of n elements taken ν at a time so that each set of μ elements shall occur together in just λ sets. (Compare Netto's *Lehrbuch der Combinatorik,* second edition, p. 325). Then a triple system is a complete 1-2-3-configuration; a quadruple system is a complete 1-3-4-configuration; and so on. A finite two-dimensional geometry $PG(2, p^n)$ is a complete 1-2-(p^n+1)-configuration. In § 8 we have shown the existence of a complete 1-4-5-configuration on 11 elements, a complete 1-5-6-configuration on 12 elements, a complete 1-5-8-configuration on 24 elements, a complete 1-4-7-configuration on 23 elements and a complete 1-3-6-configuration on

22 elements. These examples are sufficient to show the importance of complete λ-μ-ν-configurations for $\lambda = 1$. But little has been done towards a general theory of complete λ-μ-ν-configurations. In the next section we shall treat certain 2-2-k-configurations.

An infinite class of complete 2-3-4-configurations may be constructed in the following manner. Let p be any prime of the form $6m + 1$ and let ρ be a solution of the congruence $t^2 - t + 1 \equiv 0 \bmod p$. The set $\infty, 0, 1, \rho$ is transformed into itself by the group generated by the transformations

$$x' \equiv (x - 1)/x \bmod p \quad \text{and} \quad x' \equiv \rho/x \bmod p,$$

a group whose order is 12. Thence it follows readily that the set $\infty, 0, 1, \rho$ is transformed into $(p + 1)p(p - 1)/12$ quadruples by the linear fractional group modulo p, the order of which is $(p + 1)p(p - 1)$. Since this linear fractional group is triply transitive it follows that each triple of the $p + 1$ elements $\infty, 0, 1, 2, \cdots, p - 1$ occurs among the quadruples in the named set of quadruples, and indeed that each triple occurs the same number of times as any other, whence it follows that each of them occurs just twice. Thence it follows that these quadruples constitute a complete 2-3-4-configuration. In case m is odd (but not when m is even) this configuration breaks up into two equivalent configurations each of which constitutes a complete 1-3-4-configuration, a fact which one may readily verify by showing that the transformations of square determinants in the named linear fractional group then transform $\infty, 0, 1$ into every triple of the $p + 1$ elements (though as a permutation group it is only doubly transitive).

12. *Certain Complete 2-2-k-configurations.* We shall now treat those complete 2-2-k-configurations of n elements which are formed by n sets of k things each such that each two sets have just two elements in common. Since each of the $\tfrac{1}{2}n(n - 1)$ pairs of elements occurs just twice and each of the n sets of k elements contains just $\tfrac{1}{2}k(k - 1)$ pairs it follows that we must have $\tfrac{1}{2}k(k - 1)n = 2 \cdot \tfrac{1}{2}n(n - 1)$, whence it is necessary that

$$n = \tfrac{1}{2}k(k - 1) + 1.$$

The case $k = 2$ is entirely trivial. When $k = 3$ we have $n = 4$ and the configuration consists of the four triples which may be formed from four things. When $k = 4$ we have $n = 7$; then it may be shown that the configuration is that which is complementary to $PG(2, 2)$, whence it follows that the group belonging to the configuration is the doubly transitive group of degree 7 and order $7 \cdot 6 \cdot 4$.

When $k = 5$ we have $n = 11$. It is not difficult to show that there is just one configuration of degree 11 of the type now in consideration and that it is equivalent to the configuration $A_2[11]$ treated in § 9 and that the group belonging to it is doubly transitive and of order $11 \cdot 10 \cdot 6$.

When $k = 6$ we have $n = 16$. A corresponding configuration may be constructed in the following manner. By means of the adjoining scheme form 16 sets of six letters each by taking for each letter in the scheme the six which are aligned with it (excluding that letter itself). Thus corresponding to A and B we form respectively the sets $BCDEIM$ and $ACDFJN$. The 16 sets formed constitute a complete 2-2-6-configuration of the kind here in consideration, as one may readily verify. The group belonging to the configuration is the doubly transitive group of degree 16 and order $16 \cdot 15 \cdot 12 \cdot 4$.

$A \quad B \quad C \quad D$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$E \quad F \quad G \quad H$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$I \quad J \quad K \quad L$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$M \quad N \quad O \quad P$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

In this case ($k = 6$, $n = 16$) the configuration is not unique; but the total set of inequivalent configurations seems never to have been determined. (In fact, the general class of configurations treated in this section seems never to have been previously considered). A second configuration for the case $k = 6$ and $n = 16$ consists of the sets in the following sixteen columns:

A	A	A	B	B	C	B	B	C	C	D	D	A	A	A	F
B	B	C	C	D	D	E	J	E	H	F	G	E	E	I	G
C	D	D	F	E	L	F	K	G	I	M	I	F	G	J	H
H	G	F	G	H	E	N	M	O	K	O	L	K	H	L	J
L	K	J	I	I	M	O	N	J	N	H	N	I	M	M	K
P	O	N	M	J	K	L	P	P	O	P	P	P	N	O	L

To show that this is different from the foregoing 2-2-6-configuration one proves that it belongs to a different group. There exists also a complete 2-2-9-configuration (of the type here studied), consisting of the 37 sets into which the set 1, 7, 9, 10, 12, 16, 26, 33, 34 is transformed by the $37 \cdot 9$ transformations generated by $t' \equiv t + 1 \bmod 37$ and $t' \equiv 16t \bmod 37$.

The configurations which we have named are apparently all the known configurations of the class here in consideration; but there seems to be nothing known to show their non-existence for any value of k greater than unity. In particular, it seems not to be known whether such configurations exist for $k = 7$ or 8.

With every configuration of the class here in consideration one may

associate an *adjoint* configuration, in the following manner. Number the sets in the configuration from 1 to n inclusive; let $a_1, a_2, \cdots, a_n$ be the symbols appearing in the configuration. Now form a configuration of the numbers $1, 2, \cdots, n$ by taking for the i-th set the k numbers which designate the k sets in which a_i appears, doing this for $i = 1, 2, \cdots, n$. Then the n numbers appear in n sets of k numbers each. That they form a complete 2-2-k-configuration of the class in consideration is readily shown by observing that if a_i and a_j appear together in the λ-th and μ-th sets of the original configuration, then λ and μ appear together in just two sets of the new configuration, namely, in those determined by means of a_i and a_j. If the second of two configurations is adjoint to the first then the first is also adjoint to the second.

UPON A THEORY OF INFINITE SYSTEMS OF NON-LINEAR IMPLICIT AND DIFFERENTIAL EQUATIONS.

By Aurel Wintner.

Introduction. There has arisen, in recent years, a large literature upon infinite systems of non-linear implicit and differential equations.[1] The methods which are employed have become classical in the treatment of finite systems. However these methods (employed in the treatment of finite systems) may be extended only to infinite systems of very special type, restricted by inequalities, which for many reasons are not fulfilled in possible applications of the method of infinitely many variables to various classical problems in analysis. Thus the theorems which have been proved by me a few years ago[2] are the only ones to my knowledge which have found an application to concrete problems, in particular to the problems of Celestial Mechanics or the Calculus of Variations. I shall give here a comprehensive review of these questions without giving any essential extensions of my results as set forth in my previous papers and without assuming any previous knowledge of the theory of infinitely many variables. The applications will be excluded and the reader referred in this connection to some earlier papers appearing in the *Mathematische Annalen* and in the *Mathematische Zeitschrift.* A characteristic application can be found in the paper of Martin appearing in this number of the Journal.[3]

The infinite system which I shall treat is composed exclusively of power series (and not of more general functions) of the infinite sequence of variables (as is indeed always the case in concrete applications).

My problem was to introduce a method of treatment in which the power series are not subjected to inequalities which would not be fulfilled in concrete applications but which would be necessary in the classical modes of treatment. On page 252 there will be given a number of examples, of most important character, which will serve to illustrate why the usual methods *must* fail.

Inasmuch as the nature of the problems appears most clearly in its application to the complex space of Hilbert, the present paper will be restricted in its treatment to this space. It is of course possible by using the principles of General Analysis[4] to further extend the methods here set forth. There is no difficulty in using other spaces[5] than that of Hilbert. In the articles cited above also spaces other than that of Hilbert are treated; cf. the above mentioned paper of Martin.

The systems to which the existence theorems of this paper apply are simply the direct generalizations of the most general bounded (beschränkt) systems of linear equations of Hilbert, or the corresponding linear differential systems, to non-linear systems. The notion of a regular power series in infinitely many variables, which enters in the existence theorems treated in this paper, will be so defined that, in the special cases of linear and quadratic forms, the regular power series are nothing other than the bounded linear and quadratic forms of Hilbert (the coefficients of which are not necessarily real).[6]

1. *Regular power series.* By a power series in infinitely many variables $z_1, z_2, \cdots$ will be understood formally an expression

$$\Phi(z_1, z_2, \cdots) = \sum_{n=0}^{\infty} \Phi^{(n)}(z_1, z_2, \cdots), \tag{1}$$

in which

$$\Phi^{(n)}(z_1, z_2, \cdots) = \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} \cdots \sum_{\nu_n=1}^{\infty} c^{(n)}{}_{\nu_1\nu_2\ldots\nu_n} z_{\nu_1} z_{\nu_2} \cdots z_{\nu_n}, \tag{2}$$

that is $\Phi^{(n)}$ is a form of degree n. It is clear that any such form (in which the coefficients and variables may be complex) can, without loss of generality, be written so that

$$c^{(n)}{}_{\nu_1\nu_2\ldots\nu_n} = c^{(n)}{}_{\nu_2\nu_1\ldots\nu_n} = \cdots = c^{(n)}{}_{\nu_n\nu_{n-1}\ldots\nu_1}, \tag{3}$$

and this will accordingly, in the following, always be assumed to have been done.

We now define what is meant by the convergence of the power series

$$\Phi(z_1, z_2, \cdots, z_m, z_{m+1}, z_{m+2}, \cdots) = \sum_{n=0}^{\infty} \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} \cdots \sum_{\nu_n=1}^{\infty} c^{(n)}{}_{\nu_2\nu_1\ldots\nu_n} z_{\nu_1} z_{\nu_2} \cdots z_{\nu_n} \tag{4}$$

[and therefore also in the special case of a form (2)]. We put all the variables, with the exception of the first m, equal to zero and obtain a power series in the m variables $z_1, z_2, \cdots z_m$, namely

$$\begin{aligned} \Phi_{[m]}(z_1, z_2, \cdots z_m) &\equiv \Phi(z_1, z_2, \cdots, z_m, 0, 0, \cdots) \\ &= \sum_{n=0}^{\infty} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_n=1}^{m} c^{(n)}{}_{\nu_1\nu_2\ldots\nu_n} z_{\nu_1} z_{\nu_2} \cdots z_{\nu_n}. \end{aligned} \tag{5m}$$

This power series, arising from (4), will be called the m-th section (Abschnitt) of (4). We say that the power series (4) converges at a point

$$z_1, z_2, \cdots \tag{6}$$

of the space of infinitely many dimensions if first: At the point (6) every

section ($5m$) converges (in the usual sense of the theory of m-fold series) for $m = 1, 2, \cdots$ (such a restriction is obviously by its very nature fulfilled in the case of forms or polynomials in infinitely many variables) and if secondly: At the point (6) the limit

$$(7) \qquad \lim_{m\to\infty} \Phi_{[m]}(z_1, z_2, \cdots z_m)$$

exists. The sum of the series (4) at the point (6) is defined as the value of this limit.

The power series

$$(4') \qquad \sum_{n=0}^{\infty} \sum_{\nu_1=1}^{\infty} \sum_{\nu_2=1}^{\infty} \cdots \sum_{\nu_n=1}^{\infty} | c^{(n)}{}_{\nu_1\nu_2\ldots\nu_n} | z_{\nu_1} z_{\nu_2} \cdots z_{\nu_n},$$

which arises from (4) when its coefficients are replaced by their absolute values, we shall call the best majorant of (4) and will be designated by $\bar{\Phi}$. If the series (4′) converges at the point

$$(6') \qquad | z_1 |, | z_2 |, \cdots,$$

the series (4) is said to be absolutely convergent at the point (6).

With respect to a given sequence of numbers

$$(8) \qquad z_1^{(0)}, z_2^{(0)}, \cdots$$

the symbol

$$(9) \qquad \sigma_r\{z_k^{(0)}\}$$

will be understood to designate the following domain of the infinitely many independent variables (6):

$$(10) \qquad \sum_{k=1}^{\infty} | z_k - z_k^{(0)} |^2 < r^2.$$

Accordingly (9) is the interior of the complex Hilbert sphere, with radius r, taken about the point (8) as center.

In particular we shall write briefly

$$(11) \qquad \sigma_r = \sigma_r\{0\},$$

so that

$$(12) \qquad \sigma_r: \qquad \sum_{k=1}^{\infty} | z_k |^2 < r^2.$$

We say [7] that the power series (4) is regular in the domain (12) if the following three conditions are fulfilled: First, the m-th section ($5m$) of (4) shall, for any fixed m, in the sense of the theory of analytic functions of m variables be regular in the m-th "section" of σ_r, that is, in the domain

$$\sum_{k=1}^{m} | z_k |^2 < r^2. \tag{13m}$$

Second, there exists at every point of (12) the limit defined by (7); the series (4) is then, according to the above definition, convergent. Third, there exists for every positive number ϵ a positive number $M_{r-\epsilon}$ which is independent of m and such that the absolute value of the power series (5m) in the domain

$$\sum_{k=1}^{m} | z_k |^2 < (r-\epsilon)^2 \tag{13m'}$$

is not greater than $M_{r-\epsilon}$; in other words the absolute values of the function (4) in the domain $\sigma_{r-\epsilon}$ is not greater than $M_{r-\epsilon}$. It follows from the first assumption that the power series (5m) converges absolutely in the domain (13m) and uniformly in the domain (13m'). It is clear that the second assumption is not a consequence of the first. In the special cases where the power series (4) is a linear or a quadratic form Hellinger and Toeplitz [8] have shown that the third assumption is a consequence of the first two. However it is easy to show that for the general power series which contain terms of infinite degree the third assumption is independent of the first two. The power series (4) will be said to be an integral function if it is regular in a σ_r with arbitrarily large r.

The upper bound of $| \Phi(z_1, z_2, \cdots) |$ in the domain σ_ρ will be designated by $[\Phi]_\rho$. In the domain $0 \leqq \rho < r$ the upper bound $[\Phi]_\rho$ is a positive monotone function of ρ which, while it is bounded in the interval $0 \leqq \rho < r - \epsilon$ for any given value of ϵ, can become infinite if $\epsilon \to 0$ (Φ being regular in σ_r). The function Φ can be an integral function and nevertheless the series (4) not be uniformly convergent in any domain σ_δ (with arbitrarily small δ). The simplest example of an integral function of this kind is

$$\Phi = \sum_{k=1}^{\infty} z_k{}^2. \tag{14}$$

Moreover it is possible that Φ, for example even in the special case of the quadratic form

$$\Phi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} z_i z_j, \tag{15}$$

is an integral function and nevertheless the series (4) is not absolutely convergent in a domain σ_δ regardless of how small δ be chosen. We shall return to this point later (cf. page 248). In particular (4) can accordingly be an integral function without the necessary existence of a domain σ_δ in which

the best majorant is regular. If now not only Φ but also $\bar{\Phi}$ is regular in a domain σ_r we shall say Φ is absolutely regular in σ_r. A regular function which is a form is, because of its homogeneity, obviously an integral function. Inasmuch as the sum of a finite number of power series, regular in σ_r, is a power series regular in σ_r, we may say more generally that the sum of a finite number of regular forms, i. e. a regular polynomial, is likewise an integral function.

For a given function Φ regular in σ_r it is possible in the neighborhood of a given point (8) of σ_r to order formally the function according to increasing powers of the arguments $z_k - z_k^{(0)}$. One thus obtains a power series in the arguments $z_k - z_k^{(0)}$ which in the domain

$$(10') \qquad \sum_{k=1}^{\infty} | z_k - z_k^{(0)} |^2 < r'^2, \quad r'^2 = r^2 - \sum_{k=1}^{\infty} | z_k^{(0)} |^2$$

is [9] regular and for corresponding points possesses the same sum as Φ. This theorem will be called the neighborhood theorem and implies among other things that in the concept of neighborhood associated with the metric $(\sum_{i-1}^{\infty} | u_i - v_i |^2)^{1/2}$ there exists no isolated regular point. Nevertheless (because of the nature of the topology of $\sum_{i=1}^{\infty} | x_i |^2 < \delta$) it can not be expected that there exist, in the theory of functions of infinitely many variables, far reaching analogies with the theory of analytic functions of a finite number of variables. As an example, the theorem of Poincaré-Volterra, based on the theorem of Heine-Borel, is no longer true, for Hilbert gives the example [7]

$$\Phi = \sum_{\nu=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^\nu}{2^i} \binom{1/2}{\nu} z^\nu = \sum_{i=1}^{\infty} \frac{(1-z_i)^{1/2}}{2^i}$$

of an obviously regular (also in our sense) function for which the set of branches has the power of the continuum.

Given the sequence of points $\{z_k^{(0)}\}$, $\{z_k^{(1)}\}$, $\{z_k^{(2)}\}, \cdots$ in the domain σ_r in which the function Φ is defined. The function Φ will be said to be continuous at the point $\{z_k^{(0)}\}$ if for every sequence in which

$$(16) \qquad \lim_{\nu\to\infty} z_1^{(\nu)} = z_1^{(0)}, \quad \lim_{\nu\to\infty} z_2^{(\nu)} = z_2^{(0)}, \quad \lim_{\nu\to\infty} z_3^{(\nu)} = z_3^{(0)}, \cdots$$

we always have

$$(17) \qquad \lim_{\nu\to\infty} \Phi(z_1^{(\nu)}, z_2^{(\nu)}, z_3^{(\nu)}, \cdots) = \Phi(z_1^{(0)}, z_2^{(0)}, z_3^{(0)}, \cdots).$$

There exist regular, and indeed integral functions, which are continuous at no point. The simplest example is the afore-mentioned function (14).

To be sure it may be shown [10] that if Φ is regular in σ_r and (8) is a point of σ_r then (17) is valid provided that the sequence (16) converges strongly to $\{z_k^{(0)}\}$, that is if not (16), but also

$$\lim_{\nu\to\infty} \sum_{k=1}^{\infty} | z_k^{(\nu)} - z_k^{(0)} |^2 = 0 \tag{18}$$

is supposed.

If we designate by $\Phi_{z_k}(z_1, z_2, \cdots)$ the power series arising from $\Phi(z_1, z_2, \cdots)$ by term by term partial differentiation with respect to z_k, it can be shown [11] that if Φ is regular in σ_r, then also Φ_{z_k} is regular in σ_r. Moreover we have [12]

$$\sum_{k=1}^{\infty} \left| \frac{\partial \Phi(z_1, z_2, \cdots)}{\partial z_k} \right|^2 \leqq \left(\frac{[\Phi]_{r-\epsilon}}{r-\epsilon} \right)^2 = \text{const.} \quad \text{in } \sigma_{r-\epsilon} \quad (\epsilon > 0). \tag{19}$$

This inequality will be designated as the gradient inequality.

2. *The bounded forms of Hilbert considered as regular power series.* We introduce now a new complex manifold E

$$E\colon \sum_{i=1}^{\infty} | \zeta_i |^2 = 1 \tag{20}$$

in the independent variables ζ_i. The linear form

$$\sum_{i=1}^{\infty} a_i \zeta_i \tag{21}$$

is bounded in the sense of Hilbert if and only if the series [6]

$$\sum_{i=1}^{\infty} | a_i |^2 \tag{22}$$

converges. It follows from the inequality of Schwarz that the absolute value of (21) is not greater than the square root of the expression (22). That the expression (21) actually attains its maximum in E follows readily if we place in (21)

$$\zeta_i = \frac{\bar{a}_i}{\left(\sum_{k=1}^{\infty} | a_k |^2 \right)^{1/2}}. \tag{23}$$

Accordingly the linear form

$$\sum_{i=1}^{\infty} c_i z_i \tag{24}$$

is then and only then bounded in the sense of Hilbert if, considered as a power series in infinitely many variables, it is a regular function (and accordingly

an integral function) in our sense. Accordingly if (24) is a regular function in σ_ρ we have (with the use of the notation introduced on page 244)

$$[\sum_{k=1}^{\infty} c_k z_k]_\rho = \rho(\sum_{k=1}^{\infty} | c_k |^2)^{1/2}. \tag{25}$$

For brevity we write

$$[\{c_k\}] = (\sum_{k=1}^{\infty} | c_k |^2)^{1/2}. \tag{26}$$

Since the convergence of the series (26) represents the necessary and sufficient condition for the regularity of (24) it follows that (24) and

$$\sum_{i=1}^{\infty} | c_i | z_i \tag{24'}$$

are simultaneously regular. Or more concisely, a regular *linear* form is always absolutely regular.

The matrix $\| a_{ij} \|$, which may be complex and need not fulfill any symmetry conditions, is then and only then bounded in the sense of Hilbert[6] if each of the linear forms $\sum_{j=1}^{\infty} a_{ij}\zeta_j$ $(i = 1, 2, \cdots)$ is bounded in the sense of Hilbert and if in addition there exists a positive number K so that in each point of E

$$\sum_{i=1}^{\infty} | \sum_{j=1}^{\infty} a_{ij}\zeta_j |^2 \leqq K. \tag{27}$$

In order that a matrix be bounded in the sense of Hilbert it is necessary[13] that there exist a constant L independent of m so that

$$| \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\zeta_i\zeta_j | \leqq L \tag{28}$$

in every point of

$$\sum_{k=1}^{m} | \zeta_k |^2 = 1. \tag{28'}$$

It is furthermore necessary that the double series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}\zeta_i\zeta_j \tag{29}$$

converge in every point of $\boldsymbol{E}$. The sum of the series is then, from (28), obviously $\leqq L$ in absolute value. The exact upper limit $\leqq K^{1/2}$ of

$$(\sum_{i=1}^{\infty} | \sum_{j=1}^{\infty} a_{ij}\zeta_j |^2)^{1/2} \tag{27'}$$

in the domain $\boldsymbol{E}$ will be designated by $[\| a_{ij} \|]$. If $\| a_{ij} \|$ is bounded in

the sense of Hilbert it is known that the transposed matrix is also bounded in the sense of Hilbert and that $[\| a_{ij} \|] = [\| a_{ji} \|]$.

We now assume that the matrix $\| a_{ij} \|$ is symmetric $(a_{ij} = a_{ji})$ without necessarily being real. Hellinger and Toeplitz [6] have shown that in this case the existence of a positive number L, independent of m, so that (28) is valid in every point of the domain (28′) is not only necessary, but also sufficient in order that the matrix $\| a_{ij} \|$ should be bounded in the sense of Hilbert. Since we have agreed (3) to write every quadratic form in the symmetric form

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} z_i z_j \qquad (c_{ij} = c_{ji}) \tag{30}$$

(in which the coefficients may be complex) it follows from the foregoing that every form (30), that is, a not necessarily real, but symmetric matrix $\| c_{ij} \|$ is bounded in the sense of Hilbert if and only if the form (30), regarded as a power series in infinitely many variables, is a regular function (and therefore an integral function). It follows from the definitions (cf. page 244 and page 247) and from the inequality of Schwarz that

$$[\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} z_i z_j]_\rho \leqq \rho^2 [\, \| c_{ij} \| \,].$$

Toeplitz [14] has shown that there exist symmetric forms (30) which are bounded in the sense of Hilbert, although the best majorant form

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} | c_{ij} | z_i z_j \tag{30′}$$

is not bounded in the sense of Hilbert. The justification of the remarks made in connection with (15) accordingly follows from the above theorem of Hellinger and Toeplitz [the example given by Toeplitz is moreover real and not only bounded but also in the sense of Hilbert "completely continuous" (vollstetig), i. e. in the sense of the definition given in connection with (16), (17) continuous in every point of the Hilbert space].

3. *Bounded sequences and bounded matrices of regular power series.*

We will say that the vector $\{\Phi_i(z_1, z_2, \cdots)\}$ with infinitely many (not necessarily real) components

$$\Phi_1, \Phi_2, \Phi_3, \cdots \tag{31}$$

is bounded in σ_r if the following three conditions are satisfied:

(a) The power series

$$\Phi_i \equiv \Phi_i(z_1, z_2, \cdots); \qquad (i = 1, 2, \cdots) \tag{32}$$

are regular, in the sense defined above, in the domain σ_r.

(b) If $\{z_k\}$ is any arbitrarily chosen fixed point of σ_r then the linear form

$$\sum_{i=1}^{\infty} \Phi_i(z_1, z_2, \cdots)\zeta_i \tag{33}$$

is bounded in the sense of Hilbert, i. e. the number (cf. page 244)

$$[\{\Phi_i(z_1, z_2, \cdots)\}] = \sqrt{\sum_{i=1}^{\infty} |\,\Phi_i(z_1, z_2, \cdots)\,|^2}, \tag{34}$$

which depends upon $\{z_k\}$ is $< +\infty$ for any point in σ_r.

(c) The expression (34) is a bounded function of $\{z_k\}$ in the domain σ_r i. e. there exists a constant M so that for every point $\{z_k\}$ of the domain σ_r

$$[\{\Phi_i(z_1, z_2, \cdots)\}] \leqq M. \tag{34'}$$

The smallest M i. e. the upper limit of (34) in the domain σ_r will be designated by

$$[\{\Phi_i\}]_r. \tag{35}$$

In the special case when the vector components (31) are constants the above definition of a bounded vector is equivalent to Hilbert's definition of the boundedness of the linear form

$$\sum_{i=1}^{\infty} \Phi_i(0, 0, \cdots)\zeta_i. \tag{33'}$$

In an analogous manner we shall generalize the concept of boundedness, in the sense of Hilbert, for a matrix with constant elements, to a matrix whose elements are power series, as follows:

We shall say that the matrix

$$\|\,\Phi_{ij}(z_1, z_2, \cdots)\,\| \tag{36}$$

(which need neither be real nor satisfy any symmetry conditions whatsoever) is bounded in σ_r if the following three conditions are fulfilled:

(a) Each of the power series

$$\Phi_{ij}(z_1, z_2, \cdots); \qquad (i, j = 1, 2, \cdots) \tag{37}$$

is regular in σ_r.

(b) The matrix (36) is bounded in every point $\{z_k\}$ of σ_r, i. e. the number

$$[\|\,\Phi_{ij}(z_1, z_2, \cdots)\,\|], \tag{38}$$

defined on page 247, which depends upon $\{z_k\}$ remains finite for any point in σ_r.

(c) There exists a constant M so that (38) [in every point of σ_r] is not

greater than M; the smallest M [i. e. the upper limit of (38) in the domain σ_r] will be designated by

(39) $$[\| \Phi_{ij} \|]_r.$$

The following consistency theorems are now valid:

I. If the power series (4) is regular in σ_r and if ϵ designates an arbitrarily small positive number, the vector with components

(40) $$\Phi_i \equiv \Phi_{z_i} \equiv \partial\Phi/\partial z_i$$

is bounded in the domain $\sigma_{r-\epsilon}$ and satisfies the inequality

(41) $$[\{\Phi_{z_i}\}]_{r-\epsilon} \leqq \frac{[\Phi]_{r-\epsilon}}{r-\epsilon} \qquad \text{for every } \epsilon > 0.$$

II. If (31) is a bounded vector in the domain σ_r then the Jacobian matrix

(42) $$\| \partial\Phi_i/\partial z_k \|$$

is, for arbitrarily small $\epsilon > 0$, a bounded matrix in the domain $\sigma_{r-\epsilon}$.

III. If (4) is a regular power series in the domain σ_r the Hessian matrix

(43) $$\| \partial^2\Phi/\partial z_i \partial z_j \|$$

is, for arbitrarily small $\epsilon > 0$, a bounded matrix in the domain $\sigma_{r-\epsilon}$.

I follows immediately from the gradient inequality (19). II follows from I if we put $\Phi = \sum_{k=1}^{\infty} \Phi_k(z_1, z_2, \cdots)\zeta_k$ in I. III is a trivial consequence of I and II.

The multiplication theorems (Hilbertsche Faltungssätze) and related theorems of Hilbert upon matrices with constant elements can, by the usual proofs,[6] be extended to our more general case so that I shall not enter any further in these matters. However the following trivial theorem, which will be needed later, will now be formulated:

IV. Any vector (31), bounded in σ_r, is transformed by a matrix $\| a_{ij} \|$ into a vector

(44) $$\psi_1, \psi_2, \cdots; \qquad \psi_i = \sum_{j=1}^{\infty} a_{ij}\Phi_j(z_1, z_2, \cdots)$$

which is likewise bounded in σ_r.

In addition the following, easily proven, theorems are valid.[15]

V. If (31) is a bounded vector in σ_r [or if (36) is a bounded matrix in σ_r] and (8) is an arbitrarily chosen fixed point of σ_r and if one orders

every power series Φ_i of the vector (31) [or every power series Φ_{ij} of the matrix (36)] in the neighborhood of the point (8) formally according to increasing powers of the arguments $z_k - z_k^{(0)}$, one obtains a vector (or a matrix) which is bounded in the domain (10′) (cf. page 245) of the new variables $z_k - z_k^{(0)}$.

VI. The sum of two vectors (or matrices) bounded in σ_r is a vector (or matrix) bounded in σ_r.

This follows immediately with the use of the inequality of Schwarz.

4. *The existence theorems.* In the proof of existence theorems on infinite systems, the above consistency theorems permit us to take as a starting point in the proof a conveniently normalized form of the infinite system under consideration.

The fundamental existence theorem of the theory is the following:

EXISTENCE THEOREM I.[16] *If the vector* (31) *is bounded in* σ_r *the system*

$$dz_i/dt = \Phi_i(z_1, z_2, \cdots); \qquad (i = 1, 2, \cdots) \tag{45}$$

possesses in the circle

$$|t| < \frac{r}{2[\{\Phi_i\}]_r} \tag{46}$$

a holomorphic solution

$$z_i = z_i(t); \qquad (i = 1, 2, \cdots), \tag{47}$$

satisfying the initial conditions

$$z_i(0) = 0; \qquad (i = 1, 2, \cdots), \tag{48}$$

and this solution (47) *lies in the domain* σ_r *if* t *lies in the domain* (46). [The uniqueness of the solution (47), which is more interesting than in the case of a finite number of variables is treated *loc. cit.*[2], III, p. 466-467].

It follows from the Consistency Theorem V that the normalization (48) can be effected without loss of generality. In addition there is no loss in generality in assuming that the functions Φ_i of (45) do not contain the independent variable t explicitly, inasmuch as one can in this case adjoin to (45) the differential equation

$$dz_0/dt = 1 \qquad [z_0(0) = 0], \tag{45'}$$

without thereby destroying the boundedness of the vector whose components are the right hand members of (45).

For the special case of a linear system of differential equations Hart[17] has demonstrated this existence theorem by a method which cannot be ex-

tended to the non-linear case.[18] The existence domain found by Hart in the special case of the linear systems is larger than that given in (46), valid for the above more general case. In fact the existence domain [19] found by Hart can be derived as a corollary from Existence Theorem I simply by analytical continuation in which one need only use the divergence of the harmonic series.[20]

Mention has been made, in the introduction, of the failure of the usual existence proofs under our general assumptions. We intend now to discuss this matter in greater detail. First of all, the method of Arzelà and related methods,[21] in which only the continuity [or even less restrictive assumptions [22]] of the Φ_i is assumed, are not applicable. This appears immediately from the known fact that the Hilbert sphere is neither separable nor compact. The methods employed under the condition of Lipschitz are also not applicable, as is readily perceived from the integral function (14) which is neither continuous nor satisfies the condition of Lipschitz. That the majorant method of Cauchy, which assumes the existence of a single majorant equation

$$dz/dt = \phi(z), \tag{49}$$

must also fail follows from the example

$$\Phi_i = z_i; \qquad (i = 1, 2, \cdots). \tag{50}$$

The vector (31) is, in the case of (50), bounded in any σ_r. The best common majorant, with one variable, of the infinitely many power series (50) is however

$$\sum_{i=1}^{\infty} z = z \sum_{i=1}^{\infty} 1, \tag{50'}$$

a divergent series, so that the best common majorant (49) does not exist. This type of infinite differential system, in which the variable z_i dominates the function Φ_i, is moreover a typically occurring case in the applications.

The method of Cauchy is a special majorant method, as it is based on the existence of a *single* (*common*) majorant differential equation. It will now be shown that the proof of Existence Theorem I lies beyond the capabilities of the majorant method even if one replaces each of the infinitely many differential equations by its best majorant, i. e. even if one works with an infinite majorant *system*. The best majorant system of (45) is [cf. (4')]

$$dz_i/dt = \bar{\Phi}_i(z_1, z_2, \cdots). \tag{51}$$

One can now so choose the Φ_i that on one hand the vector (31) is bounded

in every domain σ_r, and consequently the system (45), according to Existence Theorem I possesses a holomorphic solution, satisfying the initial condition (48), while on the other hand the best majorant vector

$$\bar{\Phi}_1, \bar{\Phi}_2, \cdots \tag{52}$$

is not only not bounded in a sufficiently small domain σ_r but the best majorant system (51) of (45) cannot possess a holomorphic solution satisfying the initial conditions (48). [In what follows, for the sake of brevity, we shall demonstrate only the non-existence of a holomorphic solution of (51) satisfying (48)]. We infer accordingly that the best majorant of a solution is not to be confused with the solution of the best majorant system.

In order to give an example of a differential system (45) which fulfills the conditions of Existence Theorem I but which does not yield to the devices of *any* majorant method we put in (45) at first

$$\Phi_i \equiv a_i + \sum_{k=1}^{\infty} a_{ik} z_k \tag{53}$$

where the form

$$\sum_{i=2}^{\infty} \sum_{k=2}^{\infty} | a_{ik} | z_i z_k$$

is not bounded in the sense of Hilbert. The two bounded linear forms

$$\sum_{k=2}^{\infty} a_{1k} X_k, \qquad \sum_{k=2}^{\infty} a_k X_k$$

can be so chosen that [8]

$$\sum_{\nu=2}^{\infty} \sum_{\mu=2}^{\infty} | a_{\nu\mu} a_{1\nu} a_\mu | = +\infty.$$

If we now assume that

$$\sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik} z_i z_k$$

is symmetrical and bounded in the sense of Hilbert (which contains no contradiction [14]) and if we place

$$a_1 = 0, \qquad a_{i1} = 0; \qquad\qquad (i = 1, 2, \cdots),$$

the premises of the Existence Theorem I are fulfilled since the two forms

$$\sum_{i=1}^{\infty} a_i z_i, \qquad \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} z_i z_k$$

are obviously bounded in the sense of Hilbert; while the best majorant system (51) of (45), (53), namely the system

$$dz_i/dt = | a_i | + \sum_{j=1}^{\infty} | a_{ij} | z_j ; \qquad (i = 1, 2, \cdots)$$

possesses no holomorphic solution

$$z_i(t) = \sum_{k=1}^{\infty} \gamma_{ik} t^k$$

since the recursion formulas for such a holomorphic solution yield

$$3 ! \gamma_{13} = \sum_{\nu=1}^{\infty} | a_{1\nu} | \sum_{\mu=1}^{\infty} | a_{\nu\mu} | \; | a_\mu |$$

and accordingly $\gamma_{13} = +\infty$ (cf. *loc. cit.*[2], II).

From Existence Theorem I and the above consistency theorems there follows, since our concepts are the non-linear generalizations of the concepts of Hilbert, the following:

Existence Theorem II.[24] *If* (31) *is bounded in* σ_r *the implicit system*

$$z_i = t\, \Phi_i (z_1, z_2, \cdots) ; \qquad (i = 1, 2, \cdots) \tag{54}$$

possesses in the circle $| t | < \alpha$, *where* α *is a suitably chosen constant, a holomorphic solution*

$$z_i = z_i(t) \tag{55}$$

and this solution (55) *lies in* σ_r *if* t *lies in the circle* $| t | < \alpha$.

The assumption that the Φ_i are independent of t, is no restriction on the generality of system (54) as we can adjoin to (54) the equation $z_0 = t\Phi_0$ where $\Phi_0 \equiv 1$.

If one chooses in (54) the Φ_i linear functions of $z_1, z_2, \cdots$ namely

$$\Phi_i = a_i + \sum_{j=1}^{\infty} a_{ij} z_j ; \qquad (i = 1, 2, \cdots) \tag{56}$$

then the vector (31) is then and only then bounded if the matrix $\| a_{ij} \|$ and the linear form (21) are both bounded in the sense of Hilbert and (54) becomes

$$z_i - t \sum_{j=1}^{\infty} a_{ij} z_j = t a_i ; \qquad (i = 1, 2, \cdots) \tag{54'}$$

so that Existence Theorem II becomes the Theorem of Hilb[25]: If the matrix $\| a_{ij} \|$ is bounded and if $\| \delta_{ij} \|$ denotes the unit matrix the matrix $\| \delta_{ij} - t a_{ij} \|$ has for sufficiently small values of $| t |$ a bounded reciprocal matrix which is a holomorphic function of t (Neumann's series of iterations).

The method by which the proof of the existence theorems proceeds is of such character that the generalization to the case where the components of (31) contain also parameters μ_i is almost trivial. (In this respect compare *loc. cit.*[2], I).

With the use of this parametric generalization there follows from Existence Theorem II by means of a device[26] the following theorem on the regular transformations of the Hilbert space into itself:

EXISTENCE THEOREM III.[27] *If*

$$(31') \qquad \psi_1(z_1', z_2', \cdots), \qquad \psi_2(z_1', z_2', \cdots), \cdots$$

is a bounded vector in the domain

$$\sigma'_{r'}: \quad \sum_{k=1}^{\infty} |z_k'|^2 < r'^2$$

and if ψ_i contains no constant and no linear term there exists a sufficiently small r and a bounded vector (31) *of regular power series in σ_r such that the system*

$$(57) \qquad z_i = z_i' + \psi_i(z_1', z_2', \cdots); \qquad (i = 1, 2, \cdots)$$

possesses in σ_r the inverse transformation

$$(58) \qquad z_i' = \Phi_i(z_1, z_2, \cdots); \qquad (i = 1, 2, \cdots).$$

It follows from the Consistency Theorem IV that the restriction made in Existence Theorem III that the transformation of the Hilbert space into itself is to higher terms the identity transformation, is not an essential one, and Existence Theorem III is accordingly still valid if (57) be replaced by the more general system

$$(57') \qquad z_i = \sum_{j=1}^{\infty} a_{ij} z_j' + \psi_i(z_1', z_2', \cdots); \qquad (i = 1, 2, \cdots)$$

where $\| a_{ij} \|$ is a constant matrix (becoming in the normalized case (57) the unity matrix) which is bounded in the sense of Hilbert and which possesses a bounded reciprocal matrix, i. e. the linear approximation

$$(57a') \qquad z_i = \sum_{j=1}^{\infty} a_{ij} z_j'$$

of (57′) is a unique reversible transformation of the complex space of Hilbert into itself. In the case of finite many variables this assumption is obviously equivalent to the restriction that the Jacobian determinant be different from zero in the neighborhood of the origin.

In addition Existence Theorem II permits a corresponding generalization, i. e. Existence Theorem II remains true if we replace (54) by

$$(54') \qquad \sum_{j=1}^{\infty} a_{ij} z_j = t\,\Phi_i(z_1, z_2, \cdots); \qquad (i = 1, 2, \cdots)$$

provided that we assume the matrix $\| a_{ij} \|$ is bounded in the sense of Hilbert and possesses a bounded reciprocal $\| b_{ij} \|$.

Finally it should be mentioned that Existence Theorem III remains valid if the ψ_i depend not only upon the z_i' but also upon the z_i. The proof proceeds in exactly the same manner as for Existence Theorem III; as is also true for the generalization mentioned in connection with $(57a')$.

THE JOHNS HOPKINS UNIVERSITY.

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[1] The bibliography of this literature is to be found in the recently appearing book of F. R. Moulton, *Differential Equations*, New York, 1930, p. 375.

[2] In what follows, reference is made to the following papers: I. "Zur Theorie der unendlichen Differentialsysteme," *Mathematische Annalen*, Bd. 95 (1925), pp. 544-556; II. "Zur Lösung von Differentialsystemen mit unendlichvielen Veränderlichen," *Mathematische Annalen*, Bd. 98 (1928), pp. 273-280; III. "Zur Analyse im Hilbertschen Raume," *Mathematische Zeitschrift*, Bd. 28 (1928), pp. 457-470. References to papers, in which applications are made of the existence theorems mentioned in the text, are given in the above papers and in the paper of Martin referred to in footnote (3).

[3] M. Martin, "Upon the Existence and Non-Existence of Isoenergetic Periodic Perturbations of the Undisturbed Circular Motions in the Restricted Problem of Three Bodies," *American Journal of Mathematics*, Vol. 34 (1931), pp. 259-273.

[4] Cf. pp. 1471-1476 of the article by E. Hellinger and O. Toeplitz in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. $2\mathrm{III}_2$.

[5] Cf. E. Helly, "Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten," *Monatshefte für Mathematik und Physik*, Bd. 31 (1921), pp. 60-91.

[6] For the definitions of and the theory of convergence of the bounded linear forms (vectors) and bounded bilinear forms (matrices) of Hilbert, cf. E. Hellinger and O. Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, Bd. 69 (1910), pp. 289-330.

[7] This definition is not identical with the one incidentally proposed by Hilbert, "Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variablen," *Rendiconti del Circolo Matematico di Palermo*, Tomo 27 (1909), pp. 59-74, in which Hilbert considers the general power series not in the Hilbert space but in a more easily treated manifold.

[8] *Loc. cit.* (6), § 9, § 10.

[9] Cf. *loc. cit.* (2) III, pp. 454-455.

[10] Cf. *loc. cit.* (2) III, p. 457.

[11] Cf. *loc. cit.* (2) III, p. 455.

[12] Cf. *loc. cit.* (2) III, p. 456.

[13] The restriction (28), (28′) is not a sufficient condition for the boundedness of a matrix $\| a_{ij} \|$ since for example the condition is fulfilled for every skew-symmetric atrix and there exist skew-symmetric matrices which are not bounded.

One obtains a necessary and sufficient condition if one replaces (28) by [6]

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\zeta_i\bar{\zeta}_j \right| \leqq L.$$

[14] O. Toeplitz, "Über eine bei den Dirichletschen Reihen auftretende Aufgabe etc.," *Göttinger Nachrichten* (1913), pp. 417-432.

[15] Cf. *loc. cit.* ([9]).

[16] *Loc. cit.* ([2]) III, p. 464.

[17] W. L. Hart, "Linear Differential Equations in Infinitely Many Variables," *American Journal of Mathematics*, Vol. 39 (1917), pp. 407-424.

[18] For a more detailed discussion cf. *loc. cit* ([2]) II, p. 275-276; cf. also *loc. cit.* ([2]) III, p. 461, Fussnote [21].

[19] This domain is, in the case of finite differential systems, the one found by L. Fuchs.

[20] Cf. *loc. cit.* ([2]) I, p. 553-556.

[21] For a review on this subject cf. M. Müller, "Neuere Untersuchungen über den Fundamentalsatz in der Theorie der gewöhnlichen Differentialgleichungen," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Bd. 37 (1928), pp. 33-48.

[22] C. Carathéodory, *Vorlesungen über reelle Funktionen* (1918), pp. 665-668.

[23] In order to apply these methods it is necessary to define continuity by (16), (17) and not by (17), (18).

[24] *Loc. cit.* ([2]) III, p. 462.

[25] Cf. for example *loc. cit.* ([4]), p. 1431.

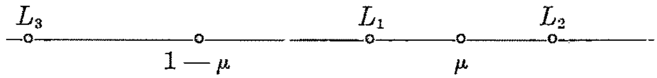
[26] Cf. *loc. cit.* ([2]) III, p. 468 or *Mathematische Annalen*, Bd. 96 (1926), p. 292 (Fussnote).

[27] Cf. a later appearing paper by Martin.

NOTE ON THE NUMERICAL VALUE OF A PARTICULAR MASS-RATIO IN THE RESTRICTED PROBLEM OF THREE BODIES.

By Jenny E. Rosenthal.*

In order to fix the positions of the collinear libration points (L_1, L_2 and L_3 in the notation of E. Strömgren †) on the line joining the masses μ and $1-\mu$ we denote ρ_1 and $\bar{\rho}_1$ the distance of L_1 from the mass μ and the mass $1-\mu$ respectively, by ρ_2 the distance of L_2 from the mass μ, and by ρ_3 the



distance of L_3 from the mass $1-\mu$. In a recent paper Martin ‡ has shown that the distances ρ_1, $\bar{\rho}_1$, ρ_2 and ρ_3 are monotone functions of μ in the interval $0 < \mu < 1$ and that they possess the following property: In this interval there exists one and only one value μ^* of μ for which

$$(1) \qquad \rho_1(\mu) \lesseqqgtr \rho^* \lesseqqgtr \rho_2(\mu) \quad \text{according as} \quad \mu \lesseqqgtr \mu^*.$$

It follows from considerations of symmetry that

$$(2) \qquad \bar{\rho}_1(\mu) \lesseqqgtr \rho^* \lesseqqgtr \rho_3(\mu) \quad \text{according as} \quad \mu \gtreqqless 1-\mu^*.$$

The equations for the determination of ρ^* and μ^* have been derived by Martin ‡ from which the author has calculated the numerical values of ρ^* and μ^* finding

$$\rho^* = 0.938933, \qquad \mu^* = 0.9992718, \qquad 1-\mu^* = 0.0007282;$$

where the results are accurate to the last significant figure indicated.

For the mass ratio of the Sun and Jupiter we have approximately $\mu = 0.9990 < \mu^*$. Since the libration points L_1, L_2 and L_3 are separated by the masses it follows from the inequalities (1) that the libration point nearest the Sun is to be found between the Sun and Jupiter.

The Johns Hopkins University.

* National Research Fellow.

† E. Strömgren, *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, Nr. 39 (1922).

‡ M. Martin, *American Journal of Mathematics*, Vol. 53 (1931), pp. 167-179.

UPON THE EXISTENCE AND NON-EXISTENCE OF ISOENERGETIC PERIODIC PERTURBATIONS OF THE UNDISTURBED CIRCULAR MOTIONS IN THE RESTRICTED PROBLEM OF THREE BODIES.

By Monroe Martin.

Introduction. In the restricted problem of three bodies if we denote the masses of the two bodies, rotating in concentric circles about their center of mass, by μ and $1-\mu$ then for $\mu = 0$ a possible orbit for the third body is a circular motion about the non-zero mass. In this paper we treat the following problem: Does there exist, for sufficiently small, positive values of μ, and for a given value of the constant of relative energy (Jacobian constant), an isoenergetic † series of periodic solutions of the equations of motion, the members of which converge to the circular motion mentioned above, and the periods of which converge to the period of this circular motion, when the parameter μ of the series converges to zero? Denoting by n the angular velocity, in the non-rotating coördinate sytem, of the third body in its circular motion for $\mu = 0$, Levi-Civita,‡ and Birkhoff,§ employing the methods of Poincaré,¶ have shown that, when n is not the ratio of two successive integers, such an isoenergetic series of periodic orbits actually exists. For n the ratio of two successive integers the problem is a resonance problem and neither the existence or non-existence of this isoenergetic series of periodic orbits has as yet been mathematically demonstrated.|| The existence of this isoenergetic series for these critical values of n requires the vanishing of

† Isoenergetic is here taken to mean that the Jacobian constant is independent of μ. For the corresponding isoperiodic case in which the Jacobian constant is a function of μ, cf. A. Wintner, "über eine Revision der Sortentheorie des restringierten Dreikörperproblems," *Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig*, Vol. 82 (1930), pp. 3-56.

‡ T. Levi-Civita, "Sopra alcuni criteri di instabilità," *Annali di Matematica* (3), Vol. 5 (1901), pp. 282-289.

§ G. D. Birkhoff, "The Restricted Problem of Three Bodies," *Rendiconti del Circolo Matematico di Palermo*, Vol. 39 (1915), § 11; and also, "Dynamical Systems," *American Mathematical Society Colloquium Publications*, Vol. 9, New York (1927), pp. 139-143.

¶ H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1 (1892), pp. 79-119.

|| Concerning a paper of Poincaré in the *Bulletin Astronomique*, cf. § 6, A. Wintner, *loc. cit.*

certain expressions, the vanishing of which expresses the fulfillment of a type or orthogonality condition (Verzweigungsgleichungen).† It will be demonstrated that for sufficiently small values of μ this condition is not fulfilled, and consequently the isoenergetic series of periodic orbits cannot exist for these critical values of n. The mathematical apparatus employed differs from that employed by Levi-Civita and Birkhoff, the method here being that of the infinitely many variables,‡ as developed for the problems of celestial mechanics in the papers of Wintner.§ In order to put in evidence the method used in the proof of the non-existence of an isoenergetic series for the critical values of n, it is necessary to present the existence proof, with the method of infinitely many variables, for the non-critical values of n.

1. *The non-linear differential equation of the normal perturbation.* If the origin of the rotating system of coördinates be taken at the mass $1-\mu$, the Lagrangian function for the third mass may be written

$$L \equiv \tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+(x\dot{y}-\dot{x}y)+F(x,y;\mu), \tag{1}$$

where we have placed

$$F(x,y;\mu) \equiv \tfrac{1}{2}[(x-\mu)^2+y^2] + \mu[(x-1)^2+y^2]^{-\frac{1}{2}}+(1-\mu)(x^2+y^2)^{-\frac{1}{2}}.$$

The dot is used here and throughout the paper to denote differentiation with respect to the time. In order to obtain the differential equation of the normal displacement we introduce polar coördinates ρ and τ and it will be convenient to place $\rho = a+\zeta$, accordingly

$$x = (a+\zeta)\cos\tau, \qquad y = (a+\zeta)\sin\tau. \tag{2}$$

The Lagrangian function and F when expressed in terms of the new

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† Cf. the papers referred to in the footnotes § and ‡ below.

‡ For another method developed by Lichstenstein in order to treat non-linear boundary value problems with the use of Greenian matrices and successive approximations and which would be in the present case equivalent, cf. L. Lichstenstein, "Zur Maxwellschen Theorie der Saturnringe," *Mathematische Zeitschrift* 17 (1923), pp. 62-110, and the further developments of this method by E. Hölder, "Mathematische Untersuchungen zur Himmelsmechanik," *Mathematische Zeitschrift*, Vol. 31 (1930), pp. 225-239.

§ Cf. for instance A. Wintner, "Über die Differentialgleichungen der Himmelsmechanik," *Mathematische Annalen*, 96 (1926), pp. 284-312.

coördinates ζ and τ will be denoted by L^* and F^* respectively. On expressing (1) in terms of ζ and τ by means of (2) we obtain

$$\begin{aligned} L^* &\equiv \tfrac{1}{2}[\dot\zeta^2 + (a+\zeta)^2\dot\tau^2] + (a+\zeta)^2\dot\tau + F^*(\zeta, \tau; \mu, a), \\ F^*(\zeta, \tau; \mu, a) &\equiv \tfrac{1}{2}(a+\zeta)^2 + (a+\zeta)^{-1} + \mu^2/2 \\ &+ \mu\{[1 - 2a\cos\tau + a^2 + 2(a - \cos\tau)\zeta + \zeta^2]^{-\frac{1}{2}} - (a+\zeta)\cos\tau - (a+\zeta)^{-1}\} \end{aligned} \tag{3}$$

from which the Lagrangian equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot\zeta} - \frac{\partial L^*}{\partial \zeta} = 0, \quad \frac{d}{dt}\frac{\partial L^*}{\partial \dot\tau} - \frac{\partial L^*}{\partial \tau} = 0, \tag{4}$$

which possess the Jacobian integral

$$\dot\zeta^2 + (a+\zeta)^2\dot\tau^2 = 2F^* - C, \tag{5}$$

may be derived.

For $\mu = 0$ equations (4) take the following simple form:

$$\ddot\zeta - (1+\dot\tau)^2(a+\zeta) + (a+\zeta)^{-2} = 0, \qquad \frac{d}{dt}[(a+\zeta)^2(1+\dot\tau)] = 0, \tag{6}$$

and possess the solution

$$\zeta \equiv 0, \qquad \tau = (n-1)t + \tau_0, \tag{7}$$

in which n (the angular velocity of the third mass in the non-rotating coördinate system) is determined from a, the radius of the circular orbit, by the third law of Kepler

$$n^2a^3 = 1. \tag{8}$$

Putting $\mu = 0$ in the Jacobian integral (5) and inserting the solution (7), the Jacobian constant C, for the circular orbits (7) is obtained in terms of a, the radius of the circular orbit, as follows:

$$C = 2a^{\frac{1}{2}} + 1/a. \tag{9}$$

For a given value of the Jacobian constant C we may employ (5) to replace (3) by a new Lagrangian function Λ, homogeneous of degree one in $\dot\zeta$ and $\dot\tau$, namely

$$\Lambda \equiv \{[2F^* - C][\dot\zeta^2 + (a+\zeta)^2\dot\tau^2]\}^{\frac{1}{2}} + (a+\zeta)^2\dot\tau. \tag{10}$$

Since Λ is homogeneous of degree one in $\dot\zeta$ and $\dot\tau$ the Lagrangian equations

$$\frac{d}{dt}\frac{\partial \Lambda}{\partial \dot\zeta} - \frac{\partial \Lambda}{\partial \zeta} = 0, \quad \frac{d}{dt}\frac{\partial \Lambda}{\partial \dot\tau} - \frac{\partial \Lambda}{\partial \tau} = 0, \tag{11}$$

are invariant under a change of independent variable

$$t = t(\theta), \qquad d\tau/d\theta > 0. \tag{12}$$

Having decided upon a suitable choice for the independent variable θ the principle of Maupertuis † states that if we write the solutions of (11) with θ as independent variable in the form

$$\zeta = \zeta(\theta), \qquad \tau = \tau(\theta), \tag{13}$$

then every solution of (4) for the given value of the Jacobian constant mentioned above, can be written in the form

$$\zeta = \zeta(\theta(t)), \qquad \tau = \tau(\theta(t)). \tag{14}$$

The function $\theta(t)$ is the inverse of the function (12) and is obtained by a quadrature from

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2F^*(\zeta, \tau; \mu, a) - C}{\left(\frac{d\zeta}{d\theta}\right)^2 + (a+\zeta)^2\left(\frac{d\tau}{d\theta}\right)^2} \tag{15}$$

Since θ is arbitrary we may take $\theta = \tau$, that is we now regard τ, originally a dependent variable, as the independent variable and seek to express the variable ζ as a function of it. The Lagrangian function, with τ the independent variable, will be denoted by Λ^* and we have from (10)

$$\Lambda^* = \{[2F^* - C][\zeta'^2 + (a+\zeta)^2]\}^{1/2} + (a+\zeta)^2; \qquad ' = d/d\tau. \tag{16}$$

The Lagrangian equation, the solution of which yields ζ as a function of τ and which will be called the equation of the normal perturbation is accordingly

$$\frac{d}{d\tau}\frac{\partial\Lambda}{\partial\zeta'} - \frac{\partial\Lambda}{\partial\zeta} = 0. \tag{17}$$

Equation (15) takes the form

$$\frac{d\tau}{dt} = \phi(\tau) \tag{18}$$

and serves to determine τ as a function of t.

It is clear, since the principle of Maupertius proceeds on the assumption that the Jacobian constant has a given value, that in order to obtain an isoenergetic series (with μ as parameter) of solutions of (4), the members of which converge to the circular motion (7) as μ converges to zero, it is not only sufficient but also necessary to obtain the solutions of (17) and (18).

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† G. D. Birkhoff, "Dynamical Systems with Two Degrees of Freedom," *Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 202-204 or F. D. Murnaghan, "The Principle of Maupertuis," *Proceedings of the National Academy of Sciences*, Vol. 17 (1931), pp. 128-132.

If one is not interested in the history of the third mass in its motion, but only the form of its orbit, equation (18) may be discarded.

2. *The calculation of the equation of the normal perturbation.* We shall now treat the equation of the normal perturbation in some detail. If we substitute the expression for Λ^* given by (16) in equation (17) we obtain

$$(19) \qquad \zeta'' - \frac{2[(a+\zeta)^2+\zeta'^2]^{3/2}}{(a+\zeta)(2F^*-C)^{1/2}}$$

$$+ \frac{F_\tau^*\,[(a+\zeta)^2+\zeta'^2]\zeta' - F_\zeta^*\,[(a+\zeta)^2+\zeta'^2]\,[a+\zeta]^2}{(2F^*-C)(a+\zeta)^2} - \frac{2\zeta'^2}{a+\zeta} = 0,$$

or

$$(19') \qquad \zeta'' = \Phi(\mu, \zeta, \zeta', \cos\tau\,; \mu, a),$$

where we understand $F_\tau^* = \partial F^*/\partial\tau$ and $F_\zeta^* = \partial F^*/\partial\zeta$.

We treat, once and for all, only those values of a for which

$$(20) \qquad 0 < a < \infty, \quad a \neq 1.$$

We have then

$$(21) \qquad 1 - 2a\cos\tau + a^2 \geqq (1-a)^2 > 0\,; \qquad -\infty < \tau < \infty,$$

and from (9) and (3)

$$(22) \quad 2F^*(0, \tau\,; 0, a) - C \equiv a^2(a^{-3/2}-1)^2 > 0\,; \qquad -\infty < \tau < +\infty.$$

We wish to emphasize here, that while Φ is a function of a, the value of a is fixed, inasmuch as the Jacobian constant C is assumed to have been given [cf. (9)]. Since a is fixed it follows from (19′), (19), (20), (21), (22) that the function Φ is regular for sufficiently small values of $|\zeta|$, $|\zeta'|$ and $|\mu|$ and for all real values of τ. Consequently there exists two positive numbers R and K such that in the domain

$$(23) \qquad |\zeta| < R, \quad |\zeta'| < R, \quad |\mu| < R, \quad -\infty < \tau < +\infty,$$

we have

$$(24) \qquad \left| \frac{\partial^2\Phi}{\partial\tau^2} \right| < K$$

and the convergent developments

$$(25) \qquad \Phi = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{jkl}\,\mu^j\zeta^k\zeta'^l,$$

$$(26) \qquad \frac{\partial^2\Phi}{\partial\tau^2} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{d^2A_{jkl}}{d\tau^2}\,\mu^j\zeta^k\zeta'^l,$$

where from (19) the coefficients A_{jkl} of (25) are obviously real and even functions of τ and can be expanded in a regular Fourier series of the form

(27) $$A_{jkl} = \sum_{\nu=-\infty}^{\infty} A_{jkl}^{(\nu)} e^{\nu\tau i}, \qquad A_{jkl}^{(\nu)} = A_{jkl}^{(-\nu)} \gtreqless 0; \quad (j, k, l = 0, 1, 2, \cdots).$$

The inequality (24) for the function (26) being valid it follows by Cauchy's theorem on convergent power series that in the domain (23) we have

(28) $$\left| \frac{d^2 A_{jkl}}{d\tau^2} \right| < KR^{-(j+k+l)}; \qquad (j, k, l = 0, 1, 2, \cdots).$$

Since (27) is a regular Fourier series, it follows on differentiation term by term

(29) $$\frac{d^2 A_{jkl}}{d\tau^2} = -\sum_{\nu=-\infty}^{\infty} \nu^2 A_{jkl}^{(\nu)} e^{\nu\tau i}; \qquad (j, k, l = 0, 1, 2, \cdots).$$

Since the absolute values of the Fourier coefficients in the development of a periodic function are not greater than the maximum of the absolute value of the function we have from (28) and (29) †

(30) $$| A_{jkl}^{(\nu)} | < \nu^{-2} KR^{-(j+k+l)}; \quad (j, k, l = 0, 1, 2, \cdots; \nu = 0, \pm 1, \pm 2, \cdots).$$

We now proceed to calculate explicitly the following coefficients in the expansion (25):

(31) $$A_{jkl} \quad \text{for} \quad j, k, l \leqq 1.$$

The labor of this calculation is lessened if instead of proceeding directly from (19) we develope the Lagrangian function Λ^* according to powers of μ, ζ and ζ' and obtain the coefficients of the terms ζ, ζ^2, ζ'^2 and $\mu\zeta$ (since Λ^* is a function of ζ'^2 it is clear at once that the coefficients of ζ', $\mu\zeta'$ and $\zeta\zeta'$ are zero). On taking the Lagrangian derivative of the development of Λ^* the coefficients (31) may be obtained immediately. From (16) and (3) we obtain

(32) $$\Lambda^* = [(A + B\mu + \mu^2)(a^2 + 2a\zeta + \zeta^2 + \zeta'^2)]^{1/2} + (a + \zeta)^2,$$

where we have placed

(33) $$A = (a + \zeta)^2 + 2(a + \zeta)^{-1} - C,$$

(34) $$B = 2[r^2 + 2(a - \cos\tau)\zeta + \zeta^2]^{-1/2} - 2(a + \zeta)\cos\tau - 2(a + \zeta)^{-1},$$

in which r denotes the distance between the mass μ and the third mass, i. e.

(35) $$r^2 = 1 - 2a\cos\tau + a^2.$$

If we develop A and B (for sufficiently small $|\zeta|$) according to powers of ζ, and replace C by (9), we obtain for the first terms mentioned above in the development of A and B respectively:

† If we agree to write $1/0 = 1$ the validity of the inequalities (30) for $\nu = 0$ follows directly from (25).

(36) $$A_2 = (a^{-\frac{1}{2}} - a)^2 + 2(a - a^{-2})\zeta + (1 + 2a^{-3})\zeta^2$$

(37) $$B_1 = 2(r^{-1} - a^{-1} - a\cos\tau) + 2[a^{-2} + (\cos\tau - a)r^{-3} - \cos\tau]\zeta.\dagger$$

As a first step in the development of (32) according to powers of μ, ζ and ζ' we have (for sufficiently small $|\zeta|$ and $|\zeta'|$)

(38) $$\Lambda^* = a(A + B\mu + \mu^2)^{\frac{1}{2}}(1 + a^{-1}\zeta + a^{-2}\zeta'^2/2) + (a + \zeta)^2 + \Lambda_1^*,$$

where Λ_1^* contains terms of order higher than the second in ζ and ζ'. On developing (for sufficiently small $|\mu|$) according to powers of μ we have on replacing A by A_2 and B by B_1

(39) $$\Lambda^* = aA_2^{\frac{1}{2}}(1 + a^{-1}\zeta + \tfrac{1}{2}a^{-2}\zeta'^2) + (a + \zeta)^2 + \tfrac{1}{2}aA_2^{-\frac{1}{2}}B_1(1 + a^{-1}\zeta + \tfrac{1}{2}a^{-2}\zeta'^2)\mu + \Lambda_2^*,$$

where Λ_2^* contains terms of order higher than the second in μ, ζ and ζ'. From (36) we have

(40) $$A_2^{\frac{1}{2}} = \mathbf{A}\{1 + a^{-2}\mathbf{A}^{-2}(a^3 - 1)\zeta + \tfrac{1}{2}[(a^3 + 2)a\mathbf{A}^2 - (a^3 - 1)^2]a^{-4}\mathbf{A}^{-4}\zeta^2\} + \cdots,$$

where we have written

(41) $$\mathbf{A} = a^{-\frac{1}{2}} - a.$$

From (37) and (40)

(42) $$\tfrac{1}{2}aA_2^{-\frac{1}{2}}B_1 = \mathbf{A}^{-1}\{ar^{-1} - a^2\cos\tau - 1 + [a^{-1} + ar^{-3}(\cos\tau - a) - a\cos\tau - (ar^{-1} - a^2\cos\tau - 1)(a^3 - 1)a^{-2}\mathbf{A}^{-2}]\zeta\} + \cdots.$$

Equation (39) may now be written

(43) $$\Lambda^* = \tfrac{1}{2}a^{-1}\mathbf{A}\zeta'^2 + \tfrac{1}{2}[2a^3\mathbf{A}^3 + 3a^4\mathbf{A}^2 - (a^3 - 1)^2]a^{-3}\mathbf{A}^{-3}\zeta^2 + [a^3 + a\mathbf{A}^2 + 2a^2\mathbf{A} - 1]a^{-1}\mathbf{A}^{-1}\zeta$$
$$+ [(a\mathbf{A}^2 - a^3 + 1)ar^{-1} + (\cos\tau - a)a^3\mathbf{A}^2r^{-3} + a^2(a^3 - 2a\mathbf{A}^2 - 1)\cos\tau + a^3 - 1]a^{-2}\mathbf{A}^{-3}\mu\zeta + \Lambda_3^*,$$

where Λ_3^* contains terms of higher order than the second in μ, ζ and ζ', together with unessential terms independent of ζ and ζ'. On multiplying (43) throughout by $a\mathbf{A}^{-1}$ we obtain

(44) $$a\mathbf{A}^{-1}\Lambda^* = \Lambda^*_0 + a\mathbf{A}^{-1}\Lambda^*_3,$$

where from (8) and (41)

(45) $$\Lambda^*_0 \equiv \tfrac{1}{2}\zeta'^2 - \tfrac{1}{2}n^2(n-1)^{-2}\zeta^2 + \mu\zeta A_{100}$$

† Since B_1 occurs in the expansion of Λ^* multiplied by μ [cf. (39) following] it is sufficient to give here only the linear and constant terms in the development of B_1 according to powers of ζ.

in which we have placed

$$(46)\quad A_{100} \equiv [(aA^2 - a^3 + 1)ar^{-1} + (\cos\tau - a)a^3A^2r^{-3} + a^2(a^3 - 2aA^2 - 1)\cos\tau + a^3 - 1]a^{-1}A^{-4}.$$

We now introduce the parameter of Hill

$$(47)\qquad m = (n-1)^{-1},$$

and note at this point that a, n and m by virtue of (8), (9) and (47) are all determined by the Jacobian constant C. With the exception of $m = 0$, -1 and $\pm\infty$, which have been excluded by (20), the integral values of m, namely,

$$(48)\qquad m = 1, \pm 2, \pm 3, \pm 4, \cdots,$$

correspond to the critical values of n mentioned in the introduction, and conversely. Expressing a and n in terms of m equations (45) and (46) become respectively:

$$(49)\qquad \Lambda^*_0 \equiv \tfrac{1}{2}\zeta'^2 - \tfrac{1}{2}(m+1)^2\zeta^2 + A_{100}\,\mu\zeta,$$

$$(50)\quad A_{100} \equiv m^{4/3}(m+1)^{2/3}\{2(m+1)r^{-1} - m^{2/3}(m+1)^{-2/3}(2m+3)\cos\tau + m^{2/3}(m+1)^{-2/3}r^{-3}\cos\tau - m^{4/3}(m+1)^{-4/3}r^{-3} - m^{-2/3}(m+1)^{2/3}(2m+1)\}.$$

The equation of normal perturbation (19) may now be written

$$(51)\qquad \zeta'' + (m+1)^2\zeta = \mu A_{100} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{jkl}\,\mu^j\zeta^k\zeta'^l,$$

or from (27)

$$(51')\quad \zeta'' + (m+1)^2\zeta = \mu\sum_{\nu=-\infty}^{\infty} A_{100}^{(\nu)}\,e^{\nu\tau i} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\nu=-\infty}^{\infty} A_{jkl}^{(\nu)}\,e^{\nu\tau i}\mu^j\zeta^k\zeta'^l,$$

where in (51) and (51)′

$$(52)\qquad j + k + l \geqq 2.$$

3. *The infinite system of conditions for the existence of a periodic solution of the equation of the normal perturbation.* We have seen in § 1 that in order to obtain an isoenergetic series (with μ as parameter) of solutions of the differential equations (4), the members of which converge to the circular motion (7) as μ converges to zero, it is both necessary and sufficient to consider the solutions of (17). If we are to treat the series of periodic solutions of (4) mentioned in the introduction it is necessary in addition, as we see from (2), that ζ be a periodic function of period 2π in τ, that is the period of ζ with respect to τ is independent of μ. On the other hand

the period of ζ with respect to t is determined from (18) and will obviously depend upon μ. Such a solution for ζ, real and periodic with period 2π in τ, will be sought for in the form of the Fourier development

$$\zeta(\tau)=\sum_{h=-\infty}^{\infty} z_h e^{h\tau i},$$

where it will be convenient to introduce the substitutions †

$$(53)\qquad z_h=x^3h^{-4}y_h,\qquad \mu=x^4;\qquad (h=0\pm 1,\pm 2,\cdots),$$

so that we have

$$(54)\qquad \zeta(\tau)=x^3\sum_{h=-\infty}^{\infty} h^{-4}y_h e^{h\tau i}.$$

The infinitely many unknowns y_k are, for a fixed value of the Jacobian constant C (and hence of a or m), functions of x which are to be so determined that for sufficiently small values of x the series (54) converges and is a solution of the differential equation (51), becoming identically zero for $x=0$.

We avail ourselves of the notation

$$(55)\qquad [\sum_{\nu=-\infty}^{\infty} a_\nu e^{\nu\tau i}]_j=a_j;\ddagger\qquad (j=0,\pm 1,\pm 2,\cdots)$$

and the infinite system of conditions which the y_k must satisfy in order that (54) be a solution of (51) may be made to take the form

$$(56)\quad \{h^{-2}(m+1)^2-1\}y_h=x\,\psi_h(x;y_0,y_1,y_{-1},\cdots);\quad (h=0,\pm 1,\pm 2,\cdots),$$

where we have placed in the notation (55)

$$(57)\qquad \psi_h=\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}' h^2x^{4j-4}[A_{jkl}\zeta^k\zeta'^l]_h;\S\qquad (h=0,\pm 1,\pm 2,\cdots)$$

and which can be expanded as follows:

$$(57')\qquad \psi_h=\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}'\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} h^2x^{4j-4}A_{jkl}^{(h-p)}[\zeta^k]_{p-q}[\zeta'^l]_q;$$
$$(h=0,\pm 1,\pm 2,\cdots).$$

The ψ_h are power series in the infinitely variables and we now proceed

† Here and throughout the remainder of the paper we shall retain the convention agreed upon in the footnote on page 264.

‡ If $f_1, f_2,\cdots,f_n$ be any n Fourier series we shall understand by the symbol $[f_i\cdot f_j]_k$ the coefficient of the k-th term in the Fourier series obtained by the multiplication of the two Fourier series f_i and f_j. If $c_1, c_2,\cdots c_n$ be any n constants the symbol $[\ ]_k$ possesses the property $[c_1f_1\cdot c_2f_2\cdots c_nf_n]_k=c_1c_2\cdots c_n[f_1\cdot f_2\cdots f_n]_k$.

§ The prime affixed to the summation sign indicates that the summation labels take the following values: $j=1,\ k=0,\ l=0;\ j+k+l\geqq 2$.

to establish an inequality for the best majorant † $\bar{\psi}_k$. Before doing so we state here a few facts on the multiplication of a special type of Fourier series.‡ In this connection we have the inequality §

$$(58) \qquad \sum_{k=-\infty}^{\infty} k^{-2}(l-k)^{-2} < 17l^{-2}; \qquad (l=0, \pm 1, \pm 2, \cdots).$$

The content of this inequality is that for n Fourier series f_j $(j=1, 2, \cdots n)$ where in the notation (55)

$$(59) \qquad |\,[f_j]_k\,| \leqq k^{-2}; \qquad (j=1, 2, \cdots n;\ k=0, \pm 1, \pm 2, \cdots),$$

the following inequalities exist

$$(60) \qquad |\,[f_i \cdot f_j]_k\,| < 17k^{-2}; \qquad (i, j=1, 2, \cdots n,\ k=0, \pm 1, \pm 2, \cdots),$$

and in general if $C_1, C_2, \cdots C_n$ denote any n constants whatsoever

$$(61) \qquad |\,[C_1 f_1 \cdot C_2 f_2 \cdots C_n f_n]_k\,| < |\,C_1 C_2 \cdots C_n\,|\,17^{n-1} k^{-2}; \qquad (k=0, \pm 1, \pm 2, \cdots)$$

because, for example

$$|\,[C_1 f_1 \cdot C_2 f_2]_k\,| = |\,C_1 C_2 [f_1 \cdot f_2]_k\,| < |\,C_1 C_2\,|\,17k^{-2} \quad (k=0, \pm 1, \pm 2, \cdots).$$

We now show that for the functions ψ_h defined by (57) there exists two positive numbers $\boldsymbol{a}$ and M_1 so that the best majorant $\bar{\psi}_h$ satisfies the inequality

$$(62) \qquad \bar{\psi}_h(\boldsymbol{a}; 1, 1, \cdots) < M_1; \qquad (h=0, \pm 1, \pm 2, \cdots).$$

We introduce the notation

$$(63) \qquad \zeta^* = |\,x\,|^3 \sum_{j=-\infty}^{\infty} |\,j^{-4} y_j\,|\, e^{j\tau i}, \quad \zeta'^* = |\,x\,|^3 \sum_{j=-\infty}^{\infty} |\,j^{-3} y_j\,|\, e^{j\tau i}$$

$$A^*_{jkl} = \sum_{\nu=-\infty}^{\infty} |\,A^{(\nu)}_{jkl}\,|\, e^{\nu\tau i}, \qquad (j, k, l=0, 1, 2, \cdots).$$

It follows from (61) that in the domain

$$(64) \qquad |\,y_j\,| \leqq 1; \qquad (j=0, \pm 1, \pm 2, \cdots),$$

the following inequalities are valid:

$$(65) \qquad [\zeta^{*k}]_j < j^{-2}\,|\,x\,|^{3k}\,17^{k-1}, \quad [\zeta'^{*l}]_j < j^{-2}\,|\,x\,|^{3l}\,17^{l-1}; \qquad (k=0, 1, 2, \cdots;\ j=0, \pm 1, \pm 2, \cdots)$$

† By the best majorant of a function defined by a power series we shall understand the function defined by the power series obtained on replacing the coefficients of the original power series by their absolute values.

‡ A. Wintner, *loc. cit.*, footnote § p. 260.

§ A. Wintner, "Über die Konvergenzefragen der Mondtheorie," *Mathematische Zeitschrift*, Vol. 30 (1929), pp. 219-220.

~d therefore from (30) and (63)

$$(66)\quad [A^{*}_{jkl}\,\zeta^{*k}\zeta'^{*l}]_h < \frac{K}{h^2}\;\frac{17^{k+l}\,|\,x\,|^{3(k+l)}}{R^{j+k+l}} \leqq \frac{K}{h^2}\left(\frac{17}{R}\right)^{j+k+l}|\,x\,|^{3(k+l)};$$
$$(j,k,l=0,1,2,\cdots;\ h=0,\pm1,\pm2,\cdots).$$

From (57) we have

$$(67)\quad \tilde{\psi}_h(x;y_0,y_1,\cdots)\leqq\psi^{*}_h(x;y_0,y_1,\cdots)$$
$$=\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}'\,h^2\,|\,x\,|^{4j-4}\,[A^{*}_{jkl}\,\zeta^{*k}\zeta'^{*l}]_h;\quad (h=0,\pm1,\pm2,\cdots),$$

and from (66)

$$(68)\quad \psi_h^{*}\,(x;1,1,\cdots)<\Psi(x)\equiv K\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}'\left(\frac{17\,|\,x\,|}{R}\right)^{j+k+l}|\,x\,|^{(3j+2k+2l-4)};$$
$$(h=0,\pm1,\pm2,\cdots).$$

It follows that $\Psi(x)$ is a regular power series in $|\,x\,|$. It follows by Cauchy's test that if

$$(69)\qquad |\,x\,|<\min\,(R/17,1)\dagger$$

and series $\Psi(x)$ is convergent. Consequently there exist two positive numbers $\boldsymbol{a}$ and M_1 so that (62) is valid.

4. *Existence proof for the non-critical case* $m\not\equiv 0\pmod 1$. Before entering upon the existence proof it is necessary to state an existence theorem on infinite systems.‡

THEOREM I. *Given the power series*

$$(70)\qquad f_{pq}(x_1,x_2,\cdots,x_s;\,y_1,y_2,\cdots;\,\lambda_1,\lambda_2,\cdots)$$
$$(p=1,2,\cdots,s;\ q=1,2,\cdots),$$

in infinitely many variables so that there exist positive numbers

$$(71)\qquad \alpha;\,\boldsymbol{a};\,b_1,b_2,\cdots;\,\Lambda_1,\Lambda_2,\cdots;\,M_1,M_2,\cdots,$$

for which, if $\tilde{f}_{pq}$ *denote the best majorant* § *of* f_{pq}, *the following inequalities exist*

$$(72)\qquad \alpha<b_j/M_j;\ \tilde{f}_{pj}\,(\boldsymbol{a},\boldsymbol{a},\cdots;\,b_1,b_2,\cdots;\,\Lambda_1,\Lambda_2,\cdots)\leqq M_j,$$
$$(p=1,2,\cdots s;\ j=1,2,\cdots),$$

then the system

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† The symbol min (a,b) signifies the minimum of the two numbers a and b.

‡ A. Wintner, *loc. cit.* (8), pp. 291-294.

§ Cf. footnote †, p. 268

(73) $$y_j = \sum_{p=1}^{s} x_p f_{pj}; \qquad (j = 1, 2, \cdots),$$

possesses in the domain

(74) $$|x_p| < \min(\alpha/s, \boldsymbol{a}); \quad |\lambda_k| < \Lambda_k; \quad (p = 1, 2, \cdots s;\ k = 1, 2, \cdots),$$

one and only one solution $y_j(x_1, x_2, \cdots, x_s;\ \lambda_1, \lambda_2, \cdots)$. *This solution is holomorphic in the domain* (74) *and satisfies the inequalities*

(75) $$|y_i(x_1, x_2, \cdots, x_s;\ \lambda_1, \lambda_2, \cdots)| < b_i \qquad (i = 1, 2, \cdots),$$

and of course

(76) $$y_i(0, 0, \cdots 0;\ \lambda_1, \lambda_2, \cdots) \equiv 0 \qquad (i = 1, 2, \cdots).$$

If the power series (70) *and the arguments are real the solution* $y_j(x_1, x_2, \cdots x_s;\ \lambda_1, \lambda_2, \cdots)$ *is also real.*

If we place $s = 1$, $M_i = M$, $b_i = b$, $(i = 1, 2, \cdots)$ and reject the parameters λ we have the following special case of Theorem I:

THEOREM Ia. *Given the power series* $f_j(x; y_1, y_2, \cdots)$ *in infinitely many variables so that there exist three positive numbers*

(77) $$\boldsymbol{a},\ b,\ M,$$

for which, if $\bar{f}_j$ *denote the best majorante of* f_j, *the following inequalities exist*

(78) $$\bar{f}_j(\boldsymbol{a};\ b, b, \cdots) < M \qquad (j = 1, 2, \cdots),$$

then the system

(79) $$y_i = x f_i; \qquad (i = 1, 2, \cdots),$$

possesses in the domain

(80) $$x < \min\,(b/M, \mathbf{a}),$$

one and only one solution $y_j(x)$. *This solution is holomorphic in the domain* (80) *and satisfies the inequalities*

(81) $$|y_j(x)| < b; \qquad (j = 1, 2, \cdots),$$

and of course

(82) $$y_j(0) = 0; \qquad (j = 1, 2, \cdots).$$

If the power series $f_j(x; y_1, y_2, \cdots)$ *and the arguments are real the solution* $y_j(x)$ *is also real.*

If $m \not\equiv 0 \pmod 1$ the infinite system (56) can be written

(83) $$y_j = x\, f_i\,(x;\ y_0, y_1, y_{-1}, \cdots); \qquad (j = 0, \pm 1, \pm 2, \cdots),$$

where we have placed

$$(84)\qquad f_j=[(m+1)^2/j^2-1]^{-1}\psi_j; \qquad (j=0,\pm 1,\pm 2,\cdots),$$

and since $m\not\equiv 0 \pmod 1$ there exists a positive number M_2 so that

$$(85)\qquad |[(m+1)^2/j^2-1]^{-1}|<M_2; \qquad (j=0,\pm 1,\pm 2,\cdots).$$

We have from (62), (84) and (85)

$$(86)\qquad \bar{f}_j(a;\, 1, 1,\cdots)<M; \qquad (j=0,\pm 1,\pm 2,\cdots),$$

on putting $M=M_1M_2$. From Theorem Ia and (86) it follows that the infinite system (83) possesses in the domain

$$(87)\qquad |x|<\min(1/M, a)=\delta,$$

one and only one solution $y_j(x)$. This solution is holomorphic in the domain (87) and in this domain satisfies the inequalities

$$(88)\qquad |y_j(x)|<1,\quad y_j(0)=0; \qquad (j=0,\pm 1,\pm 2,\cdots).$$

Accordingly from (53) and (88) the series (54) for $\zeta(\tau)$ is convergent for $|\mu|<\delta^4$ and $\zeta(\tau)$ converges identically to zero as μ converges to zero.

In order to show that the solution (54) is real for $\mu>0$ it is only necessary to write (54) in terms of sines and cosines i. e.

$$(54')\qquad \zeta(x)=b_0+\sum_{\nu=0}^{\infty}(a_\nu \sin\nu\tau+b_\nu\cos\nu\tau),$$

(where the coefficients a_ν and b_ν are real) and to obtain by substitution in the differential equation (51) a real infinite system of conditions for the a_ν and b_ν. That a unique solution of this infinite system exists for which (54′) converges (becoming identically zero for $\mu=0$) follows since (54′) is only formally different from (54). Furthermore since the infinite system for the a_ν and b_ν is real it follows from the existence theorem on infinite systems stated above that this infinite system possesses a real solution for the a_ν and b_ν. The existence of a real solution (54) is accordingly demonstrated.

5. *The critical case*, $m\equiv 0 \pmod 1$. Let m be a fixed integer.† The infinite system then takes the form

$$(89a)\quad y_j=x\,f_j(x;\, y_0, y_1, y_{-1},\cdots);\quad (j\neq\pm(m+1), j=0,\pm 1,\pm 2,\cdots),$$

$$(89b)\quad 0=A_{100}^{(m+1)}(m)+\Theta_{(m+1)}(x;\, y_0, y_1, y_{-1},\cdots),$$

† Where we have excluded $m=0,\,-1,\,\pm\infty$ [cf. (48)].

(89c) $0 = A_{100}^{-(m+1)}(m) + \Theta_{-(m+1)}(x;\ y_0, y_1, y_{-1}, \cdots),$

where we have put

(90) $$\Theta_{\pm(m+1)} \equiv \psi_{\pm(m+1)} - A_{100}^{\pm(m+1)}(m).$$

It follows from (62), (50) and (90) that there exist two positive numbers $\boldsymbol{a}$ and M_3 so that

(91) $$\bar{\Theta}_{\pm(m+1)}(\boldsymbol{a};\ 1, 1, \cdots) < M_3.$$

The power series $\Theta_{\pm(m+1)}$ in infinitely many variables are therefore uniformly convergent in the domain of infinitely many dimensions

(92) $$|x| < \boldsymbol{a}, \qquad |y_j| < 1; \qquad (j = 0, \pm 1, \pm 2, \cdots).$$

Moreover it follows from the definition of $\Theta_{\pm(m+1)}$ and (54), (55), (57′) and (52) that in the domain of infinitely many dimensions

(93) $$|y_j| < 1; \qquad (j = 0, \pm 1, \pm 2, \cdots),$$

we have

(94) $$\Theta_{\pm(m+1)}(0;\ y_0, y_1, y_{-1}, \cdots) \equiv 0.$$

If we now regard the variables y_{m+1} and $y_{-(m+1)}$ in the infinite system (89a) as parameters λ_1 and λ_2 it follows from (86) by Theorem I (§ 4) that in the domain

(95) $$|x| < \min(1/M, \boldsymbol{a}), \qquad |y_{(m+1)}| < 1, \qquad |y_{-(m+1)}| < 1,$$

the system (89a) possesses one and only one solution

(96) $$y_j = y_j(x;\ y_{(m+1)}, y_{-(m+1)}); \quad (j \neq \pm(m+1),\ j = 0, \pm 1, \pm 2, \cdots).$$

This solution is holomorphic and fulfills in the domain (95) the inequalities

(97) $$|y_j(x;\ y_{(m+1)}, y_{-(m+1)}| < 1 \quad (j \neq \pm(m+1),\ j = 0, \pm 1, \pm 2, \cdots).$$

It follows that the functions

(98) $$\theta_{\pm(m+1)}(x, y_{(m+1)}, y_{-(m+1)}) \equiv$$
$$\Theta_{\pm(m+1)}(x; y_0(x; y_{(m+1)}, y_{-(m+1)}), y_1(x;\ y_{(m+1)}, y_{-(m+1)}), y_{-1}(x;\ y_{(m+1)},\ y_{(m+1)}),$$

are holomorphic in the domain (95) of the three complex variables $y_{(m+1)}$ $y_{-(m+1)}$ and x, the power series $\Theta_{\pm(m+1)}$ in the infinitely many variables being uniformly convergent in the domain of infinitely many dimensions (92).

We can now write the remaining conditions (89b) and (89c) in the form

(89b′) $$A_{100}^{(m+1)}(m) + \theta_{(m+1)}(x; y_{(m+1)}, y_{-(m+1)}) = 0,$$

(89c′) $$A_{100}^{-(m+1)}(m) + \theta_{-(m+1)}(x; y_{(m+1)}, y_{-(m+1)}) = 0.$$

These are the so-called Verzweigungsgleichungen † and (if there exists a solution) serve to determine $y_{(m+1)}$ and $y_{-(m+1)}$ as functions of x. The remaining y_j can then, from (96), be expressed as functions of x; conversely if these equations possess no solution for $y_{(m+1)}$ and $y_{-(m+1)}$ there exists no solution of the differential equation (51) of the desired character. Now it follows from (94) and (98) that $\theta_{\pm(m+1)}(0, y_{(m+1)}, y_{(m+1)}) \equiv 0$. Accordingly if the integer m is such that the constants $A_{100}^{\pm(m+1)}(m)$ in (89b′), (89c′) are not zero there exists for sufficiently small values of x no solution of (89b′), (89c′) and hence no periodic solution of the desired character.

From (50) and (35) it is clear that $A_{100}^{\pm(m+1)}(m)$ are linear combinations, the coefficients of which are functions of m, of the coefficients of Laplace (which themselves are functions of m). Accordingly, for a given value of m, the numerical values of $A_{100}^{\pm(m+1)}(m)$ can be calculated from tables of the Laplace coefficients.‡ The author has calculated § these coefficients for $|m| \leqq 4$ and obtains

$$(99) \qquad A_{100}^{\pm(m+1)}(m) \neq 0 \qquad (m = 1, \pm 2, \pm 3, \pm 4),$$

which is sufficient to show that for sufficiently small values of x the equations (89b′) (89c′) possess no solution for $y_{(m+1)}$ and $y_{-(m+1)}$.

THE JOHNS HOPKINS UNIVERSITY.

† Cf. footnote †, p. 260.

‡ Tables giving the numerical values for the coefficients of Laplace for various values of a (and therefore of m) are given for instance by Runkle. Not having access to these tables, recourse was had to a memoir of Leverrier in the second volume of the *Annales de l'Observatoire de Paris*, in which he gives recursion formulas for the coefficients of Laplace and their well known expansion in a hypergeometric series according to powers of a, in a form suitable for calculation.

§ The inequality (99) can be established for very large values of m by means of well known asymptotic methods.

NOTE ON THE CONSTANTS OF THE DISTURBING FUNCTION.

By K. P. Williams.

The method for obtaining the quantities $c_n^{(i,j)}$ that appear in Newcomb's development of the disturbing function that was given in this journal recently * had in view primarily the case where the ratio $s = a/a'$ is large. For small values of s the work can be greatly shortened by using the following method for finding the $c_1^{(i,j)}$, $j > 0$.

Making use of (16) and the series (19) for H, we have

$$\frac{c_1^{(i,j+1)}}{c_1^{(i,j)}} = \frac{2(2j+1)(2j+2i+1)}{j+i+1}\left(\frac{s}{2}\right)^2 [1 + b_1 y + \cdots].$$

When the value for a_1 is used we find

$$b_1 = -1 + \frac{1+2i}{4(i+j+1)(i+j+2)}.$$

Hence

$$\frac{c_1^{(i,j+1)}}{c_1^{(i,j)}} = s^2(2j+1)M,$$

where M is a number not much different from unity.

If s is small we can easily observe by this formula, after the $c_1^{(i)}$ have been found, the lowest element $c_1^{(k,1)}$ in the second column that is not negligible. Putting $c_1^{(k+1,1)} = 0$ we have from (45)

$$c_1^{(k,1)} = s c_1^{(k+1)}.$$

The column can be completed as before. Similar remarks apply to succeding columns.

The following errata appear in the numerical series given at the end of the article cited:

The numerical coefficient in $c_1^{(9)}$ should be $2 \cdot 5 \cdot 11 \cdot 13 \cdot 17$; that of $c_1^{(10,3)}$ should be $1280 \cdot 17 \cdot 19 \cdot 21 \cdot 23 \cdot 25$.

Indiana University,
Bloomington, Indiana.

* K. P. Williams, "The Constants of the Disturbing Function," *American Journal of Mathematics*, Vol. 52, pp. 571-584. References are to this article.

CONDUCTORS IN AN ELECTROMAGNETIC FIELD (E^0e^{pt}, H^0e^{pt}).

By F. H. Murray.

The physical problem of a system of conductors in an impressed electromagnetic field of the form (E^0e^{pt}, H^0e^{pt}) leads to the well-known field equations in space and in the conductors, with boundary conditions at their surfaces. It is proposed here to develop certain formulas required in the mathematical discussion of the general problem; by means of certain vector identities it is shown that the field which must be added to the impressed field exterior to the conductors can be represented by integrals over the surfaces of the conductors, which are equivalent to a representation in terms of surface distributions of electric and magnetic doublets.* Part I is devoted to the derivation and discussion of these identities; in Part II a discussion is given of the special case of perfect conductors. The representation leads to a system of integral equations which can be reduced to the type of Fredholm, and it is shown that the exceptional values of p, for which the equations corresponding to an arbitrary impressed field do not possess a solution, must be the values which correspond to cavity-radiation for at least one conductor. While the discussion of certain points could be abbreviated by appeals to physical intuition, it appeared desirable, for possible applications to high frequencies, to give formal proofs of all propositions stated.

From the general formulas of Part I can be derived the equations for the case of wire conductors, in a high-frequency radiation field; a discussion of these equations and applications will be given in another paper.

Part I. General Formulas.

1. Let the system of surfaces $S_1, S_2 \cdots S_n$ be denoted by S; each surface is assumed to have a continuous tangent plane and to be such that if an arbitrary point (x_0, y_0, z_0) on S is given, a transformation of the coördinate axes can be made such that all points of S in some neighborhood of this point can be represented in the form $z = f(x, y)$, where f possesses continuous partial derivatives of the first three orders, and

$$0 = f(0,0) = \partial f/\partial x \,\big|_{x=y=0} = \partial f/\partial y \,\big|_{x=y=0}.$$

* Related formulae have been developed by Hasenörl, *Physikalische Zeitschrift*, Band 7 (1906), p. 37. For the special case of perfect conductors see MacDonald, *Electric Waves*, p. 15, also *Proceedings of the London Mathematical Society* (2), Vol. 10 (1911), p. 91.

Let μ, σ, ϵ denote the permeability, conductivity, and specific inductive capacity, respectively, of the medium exterior to S (exterior to every S_i), while μ', σ', ϵ' denote the constants of the interior, fixed for any conductor, but not necessarily the same for different conductors.

If the impressed and added fields are written in the form

$$\begin{array}{ll} \boldsymbol{E}^0(xyz)e^{pt}, & \boldsymbol{H}^0(xyz)e^{pt} \\ \boldsymbol{E}\ (xyz)e^{pt}, & \boldsymbol{H}\ (xyz)e^{pt} \end{array}$$

and if

$$\lambda = 1/c(4\pi\sigma + \epsilon p), \quad \kappa = -\mu p/c, \quad h = (-\kappa\lambda)^{1/2}$$

the equations of Maxwell become,

$$(1.1) \qquad \begin{array}{ll} \lambda\boldsymbol{E} = \operatorname{curl} \boldsymbol{H} & \lambda'\boldsymbol{E}' = \operatorname{curl} \boldsymbol{H}' \\ \kappa\boldsymbol{H} = \operatorname{curl} \boldsymbol{E} & \kappa'\boldsymbol{H}' = \operatorname{curl} \boldsymbol{E}' \end{array}$$

primes denoting interior values. The components of $\boldsymbol{E}$, $\boldsymbol{H}$ satisfy the wave equation

$$(1.2) \qquad (\Delta - h^2)\phi = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - h^2)\phi = 0.$$

This equation has the fundamental solution

$$\phi = (1/r)e^{-hr}, \qquad r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}.$$

The real part of h is assumed positive or zero; it is at once seen that if p lies on the right of the imaginary axis in its complex plane, a branch of h which is positive when p is positive real will have its real part positive and this branch of the function will be used throughout. The boundary conditions on S are that the tangential components of $\boldsymbol{E}^0 + \boldsymbol{E}$ are equal to the corresponding tangential components of $\boldsymbol{E}'$, similarly, $(\boldsymbol{H}^0 + \boldsymbol{H})_{\text{tang.}} = \boldsymbol{H}'_{\text{tang.}}$.

2. Let Σ be the surface of a large sphere enclosing S, while γ is the surface of a small sphere enclosing a point $(x_0y_0z_0)$ exterior to S, but interior to Σ; if $\boldsymbol{u}$, $\boldsymbol{v}$ are two vectors which are continuous, with their partial derivatives of the first two orders in the interior of Σ, exterior to S and γ, we have the identity *

$$[\boldsymbol{u}, \operatorname{curl} \boldsymbol{v}]_n + \boldsymbol{u}_n \operatorname{div} \boldsymbol{v} - [\boldsymbol{v}, \operatorname{curl} \boldsymbol{u}]_n - \boldsymbol{v}_n \operatorname{div} \boldsymbol{u} = \boldsymbol{u} d\boldsymbol{v}/dn - \boldsymbol{v} d\boldsymbol{u}/dn + \operatorname{curl}_n [\boldsymbol{u}, \boldsymbol{v}].$$

From Green's formula in vector notation

$$\iiint_{(S+\Sigma+\gamma)} \{\boldsymbol{u}\Delta\boldsymbol{v} - \boldsymbol{v}\Delta\boldsymbol{u}\}\, dx\, dy\, dz = -\iint_{S+\Sigma+\gamma} (\boldsymbol{u} d\boldsymbol{v}/dn - \boldsymbol{v} d\boldsymbol{u}/dn)\, dS.$$

* Weyl, "Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spectralgesetze, *Crelle's Journal*, Band 143 (1913), p. 182.

Also,

$$0 = \iint_{S+\Sigma+\gamma} \operatorname{curl}_n [\boldsymbol{u}, \boldsymbol{v}]\, dS$$

consequently

$$\iiint_{(S+\Sigma+\gamma)} \{\boldsymbol{u}\Delta \boldsymbol{v} - \boldsymbol{v}\Delta \boldsymbol{u}\}\, dx\, dy\, dz = \iint_{S+\Sigma+\gamma} \left\{ \begin{array}{l} -[\boldsymbol{u}, \operatorname{curl} \boldsymbol{v}]_n - \boldsymbol{u}_n \operatorname{div} \boldsymbol{v} \\ +[\boldsymbol{v}, \operatorname{curl} \boldsymbol{u}]_n + \boldsymbol{v}_n \operatorname{div} \boldsymbol{u} \end{array} \right\} dS.$$

If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of the equation (1.2), the integral on the left vanishes; if in addition the components of $\boldsymbol{u}$ have continuous partial derivatives of the first two orders interior to γ, while $\boldsymbol{v}$ is one of the vectors

$$\boldsymbol{v}' = (\phi, 0, 0), \quad \boldsymbol{v}'' = (0, \phi, 0), \quad \boldsymbol{v}''' = (0, 0, \phi),$$

from Stokes' theorem

$$\iint_{\gamma} \operatorname{curl}_n [\boldsymbol{u}, \boldsymbol{v}]\, dS = 0.$$

Consequently, if the vectors above are substituted in succession in the identity

$$-\iint_{\gamma} \operatorname{curl}_n [\boldsymbol{u}, \boldsymbol{v}]\, dS - \iint_{\gamma} \left(\boldsymbol{u}\frac{d\boldsymbol{v}}{dn} - \boldsymbol{v}\frac{d\boldsymbol{u}}{dn}\right) dS$$
$$= \iint_{S+\Sigma} \left\{ \begin{array}{l} [\boldsymbol{u}, \operatorname{curl} \boldsymbol{v}]_n + \boldsymbol{u}_n \operatorname{div} \mathrm{v} \\ -[\boldsymbol{v}, \operatorname{curl} \boldsymbol{u}]_n - \boldsymbol{v}_n \operatorname{div} \mathrm{u} \end{array} \right\} dS$$

and the radius of γ is made to approach zero, the result is,

$$4\pi \boldsymbol{u}_x = \iint_{S+\Sigma} \{[\boldsymbol{n}, \boldsymbol{u}] \cdot \operatorname{curl} \boldsymbol{v}' + \boldsymbol{u}_n \operatorname{div} \boldsymbol{v}' - [\boldsymbol{n}, \boldsymbol{v}'] \operatorname{curl} \boldsymbol{u} - \boldsymbol{v}_n' \operatorname{div} \boldsymbol{u}\} dS.$$

The components $\boldsymbol{u}_y$, $\boldsymbol{u}_z$ have the same representation in terms of $\boldsymbol{v}''$, $\boldsymbol{v}'''$, respectively.

If $\boldsymbol{u} = \boldsymbol{E}$, since $\operatorname{div} \boldsymbol{E} = 0$, one obtains from the field equations

$$4\pi E_x = \iint_{S+\Sigma} \{[\boldsymbol{n}, \boldsymbol{E}] \cdot \operatorname{curl} \boldsymbol{v}' + \boldsymbol{E}_n \operatorname{div} \boldsymbol{v}' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\boldsymbol{v}'\} dS.$$

If the real part of h is not zero, the integral over Σ approaches zero if $\boldsymbol{E}$ is bounded at infinity; if the real part of h is zero, but $h \neq 0$, while $(\boldsymbol{E}, \boldsymbol{H})$ behaves at infinity like the field of a system of diverging spherical waves

$$\lim_{R\to\infty} Re^{hR}\boldsymbol{E} = \boldsymbol{E}_1(\theta, \phi), \quad \boldsymbol{E} \sim \boldsymbol{E}_1(\theta, \phi)\,(1/R)\,e^{-hR}$$

and if the approach to the limiting value is uniform with respect to the polar angles θ, ϕ,

$$\iint_{\Sigma} \{[\boldsymbol{n}, \boldsymbol{E}] \operatorname{curl} \boldsymbol{v}' + \boldsymbol{E}_n \operatorname{div} \boldsymbol{v}' + \kappa[\boldsymbol{n}, \boldsymbol{H}] \cdot \boldsymbol{v}'\} dS \sim Ce^{-2hR}.$$

Consequently, if $C \neq 0$, $4\pi \boldsymbol{E}(x_0, y_0, z_0)$ does not approach a limit as $R \to \infty$. But since $\boldsymbol{E}$ is independent of R, $C = 0$.
Hence

$$(1.3)\quad \begin{aligned} 4\pi \boldsymbol{E}_x &= \iint_S \{[\boldsymbol{n}, \boldsymbol{E}] \cdot \operatorname{curl} \boldsymbol{v}' + \boldsymbol{E}_n \operatorname{div} \boldsymbol{v}' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\boldsymbol{v}'\} dS \\ 4\pi \boldsymbol{E}_y &= \iint_S \{[\boldsymbol{n}, \boldsymbol{E}] \cdot \operatorname{curl} \boldsymbol{v}'' + \boldsymbol{E}_n \operatorname{div} \boldsymbol{v}'' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\boldsymbol{v}''\} dS \\ 4\pi \boldsymbol{E}_z &= \iint_S \{[\boldsymbol{n}, \boldsymbol{E}] \cdot \operatorname{curl} \boldsymbol{v}''' + \boldsymbol{E}_n \operatorname{div} \boldsymbol{v}''' + \kappa[\boldsymbol{n}, \boldsymbol{H}]\boldsymbol{v}'''\} dS. \end{aligned}$$

The corresponding representation of $\boldsymbol{H}$ is obtained by interchanging $\boldsymbol{E}$ and $\boldsymbol{H}$ everywhere, while κ is replaced by λ.

If the definitions of $\boldsymbol{v}'$, $\boldsymbol{v}''$, $\boldsymbol{v}'''$ are employed, the result is,

$$(1.4)\quad \begin{aligned} 4\pi \boldsymbol{E} &= \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{E}]\, \phi dS - \operatorname{grad} \iint_S \boldsymbol{E}_n \phi dS + \kappa \iint_S [\boldsymbol{n}, \boldsymbol{H}]\, \phi dS \\ 4\pi \boldsymbol{H} &= \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{H}]\, \phi dS - \operatorname{grad} \iint_S \boldsymbol{H}_n \phi dS + \lambda \iint_S [\boldsymbol{n}, \boldsymbol{E}]\, \phi dS. \end{aligned}$$

If the point (x_0, y_0, z_0) had been taken interior to any S_i, the integral over γ could have been omitted in the discussion, and the left-hand side of (1.4) would be zero. The normal indicated above is the exterior normal; the formulas for the interior of any surface S_i are the same as above, with $\boldsymbol{n}$ indicating the interior normal, S replaced by S_i, and ϕ replaced by ϕ' constructed with the constants of the conductor.

The impressed field $(\boldsymbol{E}^0, \boldsymbol{H}^0)$ has no singularities interior to any S_i, hence if (x, y, z) is exterior to S_i,

$$0 = \operatorname{curl} \iint_{S_i} [\boldsymbol{n}', \boldsymbol{E}^0]\, \phi dS - \operatorname{grad} \iint_{S_i} \boldsymbol{E}_{n'}{}^0 \phi dS + \kappa \iint_{S_i} [\boldsymbol{n}', \boldsymbol{H}^0]\, \phi dS.$$

Replacing $\boldsymbol{n}'$ by $-\boldsymbol{n}$, and observing that the tangential components of $\boldsymbol{E}^0$, $\boldsymbol{H}^0$ are the same on both sides of S_i, $\boldsymbol{E}_{n'}{}^0 = -\boldsymbol{E}_n{}^0$, this can be written

$$0 = \operatorname{curl} \iint_{S_i} [\boldsymbol{n}, \boldsymbol{E}^0]\, \phi dS - \operatorname{grad} \iint_{S_i} \boldsymbol{E}_n{}^0 \phi dS + \kappa \iint_{S_i} [\boldsymbol{n}, \boldsymbol{H}^0]\, \phi dS.$$

Summing with respect to i and adding the result to the first equation (1.4), and repeating the process for the second equation,

(1.5)

$$\begin{aligned} 4\pi \boldsymbol{E} &= \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{E} + \boldsymbol{E}^0] \phi dS - \operatorname{grad} \iint_S (\boldsymbol{E}_n + \boldsymbol{E}_n{}^0) \phi dS + \kappa \iint_S [\boldsymbol{n}, \boldsymbol{H} + \boldsymbol{H}^0] \phi dS \\ 4\pi \boldsymbol{H} &= \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{H} + \boldsymbol{H}^0] \phi dS - \operatorname{grad} \iint_S (\boldsymbol{H}_n + \boldsymbol{H}_n{}^0) \phi dS + \lambda \iint_S [\boldsymbol{n}, \boldsymbol{E} + \boldsymbol{E}^0] \phi dS. \end{aligned}$$

From the boundary conditions

$$[\boldsymbol{n}, \boldsymbol{E} + \boldsymbol{E}^0] = [\boldsymbol{n}, \boldsymbol{E}'], \quad [\boldsymbol{n}, \boldsymbol{H} + \boldsymbol{H}^0] = [\boldsymbol{n}, \boldsymbol{H}'],$$

and the derived conditions*

$$\lambda(E_n{}^0 + E_n) = \lambda' E_n', \quad \mu(H_n{}^0 + H_n) = \mu' H_n'$$

and (1.5) becomes

$$(1.6)\quad \begin{aligned} 4\pi E &= \operatorname{curl}\iint_S [n, E']\phi dS - \operatorname{grad}\iint_S (\lambda'/\lambda) E_n'\phi dS + \kappa\iint_S [n, H']\phi dS \\ 4\pi H &= \operatorname{curl}\iint_S [n, H']\phi dS - \operatorname{grad}\iint_S (\kappa'/\kappa) H_n'\phi dS + \lambda\iint_S [n, E']\phi dS. \end{aligned}$$

3. If (x_1, y_1, z_1) represents a point on S in the formulas (1.4), while (x, y, z) denotes a point exterior to S, and if $J = [n, H]$, the condition $\operatorname{div} E = 0$ becomes,

$$-\iint_S E_n \Delta\phi dS + \kappa\iint_S (J_{x_1}\partial\phi/\partial x + J_{y_1}\partial\phi/\partial y + J_{z_1}\partial\phi/\partial z) dS = 0$$

or since $h^2 = -\kappa\lambda$,

$$\iint_S (-\lambda E_n\phi + J_{x_1}\partial\phi/\partial x_1 + J_{y_1}\partial\phi/\partial y_1 + J_{z_1}\partial\phi/\partial z_1) dS = 0.$$

On the surface S let (u, v) be isothermal Gaussian coördinates for a small part of the surface Ω bounded by an ordinary curve C; if $\phi_x = \partial\phi/\partial x$ etc., while (u, v, n) forms a right-handed system; then

$$\begin{array}{lll} \alpha' = \partial x_1/\partial u & \alpha'' = \partial x_1/\partial v & l = \partial x_1/\partial n\,|_{n=0} \\ \beta' = \partial y_1/\partial u & \beta'' = \partial y_1/\partial v & m = \partial y_1/\partial n\,|_{n=0} \\ \gamma' = \partial z_1/\partial u & \gamma'' = \partial z_1/\partial v & n = \partial z_1/\partial n\,|_{n=0} \end{array} \qquad \begin{vmatrix} \alpha' & \alpha'' & l \\ \beta' & \beta'' & m \\ \gamma' & \gamma'' & n \end{vmatrix} = 1$$

$$\begin{array}{ll} H_u = \alpha' H_{x_1} + \beta' H_{y_1} + \gamma' H_{z_1}, & H_{x_1} = \alpha' H_u + \beta'' H_v + l H_n \\ H_v = \alpha'' H_{x_1} + \beta'' H_{y_1} + \gamma'' H_{z_1}, & H_{y_1} = \beta' H_u + \alpha'' H_v + m H_n \\ H_n = l H_{x_1} + m H_{y_1} + n H_{z_1}, & H_{z_1} = \gamma' H_u + \gamma'' H_v + n H_n. \end{array}$$

Then

$$\begin{aligned} J_{x_1}\partial\phi/\partial x_1 + J_{y_1}\partial\phi/\partial y_1 + J_{z_1}\partial\phi/\partial z_1 &= H_u\partial\phi/\partial v - H_v\partial\phi/\partial u. \\ &= -\partial/\partial u(H_v\phi) + \partial/\partial v(H_u\phi) + \phi\,[\partial H_v/\partial u - \partial H_v/\partial v]. \end{aligned}$$

Consequently

$$\begin{aligned} &\iint_\Omega (J_{x_1}\partial\phi/\partial x_1 + J_{y_1}\partial\phi/\partial y_1 + J_{z_1}\partial\phi/\partial z_1) dS \\ &= \iint_\Omega [\partial/\partial v(H_u\phi) - \partial/\partial u(H_v\phi)]\, du\, dv + \iint_\Omega \phi(\partial H_v/\partial u - \partial H_u/\partial v) du\, dv \\ &= -\int_C \phi\,[H_u du + H_v dv] + \iint_\Omega \phi(\operatorname{curl} H)_n dS \end{aligned}$$

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* These equations result from the fact that the identity $E_t' = (E + E^0)_t$ implies that an identity is obtained by differentiating each side tangentially; the normal component of $(H + H^0)$ is expressed in terms of the tangential derivatives of $E + E^0$, hence the second relation. The first is obtained in the same manner.

from Stokes' theorem in the (u, v) plane. Now if each surface S_i is divided up into a number of parts Ω, each bounding arc of the curves C occurs twice, the integration being in opposite directions, while the integrand of the line integrals is independent of the choice of Gaussian coördinates; the sum of the integrals over the curves C vanishes. Hence

$$\iint_{S_i} (J_{x_1}\partial\phi/\partial x_1 + J_{y_1}\partial\phi/\partial y_1 + J_{z_1}\partial\phi/\partial z_1)\, dS = \iint_{S_i} \phi(\operatorname{curl} H)_n dS = \iint_{S_i} \lambda E_n \phi dS.$$

Hence $\operatorname{div} E = 0$; similarly, $\operatorname{div} H = 0$.

A representation of the field (E, H) in terms of surface distributions of electric and magnetic doublets results immediately. Let

$$4\pi H_1 = \operatorname{curl} \iint_S [n, H]\, \phi dS, \qquad \Phi = \iint_S E_n \phi dS.$$

The electric field corresponding to H_1 results from the field equations:

$$E_1 = 1/\lambda \operatorname{curl} H_1 = (1/4\pi\lambda)\{\operatorname{grad} \operatorname{div} \iint_S [n, H]\, \phi dS - h^2 \iint_S [n, H]\, \phi dS\}.$$

Since

$$\operatorname{div}\{-\operatorname{grad}\Phi + \kappa \iint_S [n, H]\, \phi dS\} = 0,$$

we have

$$h^2\Phi = \operatorname{div} \kappa \iint_S [n, H]\, \phi dS. \tag{3.1}$$

Hence

$$4\pi E_1 = -\operatorname{grad}\Phi + \kappa \iint_S [n, H]\, \phi dS.$$

Similarly, if

$$\psi = \iint_S H_n \phi dS,$$

the vectors

$$4\pi E_2 = \operatorname{curl} \iint_S [n, E]\, \phi dS,$$
$$4\pi H_2 = -\operatorname{grad}\psi + \lambda \iint_S [n, H]\, \phi dS$$

form a system satisfying the field equations; since

$$E = E_1 + E_2, \quad H = H_1 + H_2$$

it follows that an arbitrary diverging field (E, H) can be represented as the sum of surface distributions of electric and magnetic doublets.

4. Another representation of the field (E, H) in terms of surface and volume integrals, which reduces to a well-known representation when h^2 is neglected in comparison with h'^2, can be obtained as follows. As before,

(E', H') denote the field interior to any conductor; if v is one of the vectors v', v'', v''', an identity already used gives the equation, in which n' the interior normal,

$$\iiint_{(S_i)\,\text{int.}} (E'\Delta v - v\Delta E')\,dx_1dy_1dz_1 = -\iint_{S_i} \{[n', E']\,\text{curl}\,v + E'_{n'}\,\text{div}\,v - [n', v]\,\text{curl}\,E'\}dS.$$

Since
$$\Delta v = h^2 v, \quad \Delta E' = h'^2 E',$$
this becomes

$$\iiint_{(S_i)\,\text{int.}} (h^2 - h'^2)E'v\,dx_1dy_1dz_1 = -\iint_{S_i} \{[n', E']\,\text{curl}\,v + E'_{n'}\,\text{div}\,v + \kappa'\,[n', H']\,v\}dS.$$

Replacing n' by $-n$, and summing with respect to all conductors,

$$\iiint_{(S)\,\text{int.}} (h^2 - h'^2)E'v\,dx_1dy_1dz_1 = \iint_S \{[n, E']\,\text{curl}\,v + E_n'\,\text{div}\,v + \kappa'\,[n, H']\,\text{v}\}dS$$

which is easily transformed into

$$\iiint_{(S)\,\text{int.}} (h^2 - h'^2)E'\phi\,dx_1dy_1dz_1 = \text{curl}\iint_S [n, E']\,\phi dS - \text{grad}\iint_S E_n'\phi dS + \iint_S \kappa'\,[n, H']\,\phi dS.$$

Combining this identity with the first equation of (1.6), one obtains

$$(4.1)\quad 4\pi E = \iiint_{(S)\,\text{int.}} (h^2 - h'^2)E'\phi dx_1dy_1dz_1 - \text{grad}\iint_S [(\lambda' - \lambda)/\lambda]\,E_n'\phi dS + \iint_S (\kappa - \kappa')[n, H']\,\phi dS.$$

From the field equations

$$(4.2)\quad 4\pi H = \text{curl}\iiint_{(S)\,\text{int.}} [(h^2 - h'^2)/\kappa]\,E'\phi dx_1dy_1dz_1 + \text{curl}\iint_S [(\kappa - \kappa')/\kappa]\,[n, H']\,\phi dS.$$

These formulas bring into evidence the vanishing of the field (E, H) if $\lambda = \lambda'$, $\kappa = \kappa'$.

Part II. Perfect Conductors.

1. If all the surfaces S_i are assumed to bound perfect conductors, on each $(E^0 + E)_{\text{tang.}} = 0$; the representation of the magnetic field * reduces to

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* If $\sigma \neq 0$ exterior to S, another treatment of the problem is possible; see "The Electromagnetic Field Exterior to a System of Perfectly Reflecting Surfaces," *Proceedings of the National Academy of Sciences*, Vol. 16, No. 5 (May, 1930), pp. 353-357.

$$4\pi\boldsymbol{H} = \text{curl} \iint\limits_S [\boldsymbol{n}, \boldsymbol{H}'] \,\phi dS.$$

Let $\boldsymbol{J} = [\boldsymbol{n}, \boldsymbol{H}']$. Introducing the isothermal coördinate system (u, v) the surface, and the direction cosines of the u, v curves $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, respectively, let components at a fixed point be denoted by the subscript 0; then if (x, y, z) is a point on the normal at (x_0, y_0, z_0), we have

$$4\pi(\alpha_0'\boldsymbol{H}_x + \beta_0'\boldsymbol{H}_y + \gamma_0'\boldsymbol{H}_z) = -\iint\limits_S \begin{vmatrix} \alpha_0' & \beta_0' & \gamma_0' \\ \partial\phi/\partial x_1 & \partial\phi/\partial y_1 & \partial\phi/\partial z_1 \\ \boldsymbol{J}_{x_1} & \boldsymbol{J}_{y_1} & \boldsymbol{J}_{z_1} \end{vmatrix} dS.$$

On the surface S,

$$\begin{aligned} \partial\phi/\partial x &= \alpha'\partial\phi/\partial u + \alpha''\partial\phi/\partial v + l\partial\phi/\partial n \\ \partial\phi/\partial y &= \beta'\partial\phi/\partial u + \beta''\partial\phi/\partial v + m\partial\phi/\partial n \\ \partial\phi/\partial z &= \gamma'\partial\phi/\partial u + \gamma''\partial\phi/\partial v + n\partial\phi/\partial n \end{aligned}$$

consequently

$$\begin{aligned} &4\pi\boldsymbol{H}_{u_0}(u_0, v_0, n) \\ &\quad = -\iint\limits_S \begin{vmatrix} \alpha' & \beta' & \gamma' \\ \alpha''\partial\phi/\partial v + l\partial\phi/\partial n & \beta''\partial\phi/\partial v + m\partial\phi/\partial n & \gamma''\partial\phi/\partial u + n\partial\phi/\partial n \\ \boldsymbol{J}_x & \boldsymbol{J}_y & \boldsymbol{J}_z \end{vmatrix} dS \\ &\quad + \iint\limits_S \{(\alpha_0' - \alpha')(\boldsymbol{J}_z\partial\phi/\partial y - \boldsymbol{J}_y\partial\phi/\partial z) + (\beta_0' - \beta')(\boldsymbol{J}_x\partial\phi/\partial z - \boldsymbol{J}_z\partial\phi/\partial x) \\ &\qquad\qquad + (\gamma_0' - \gamma')(\boldsymbol{J}_y\partial\phi/\partial x - \boldsymbol{J}_x\partial\phi/\partial y)\} dS. \end{aligned}$$

Now

$$\boldsymbol{J}_x = \alpha'\boldsymbol{J}_u + \alpha''\boldsymbol{J}_v, \quad \boldsymbol{J}_y = \beta'\boldsymbol{J}_u + \beta''\boldsymbol{J}_v, \quad \boldsymbol{J}_z = \gamma'\boldsymbol{J}_u + \gamma''\boldsymbol{J}_v,$$

from which

$$4\pi\boldsymbol{H}_{u_0}(u_0, v_0, n) = \iint\limits_S (\boldsymbol{J}_v d\phi/dn) dS + \iint\limits_S (A\boldsymbol{J}_u + B\boldsymbol{J}_v) dS.$$

Since

$$\boldsymbol{J}_v = \boldsymbol{H}_u', \qquad \boldsymbol{J}_u = -\boldsymbol{H}_v'$$

the preceding equation becomes,

$$(1.1) \qquad 4\pi\boldsymbol{H}_{u_0}(u_0, v_0, n) = \iint\limits_S (\boldsymbol{H}_u' d\phi/dn) dS + \iint\limits_S (B\boldsymbol{H}_u' - A\boldsymbol{H}_v') dS.$$

Now the function ϕ becomes infinite like $1/r$, and from the theory of the Newtonian potential,

$$\begin{aligned} 4\pi\boldsymbol{H}_{u_0}(u_0, v_0, 0) &= 2\pi\boldsymbol{H}_{u_0}'(u_0, v_0, 0) \\ &\quad + \iint\limits_S \boldsymbol{H}_u'[d\phi/dn] dS + \iint\limits_S (B\boldsymbol{H}_u' - A\boldsymbol{H}_v') dS \end{aligned}$$

since A, B become infinite only to the first order on the surface. This equation may also be written

$$(1.2)\quad 2\pi \boldsymbol{H}_{u_0}' = 4\pi \mathbf{H}_{u_0}{}^0 - \iint_S \begin{vmatrix} \alpha_0' & \beta_0' & \gamma_0' \\ [\partial\phi/\partial x]_0 & [\partial\phi/\partial y]_0 & [\partial\phi/\partial z]_0 \\ [\boldsymbol{n}, \boldsymbol{H}']_x & [\boldsymbol{n}, \boldsymbol{H}']_y & [\boldsymbol{n}, \boldsymbol{H}']_z \end{vmatrix} dS$$

the bracket indicating the values on the surface. Similarly,

$$2\pi \boldsymbol{H}_{v_0}' = 4\pi \boldsymbol{H}_{v_0}{}^0 - \iint_S \begin{vmatrix} \alpha_0'' & \beta_0'' & \gamma_0'' \\ [\partial\phi/\partial x]_0 & [\partial\phi/\partial y]_0 & [\partial\phi/\partial z]_0 \\ [\boldsymbol{n}, \boldsymbol{H}']_x & [\boldsymbol{n}, \boldsymbol{H}']_y & [\boldsymbol{n}, \boldsymbol{H}']_z \end{vmatrix} dS.$$

These equations form a system of integral equations which can be reduced to the type of Fredholm by iteration, and consequently possess a unique solution unless the homogeneous equations, corresponding to no impressed field, have a solution. In this case the solution of the homogeneous equations defines a field $(\boldsymbol{E}, \boldsymbol{H})$ by means of

$$4\pi \boldsymbol{H} = \text{curl} \iint_S [\boldsymbol{n}, \bar{\boldsymbol{H}}]\, \phi dS,$$

$$\boldsymbol{E} = 1/\lambda \,\text{curl}\, \boldsymbol{H}.$$

The integral equations express that the condition $[\boldsymbol{n}, \boldsymbol{H}] \rightarrow [\boldsymbol{n}_0, \bar{\boldsymbol{H}}]$ is satisfied, hence $\boldsymbol{H}$ is expressed in terms of its tangential components. Hence in the equations (1.4), Part I, the first integral representing $\boldsymbol{H}$ is the same. This is only possible if

$$0 = -\text{grad} \iint_S \boldsymbol{H}_n \phi dS + \lambda \iint_S [\boldsymbol{n}, \boldsymbol{E}]\, \phi dS$$

and

$$4\pi \boldsymbol{E} = -\text{grad} \iint_S \boldsymbol{E}_n \phi dS + \kappa \iint_S [\boldsymbol{n}, \boldsymbol{H}]\, \phi dS.$$

The surface integrals which represent $(\boldsymbol{E}, \boldsymbol{H})$ exterior to S continue to define a solution of the field equations, hence a field, in the interior of any S_i; but since a potential of a double layer

$$f(xyz) = \iint_S \mu(S)\,[d(1/r)\,dn]\, dS$$

has the property that

$$f_{n=+0} - f_{n=-0} = 4\pi\mu(S_0)$$

it results that if $\boldsymbol{n}$ continues to represent the exterior normal, instead of $4\pi \boldsymbol{H}_{u_0}$ on the left as in (1.1),

$$0 = \lim_{n=-0} \iint_S (\boldsymbol{H}_u' d\phi/dn)\, dS + \iint_S (B\boldsymbol{H}_u' - A\boldsymbol{H}_v')\, dS.$$

Hence the field defined in the interior of any S_i must satisfy the condition $\boldsymbol{H}_v = \boldsymbol{H}_u = 0$. The tangential components of $\boldsymbol{E}$ are the same on both sides of S_i, hence unless $[\boldsymbol{n}, \boldsymbol{E}]$ vanishes identically exterior to each S_i, on at leas one surface the interior tangential components of $\boldsymbol{E}$ are not identically zero; let this surface be S_k. Interior to S_k the tangential components of $\boldsymbol{H}$ are zero, while $[\boldsymbol{n}, \boldsymbol{E}] \neq 0$. Now let the field equations be written

$$\lambda \boldsymbol{E} = \operatorname{curl} \boldsymbol{H}, \qquad \kappa \boldsymbol{H} = \operatorname{curl} \boldsymbol{E}.$$

Then

$$\kappa\lambda \boldsymbol{E} = \operatorname{curl} \kappa \boldsymbol{H}, \qquad \lambda\kappa \boldsymbol{H} = \operatorname{curl} \lambda \boldsymbol{E}$$

or if

$$\boldsymbol{E}_1 = \kappa \boldsymbol{H}, \qquad \boldsymbol{H}_1 = \lambda \boldsymbol{E},$$

the first equations are transformed into

$$\kappa \boldsymbol{H}_1 = \operatorname{curl} \boldsymbol{E}_1, \qquad \lambda \boldsymbol{E}_1 = \operatorname{curl} \boldsymbol{H}_1.$$

Hence a field $(\boldsymbol{E}_1, \boldsymbol{H}_1)$ exists, under the preceding conditions, such that interior to S_k the tangential components of $\boldsymbol{H}_1$ are not identically zero, while the tangential components of $\boldsymbol{E}_1$ vanish. This is only possible if the time-constant p is one of the set of discrete values for which cavity-radiation (Hohlraumstrahlung) exists, while the conductivity σ of the exterior region, also of the interior of S_k which is now merely a geometrical surface, is zero.

If p is not one of the set corresponding to cavity-radiation for any surface S_i, the assumption that $[\boldsymbol{n}, \boldsymbol{E}] \neq 0$ exterior to each S_i is reduced to an absurdity.

The assumption that the tangential vector $[\boldsymbol{n}, \boldsymbol{E}]$ defined by a solution of the homogeneous integral equations is identically zero on each S_i also leads to a contradiction; to show this it is necessary to derive expressions for the field at infinity.

2. From (1.4), Part I and § 3, a diverging field $(\boldsymbol{E}, \boldsymbol{H})$ can be represented in the form

$$4\pi \boldsymbol{E} = \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{E}]\, \phi dS + 1/\lambda \operatorname{curl}\operatorname{curl} \iint [\boldsymbol{n}, \boldsymbol{H}]\, \phi dS$$

$$4\pi \boldsymbol{H} = \operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{H}]\, \phi dS + 1/\kappa \operatorname{curl}\operatorname{curl} \iint_S [\boldsymbol{n}, \boldsymbol{E}]\, \phi dS.$$

Let (xyz) denote a point in space, while (x_1, y_1, z_1) is a point on S. From the definition,

$$\phi = e^{-hr}/r, \qquad r = (R^2 + r_1{}^2 - 2Rr_1 \cos\psi)^{1/2},$$

$$r_1 = (x_1{}^2 + y_1{}^2 + z_1{}^2)^{1/2}, \quad R + (x^2 + y^2 + z^2)^{1/2}, \quad \cos\psi = (xx_1 + yy_1 + zz_1)/r_1 R.$$

Then

$$r = R - r_1 \cos\psi + (\cdot\ \cdot\ \cdot)/R.$$

Let

$$\cos\alpha = x/R, \quad \cos\beta = y/R, \quad \cos\gamma = z/R.$$

hen the limits are easily obtained

$$\lim_{R\to\infty} Re^{hR}\phi = e^{hr_1\cos\psi},$$
$$\lim_{R\to\infty} Re^{hR}\partial\phi/\partial x = -h\cos\alpha e^{hr_1\cos\psi},$$
$$\lim_{R\to\infty} Re^{hR}\partial^2\phi/\partial x^2 = h^2\cos^2\alpha e^{hr_1\cos\psi},$$
$$\lim_{R\to\infty} Re^{hR}\partial^2\phi/\partial x\partial y = h^2\cos\alpha\cos\beta e^{hr_1\cos\psi}, \quad \text{etc.}$$

Let

$$\mathbf{U} = 1/4\pi\iint_S [\mathbf{n},\mathbf{H}]e^{hr_1\cos\psi}dS, \qquad \mathbf{V} = 1/4\pi\iint_S [\mathbf{n},\mathbf{E}]\,e^{hr_1\cos\psi}dS.$$

With the preceding limits,

$$\lim_{R\to\infty} Re^{hR}\,\text{curl}\iint_S \{[\mathbf{n},\mathbf{H}]/4\pi\}\phi dS$$
$$= -h\{i(\mathbf{U}_z\cos\beta - \mathbf{U}_y\cos\gamma) + j(\mathbf{U}_x\cos\gamma - \mathbf{U}_z\cos\alpha) + k(\mathbf{U}_y\cos\alpha - \mathbf{U}_x\cos\beta)\}.$$

Let Ω be a sphere of unit radius about the origin, while Σ is a sphere with center at the origin, and radius R. The exterior normal on each is $\bar{\mathbf{n}} = (\cos\alpha, \cos\beta, \cos\gamma)$; the right-hand member above is equal to

$$-h\,[\bar{\mathbf{n}}, \mathbf{U}].$$

If a vector $\mathbf{F}$ satisfies the equation $(\Delta - h^2)\mathbf{F} = 0$,

$$\text{curl curl}\,\mathbf{F} = \text{grad div}\,\mathbf{F} - h^2\mathbf{F};$$

consequently from the limits preceding it is found that

$$\lim_{R\to\infty} Re^{hR}\mathbf{E} = -h[\bar{\mathbf{n}},\mathbf{V}] - \kappa\{\bar{\mathbf{n}}(\bar{\mathbf{n}}\mathbf{U}) - \mathbf{U}\} = \mathbf{E}_1$$

$$\lim_{R\to\infty} Re^{hR}\mathbf{H} = -h[\bar{\mathbf{n}},\mathbf{U}] - \lambda\{\bar{\mathbf{n}}(\bar{\mathbf{n}}\,\mathbf{V}) - \mathbf{V}\} = \mathbf{H}_1.$$

Also,

$$\lim_{R\to\infty} Re^{hR}[\bar{\mathbf{n}},\mathbf{E}] = \kappa\,[\bar{\mathbf{n}},\mathbf{U}] + h\mathbf{V} - h\bar{\mathbf{n}}(\bar{\mathbf{n}}\mathbf{V}) = [\bar{\mathbf{n}},\mathbf{E}_1]$$

$$\lim_{R\to\infty} Re^{hR}[\bar{\mathbf{n}},\mathbf{H}] = \lambda\,[\bar{\mathbf{n}},\mathbf{V}] + h\mathbf{U} - h\bar{\mathbf{n}}(\bar{\mathbf{n}}\mathbf{U}) = [\bar{\mathbf{n}},\mathbf{H}_1]$$

and from the relation $h^2 = -\kappa\lambda$,

$$h[\bar{\mathbf{n}},\mathbf{E}_1] = -\kappa\mathbf{H}_1, \qquad h[\bar{\mathbf{n}},\mathbf{H}_1] = -\lambda\mathbf{E}_1.$$

Let the conductivity of the exterior region be zero, while $p = i\omega$; then $\lambda = \epsilon i\omega/c$, $\kappa = -\mu i\omega/c$, $h = i\omega(\epsilon\mu)^{1/2}/c$. The preceding identities become

$$(\epsilon\mu)^{1/2}[\bar{\boldsymbol{n}}, \boldsymbol{E}_1] = \mu \boldsymbol{H}_1, \qquad (\epsilon\mu)^{1/2}[\bar{\boldsymbol{n}}, \boldsymbol{H}_1] = -\epsilon \boldsymbol{E}_1.$$

Now the energy flow outward across the sphere Σ is equal to

$$c/4\pi \iint_{\Sigma} \{R[\boldsymbol{E}e^{i\omega t}], \quad R[\boldsymbol{H}e^{i\omega t}]\}_{\bar{n}}\, dS$$

if only the real parts of the field are considered; omitting the factor $c/4\pi$, the average value of this expression over a period $T = 2\pi/\omega$ is half the quantity

$$\begin{aligned} K_R &= \tfrac{1}{2} \iint_{\Sigma} \{[\boldsymbol{E}, \bar{\boldsymbol{H}}]_{\bar{n}} + [\bar{\boldsymbol{E}}, \boldsymbol{H}]_{\bar{n}}\} dS \\ &= -\tfrac{1}{2} \iint_{\Sigma} \{\boldsymbol{E}[\bar{\boldsymbol{n}}, \bar{\boldsymbol{H}}] + \bar{\boldsymbol{E}}[\bar{\boldsymbol{n}}, \boldsymbol{H}]\} dS \\ &= -\tfrac{1}{2} \iint_{\Omega} \{Re^{hR}\boldsymbol{E}[\bar{\boldsymbol{n}}, Re^{-hR}\bar{\boldsymbol{H}}] + Re^{-hR}\bar{\boldsymbol{E}}\,[\bar{\boldsymbol{n}}, Re^{hR}\boldsymbol{H}]\} d\omega. \end{aligned}$$

From the identities developed above it is seen that

$$\begin{aligned} \lim_{R\to\infty} K_R &= -\tfrac{1}{2} \iint_{\Omega} \{\boldsymbol{E}_1[\bar{\boldsymbol{n}}, \boldsymbol{H}_1] + \boldsymbol{E}_1[\bar{\boldsymbol{n}}, \boldsymbol{H}_1]\} d\omega \\ &= (\epsilon/\mu)^{1/2} \iint_{\Omega} |\boldsymbol{E}_1|^2 \, d\omega = (\mu/\epsilon)^{1/2} \iint_{\Omega} |\boldsymbol{H}_1|^2 \, d\omega. \end{aligned}$$

It is an immediate consequence that if the mean energy flow at infinity is zero, the functions $\boldsymbol{E}_1$, $\boldsymbol{H}_1$ vanish identically. This condition is satisfied if the tangential components of $\boldsymbol{E}$ or of $\boldsymbol{H}$ are zero exterior to the system of surfaces S; for in the identity*

$$\begin{aligned} &\iiint_{(S+\Sigma)} \{\text{curl}\, \boldsymbol{u}\, \text{curl}\, \boldsymbol{v} + \text{div}\, \boldsymbol{u}\, \text{div}\, \boldsymbol{v} + \boldsymbol{u}\Delta\boldsymbol{v}\} dx\, dy\, dz \\ &\qquad = -\iint_{(S+\Sigma)} \{[\boldsymbol{n}, \boldsymbol{u}]\ \text{curl}\, \boldsymbol{v} + \boldsymbol{u}_n\, \text{div}\, \boldsymbol{v}\} dS \end{aligned}$$

replace $\boldsymbol{u}$ by $\boldsymbol{H}$, $\boldsymbol{v}$ by $\bar{\boldsymbol{H}}$; from the result subtract the corresponding members of the identity with $\boldsymbol{u} = \bar{\boldsymbol{H}}$, $\boldsymbol{v} = \boldsymbol{H}$. Since

$$\text{curl}\, \boldsymbol{H} = -\lambda \bar{\boldsymbol{E}},$$

the right-hand member becomes

$$\lambda \iint \{\bar{\boldsymbol{E}}[\boldsymbol{n}, \boldsymbol{H}] + \boldsymbol{E}[\boldsymbol{n}, \bar{\boldsymbol{H}}]\} dS$$

and vanishes since the left-hand member is zero. Consequently if the inte-

* Weyl, *loc. cit.*

grand of this expression, which depends only on the tangential components of $E, \bar{E}, H, \bar{H}$ vanishes on S, the integral is also zero on Σ, for every $R > R_0$.

If $E_1 = H_1 = 0$, it results that the field (E, H) vanishes identically exterior to S. For the components of the field satisfy the wave equation, which becomes in spherical coördinates

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}-h^2u=0.$$

Let Y_n be an arbitrary spherical harmonic of order n; the integral

$$W_n(R)=\int_0^{2\pi}d\phi\int_0^{\pi}E_x(R,\theta,\phi)Y_n(\theta,\phi)\sin\theta d\theta$$

satisfies the differential equation

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dW_n}{dR}\right)+\left(-h^2-\frac{n(n+1)}{R^2}\right)W_n=0$$

as is seen from the equation in Y_n,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\phi^2}+n(n+1)Y_n=0$$

and an integration by parts.*

Consequently

$$\begin{aligned}W_n &= C_1K_{n+1/2}\ (hR)/(R)^{1/2}+C_2I_{n+1/2}\ (hR)/(R)^{1/2}\\ &= C_1/(R)^{1/2}\,(\pi/2hR)^{1/2}\,e^{-hR}\,[1+1/R(\cdot\cdot\cdot)]\end{aligned}$$

since a limit must exist, as $R\to\infty$, for the quantity $Re^{hR}W_n$.

$$\lim_{R\to\infty} Re^{hR}W_n=C_1{}^{(n)}(\pi/2h)^{1/2}.$$

Since the products $Re^{hR}E_x$, etc., approach their limits uniformly with respect to the spherical angles θ, ϕ,

$$\begin{aligned}&\lim_{R\to\infty} Re^{hR}\iint_\Omega E_x(R,\theta,\phi)Y_n(\theta,\phi)\sin\theta\,d\theta d\phi\\ &\qquad=\iint_\Omega Y_n(\theta,\phi)E_{1x}(\theta,\phi)\sin\theta\,d\theta d\phi=C_1{}^{(n)}(\pi/2h)^{1/2}.\end{aligned}$$

If $E_1=0$, $C_1=0$. But Y_n was an arbitrary spherical harmonic, hence the expansions of the components of E in spherical harmonics on the surface Σ vanish identically. Since E is analytic on Σ, provided R is so large that the surfaces S_i lie entirely in the interior of Σ, E must vanish identically

* Carleman, *Sur les équations intégrales singulières á noyau réel et symetrique,* Upsala, 1923, pp. 181-183.

on Σ, likewise $\boldsymbol{H}$. If the surfaces S_i are ordinary surfaces, the values of $\boldsymbol{E}$ and $\boldsymbol{H}$ at an ordinary point exterior to S can be obtained by analytic continuation from their values on some Σ, hence the field $(\boldsymbol{E}, \boldsymbol{H})$ vanishes identically exterior to S.

Returning to the homogeneous equations assumed satisfied when the surfaces S_i bound perfect conductors, it has been found that the field $(\boldsymbol{E}, \boldsymbol{H})$ defined by a solution of these equations must vanish identically exterior to S, if p is not one of the exceptional values. But if $\boldsymbol{H}$ vanishes identically exterior to S, the tangential components of $\boldsymbol{H}$ on S vanish identically; the integral equations express that the vector $[\boldsymbol{n}, \boldsymbol{H}]$ appearing in the representation of $\boldsymbol{H}$ is formed from these tangential components, hence vanishes identically. The assumption that a solution of the homogeneous equations exists leads therefore to a contradiction unless p is one of the exceptional values corresponding to cavity-radiation in the interior of some S_i.

From the theorems of Fredholm it results that the equations for the tangential components of $(\boldsymbol{E}, \boldsymbol{H})$ when an impressed field is present have a unique solution, if p is not one of these exceptional values.

THE GRAVITATIONAL FIELD OF A BODY WITH ROTATIONAL SYMMETRY IN EINSTEIN'S THEORY OF GRAVITATION.

By P. Y. Chou.

Introduction.

The present paper is an attempt to solve rigorously the problem of the static gravitational field of a body whose mass is distributed symmetrically around an axis in Einstein's theory of gravitation. In § 1 Einstein's field equations in vacuo *

$$(0.1) \qquad G_{\mu\nu} = 0$$

are set up and reduced in § 2 to a form such that simple problems like the sphere (§ 4) and the plane (§ 5) can be solved. In the general problem there is a fundamental difficulty which will be avoided by the introduction of the Newtonian potential (§ 7). The solution of the whole problem then depends upon the solution of the well known Laplace's equation and a partial differential equation of the second order which is not linear. Finally the gravitational fields of spheroidal homoeoids (§ 8, § 9) are given as illustrations of the present investigation and the motion of a particle in the field of an oblate spheroidal homoeoid is discussed (§ 10). The paper also contains a critical examination of earlier works upon the problem notably those of Prof.'s Weyl and Levi-Civita (§ 3).

I. Einstein's Law of Gravitation.

1. *The field equations.* We consider the static gravitational field outside of a body whose mass is distributed symmetrically about an axis. Hence the $g_{\mu\nu}$'s do not vary with respect to time. The most general fundamental quadratic differential form in such a field appears to be

$$(1.1) \quad ds^2 = -(g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2) - g_{33}dx_3^2 + g_{44}dx_4^2$$

where x_1, x_2 are any two coördinates in the meridianal plane containing the z-axis, $x_3 = \phi$, the azimuthal angle, $x_4 = t$, the time coördinate, the unit of time being so chosen that the velocity of light in vacuo is unity. The $g_{\mu\nu}$'s in (1.1) are functions of x_1 and x_2 only.

* A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd Ed. (1924), p. 81. Eddington's notation with slight modifications will be followed throughout the present paper.

We assume that the values of the $g_{\mu\nu}$'s exist. From a well-known theorem * on positive definite quadratic differential forms of two variables such as the form in the parenthesis of (1.1), it is always possible when g_{11} g_{12}, g_{22} are explicitly given, to make a real single-valued, continuous transformation from x_1 and x_2 to u and v by

$$(1.2) \qquad x_1 = x_1(u, v), \quad x_2 = x_2(u, v), \quad \text{where} \quad J \equiv [\partial(x_1, x_2)/\partial(u, v)] \neq 0$$

such that the following identity is true,

$$(1.3) \qquad g_{11}dx_1{}^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2{}^2 \equiv e^{2m}(du^2 + dv^2).$$

Hence (1.1) becomes

$$(1.4) \qquad ds^2 = -e^{2m}(du^2 + dv^2) - e^{2n}dx_3{}^2 + e^{2\nu}dx_4{}^2$$

where m, n, ν are functions of u and v to be determined. Let

$$u = x_1, \quad v = x_2.$$

Then

$$(1.5) \qquad \begin{aligned} &g_{11} = g_{22} = -e^{2m}, \quad g_{33} = -e^{2n}, \quad g_{44} = e^{2\nu}, \\ &g = g_{11}g_{22}g_{33}g_{44} = -e^{4m+2n+2\nu} \\ &g^{11} = g^{22} = -e^{-2m}, \quad g^{33} = -e^{-2n}, \quad g^{44} = e^{-2\nu}. \end{aligned}$$

Now (1.4) is an orthogonal quadratic differential form. The general expressions of the Christoffel symbols of the second kind for such forms are well known.† In the present problem the non-vanishing symbols are

$$(1.6) \qquad \begin{array}{ll} \{11, 1\} = m_u & \{11, 2\} = -m_v \\ \{12, 1\} = m_v & \{12, 2\} = m_u \\ \{22, 1\} = -m_u & \{22, 2\} = m_v \\ \{33, 1\} = -e^{2n-2m}n_u & \{33, 2\} = -e^{2n-2m}n_v \\ \{44, 1\} = e^{2\nu-2m}\nu_u & \{44, 2\} = e^{2\nu-2m}\nu_v \\ \{13, 3\} = n_u & \{14, 4\} = \nu_u \\ \{23, 3\} = n_v & \{24, 4\} = \nu_v \end{array}$$

where the subscripts mean partial differentiations for simplicity.

Written out in full Einstein's field equations in vacuo are

$$(1.7) \qquad \begin{aligned} G_{\mu\nu} \equiv -\partial/\partial x_\alpha\{\mu\nu, \alpha\} + \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} \\ \partial^2/\partial x_\mu \partial x_\nu \log(-g)^{1/2} - \{\mu\nu, \alpha\}\,\partial/\partial x_\alpha \log(-g)^{1/2} = 0. \end{aligned}$$

If we substitute for the three-index Christoffel symbols of the second kind

* L. Bianchi-Lucat, *Vorlesungen über Differentialgeometrie,* (1910), pp. 69.

† A. S. Eddington, *loc. cit.*, pp. 83.

from their values (1. 6) into (1. 7), we obtain the following five non-vanishing components,

$$(1.8) \quad G_{11} \equiv m_{uu} + m_{vv} + n_{uu} + \nu_{uu} + n_u^2 + \nu_u^2 - m_u(n_u + \nu_u) + m_v(n_v + \nu_v)$$

$$(1.9) \quad G_{12} \equiv n_{uv} + \nu_{uv} + n_u n_v + \nu_u \nu_v - m_v(n_u + \nu_u) - m_u(n_v + \nu_v) = 0,$$

$$(1.10) \quad G_{22} \equiv m_{uu} + m_{vv} + n_{vv} + \nu_{vv} + n_v^2 + \nu_v^2 + m_u(n_u + \nu_u) - m_v(n_v + \nu_v)$$

$$(1.11) \quad G_{33} \equiv e^{2n-2m}\left[n_{uu} + n_{vv} + n_u(n_u + \nu_u) + n_v(n_v + \nu_v)\right] = 0,$$

$$(1.12) \quad G_{44} \equiv - e^{2\nu-2m}\left[\nu_{uu} + \nu_{vv} + \nu_u(n_u + \nu_u) + \nu_v(n_v + \nu_v)\right] = 0.$$

2. *Reduction of the field equations.* By putting

$$(2.1) \qquad \chi = n + \nu,$$

and adding the expressions in the square brackets of (1. 11) and (1. 12) we get

$$(2.2) \qquad \chi_{uu} + \chi_{vv} + \chi_u^2 + \chi_v^2 = 0,$$

which becomes Laplace's equation in the uv-plane,

$$(2.3) \qquad \Phi_{uu} + \Phi_{vv} = 0, \quad \text{on setting} \quad \Phi = e^{\chi} = e^{n+\nu}.$$

It is well known that the solution of (2. 3) is unique, if the boundary value of Φ be given in the uv-plane. Then $G_{44} = 0$ becomes

$$(2.4) \qquad \nu_{uu} + \nu_{vv} + \chi_u \nu_u + \chi_v \nu_v = 0,$$

which determines ν. We obtain n by (2. 1).

To get the unknown function, m, we use (1. 8), (1. 9), (1. 10). Write

$$(2.5) \qquad \begin{aligned} &G_{12} = 0: \\ &\qquad \chi_v m_u + \chi_u m_v = \chi_{uv} + n_u n_v + \nu_u \nu_v \equiv A, \\ &G_{11} - G_{22} = 0: \\ &\qquad -\chi_u m_u + \chi_v m_v = \tfrac{1}{2}\{-\chi_{uu} + \chi_{vv} - n_u^2 - \nu_u^2 + n_v^2 + \nu_v^2\} \equiv B. \end{aligned}$$

Then by solving m_u and m_v simultaneously from (2. 5) we get

$$(2.6) \quad m_u = (\chi_u^2 + \chi_v^2)^{-1}\{\chi_v A - \chi_u B\}, \quad m_v = (\chi_u^2 + \chi_v^2)^{-1}\{\chi_u A + \chi_v B\}.$$

It can be shown by direct differentiation and the aid of (1. 11) and (1. 12) that

$$(2.7) \qquad dm \equiv m_u du + m_v dv$$

where m_u and m_v are given in (2. 6) is an exact differential and secondly that m satisfy (1. 8) and (1. 10). This completes the proof that the functions, m, n, ν, thus obtained satisfy every component of Einstein's field equations (1. 7).

3. *On Weyl-Levi-Civita's solution.* The problem under consideration was first attacked by H. Weyl.* His result was criticized by T. Levi-Civita † as being incomplete due to the incomplete use of a variational principle. T latter started with (0. 1) and gave a complete though restricted set of solutions. Consider (2. 3) and set $\Phi=\rho$. Let z be the conjugate function of ρ. Then

$$\rho+iz=f(u+iv) \tag{3.1}$$

where $f(u+iv)$ is analytic in $u+iv$. From this it follows that

$$d\rho^2+dz^2=f'(u+iv)f'(u-iv)(du^2+dv^2) \tag{3.2}$$

namely, one set of coördinates (say u, v) is conformally transformed into the other (say ρ, z). In order to avoid cumbersome mathematical manipulations in § 2, both Weyl and Levi-Civita assume initially that

$$\rho=u, \quad z=v. \quad \text{Then} \quad e^{2n}=\rho^2e^{-2\nu} \quad \text{and} \tag{3.3}$$

$$ds^2=-e^{2m}(d\rho^2+dz^2)-\rho^2e^{-2\nu}d\phi^2+e^{2\nu}dt^2. \tag{3.4}$$

Moreover, $G_{44}=0$ and dm become respectively

$$\partial^2\nu/\partial\rho^2+\partial^2\nu/\partial z^2+(1/\rho)\partial\nu/\partial\rho=0, \tag{3.5}$$

$$dm=-d\nu+\rho[(\partial\nu/\partial\rho)^2-(\partial\nu/\partial z)^2]\,d\rho+2\rho\,(\partial\nu/\partial\rho)(\partial\nu/\partial z)dz. \tag{3.6}$$

We recognize (3. 5) as Laplace's equation in cylindrical coördinates (ρ, z, ϕ) independent of ϕ. Weyl calls (ρ, z) in (3. 3) the "canonical cylindrical coördinates" which are apparently different from the ordinary cylindrical coördinates used in solving Newtonian potential problems. He then emphasizes the fact ‡ that if the distribution of mass of a given body in our space-time manifold is known in terms of this set of configurational canonical coördinates, the problem is reduced to the solution of (3. 5). He shows § that Schwarzschild's solution in isotropic coördinates of a body with mass, m, having spherical symmetry, corresponds to that of a finite line segment of length $2m$, with constant linear density, lying on the z-axis of the configurational canonical space-time manifold. But he does not make clear that it is almost impossible to know the corresponding distribution of mass in this

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* H. Weyl, *Annalen der Physik*, Bd. 54 (1918), pp, 134.

† T. Levi-Civita, *Rendiconti Accademie dei Lincei*, Vol. 28, i (1919), pp. 9.

‡ H. Weyl, *Raum, Zeit, Materie*, 5th ed. (1923), p. 266.

§ H. Weyl, *Annalen der Physik*, Bd. 54 (1918), p. 140. Schwarzschild's solution is not necessarily limited to a particle. It can be applied equally well to a spherical shell. Cf. J. T. Combridge, *Philosophical Magazine* (7), Vol. 1 (1926), pp. 276.

canonical coördinate system when the distribution of mass in our space-time coördinates is given. This difficulty is clearly brought out by the following argument.

When we carry out the transformation from (x_1, x_2, ϕ, t) to (ρ, z, ϕ, t) by (1. 2) and (3. 3) we assume only the existence of the values of g_{11}, g_{12}, g_{22} in (1. 1) so that the transformation is possible, but their explicit forms are not given *a priori* and consequently (1. 2) is not explicitly known. Although we know the boundary values of $g_{\mu\nu}$ in the original (x_1, x_2, ϕ, t) system, we do not know the corresponding boundary conditions in the (u, v, ϕ, t) system on account of the uncertainty of (1. 2). Since (3. 5) has an infinite number of solutions if the boundary value of ν is not specified, the solution obtainable from (3. 5) and (3. 6) will not be unique, and consequently it is indeterminate.

The same difficulty arises if we do not assume the solution of Φ in (3. 3). Here we do not know which solution of (2. 3) we should take in order to solve (2. 4). The complexity of the situation is enhanced further by the uncertainty of the boundary conditions of ν in the uv-coördinates.

An alternative procedure to get a solution for the original physical problem from (3. 5) and (3. 6) is to choose a solution of (3. 5) in terms of the canonical coördinates first and then try to interpret it in the (x_1, x_2, ϕ, t) system by a transformation (1. 2). The $g_{\mu\nu}$'s thus obtained must satisfy the original boundary conditions in terms of (x_1, x_2, ϕ, t) given initially. The question whether this procedure will lead to a unique transformation (1. 2) needs further investigation. It appears not to have been considered in the literature.

II. FIELDS OF SPHERE AND PLANE.

4. *Schwarzschild's solution.* As the first application of the results in § 2 let us consider Schwarzschild's solution. The arc element in the gravitational field outside a body with spherical symmetry is

$$(4.1)\qquad ds^2 = -e^{2\lambda}dr^2 - e^{2\mu}(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + e^{2\nu}dt^2,$$

where (r, θ, ϕ) denote spherical polar coördinates and λ, μ, ν are functions of r only. (4. 1) may be put in the form of (1. 4),

$$(4.2)\qquad ds^2 = -e^{2m}(du^2 + dv^2) - e^{2n}d\phi^2 + e^{2\nu}dt^2, \quad \text{where}$$

$$(4.3)\qquad du = r^{-1}e^{\lambda-\mu}dr, \quad v = \theta, \quad e^m = re^{\mu}, \quad e^n = r\sin\theta e^{\mu},$$

m, λ, ν being then functions of u. Let $\Phi = e^{n+\nu} \equiv R \sin v$. Then (2. 3) is

$$(4.4)\qquad d^2R/du^2 - R = 0.$$

Integrating (4. 4), we obtain

$$R = re^{\mu+\nu} = c_1 \sinh u. \tag{4.5}$$

Then (2. 4) becomes $d^2\nu/du^2 + \coth u(d\nu/du) = 0$ giving

$$\exp(\nu/c_2) = c_3(\coth u - \operatorname{csch} u). \tag{4.6}$$

Eliminating u between (4. 5) and (4. 6) we obtain

$$\exp(\nu/c_2) = c_3 r^{-1} e^{-(\mu+\nu)} \{[c_1^2 + r^2 e^{2(\mu+\nu)}]^{1/2} - c_1\}. \tag{4.7}$$

By using the boundary condition on μ and ν that both of them tend toward zero as r increases indefinitely we get $c_3 = 1$. Solving ν from (4. 7), we get

$$\sinh(\nu/c_2) = -c_1 r^{-1} e^{-(\mu+\nu)} = -\operatorname{csch} u. \tag{4.8}$$

From (4. 3) we see that m is a function of u only. In (2. 6) we must have $m_\nu = 0$ which together with (4. 6) gives

$$c_2 = \pm 1. \tag{4.9}$$

Take $c_2 = 1$. Then from (4. 8) we have

$$e^{2\nu} = 1 - (2c_1/r) e^{-\mu}. \tag{4.10}$$

Eliminating u between (4. 3) and (4. 6), we obtain

$$e^{\lambda} = -r \operatorname{csch} \nu (d\nu/dr) e^{\mu}. \tag{4.11}$$

The second case $c_2 = -1$ only changes c_1 to $-c_1$.

Relations (4. 10) and (4. 11) connect the three unknown functions λ, μ, ν. Consequently an infinite number of solutions arises. To obtain Schwarzschild's solution we set $\mu = 0$. Then (4. 10) becomes

$$g_{44} = e^{2\nu} = 1 - 2c_1/r \tag{4.12}$$

where c_1 may be identified as the mass of the body from Newton's theory. From (4. 10) and (4. 11) it follows that $\lambda = -\nu$. The same result can be also obtained by assuming that g_{44} is $1 - 2V$ to start with where V is the Newtonian potential of the body.

A second solution of interest is the one in isotropic coördinates where the velocity of light is independent of direction. Putting $\lambda = \mu$ in (4. 11) and integrating, we get

$$\sinh \nu = 2c_4 r (r^2 - c_4^2)^{-1}. \tag{4.13}$$

To determine the constant of integration, c_4, we use (4. 10) and let r tend toward infinity. This gives

(4. 14) $$2c_4 = -c_1.$$

Solving for e^{ν} and rejecting the negative root of e^{ν} which is essentially positive from (4. 13), we find

(4. 15) $$g_{44} \equiv e^{2\nu} = (2r - c_1)^2/(2r + c_1)^2 \quad \text{and} \quad e^{2\mu} = (1 + c_1/2r)^4.$$

This result was also obtained by a transformation of r in Schwarzschild's solution.*

5. *Infinite plane: a. Gravitational field.* Let the xy-plane be the given plane. From symmetry considerations around any line parallel to the z-axis the most general fundamental quadratic differential form appears to be

(5. 1) $$ds^2 = -e^{2\lambda}(d\rho^2 + \rho^2 d\phi^2) - e^{2\mu}dz^2 + e^{2\nu}dt^2$$

where λ, μ, ν are functions of z only. (5. 1) can be put in the form (1. 4),

(5. 2) $$ds^2 = -e^{2m}(du^2 + dv^2) - e^{2n}d\phi^2 + e^{2\nu}dt^2 \quad \text{with}$$

(5. 3) $$du = e^{\mu-\lambda}dz, \quad \rho = v, \quad m = \lambda, \quad e^{2n} = \rho e^{\lambda}.$$

In the present case $\Phi = \rho e^{\lambda+\nu} \equiv \rho R$ and (2. 3) becomes

(5. 4) $$d^2R/du^2 = 0. \quad \text{Hence}$$

(5. 5) $$R = e^{\lambda+\nu} = c_1 u.$$

Then (2. 4) becomes $d^2\nu/du^2 + (1/u)d\nu/du = 0$ giving

(5. 6) $$\nu = \log c_3 + c_2 \log u.$$

From (5. 3) and $n = \chi - \nu$, we find

(5. 7) $$\lambda = (1 - c_2)\log u + \log c_4, \qquad (c_4 = c_1/c_3).$$

By (2. 6), $m_v = \lambda_v = 0$, we get

(5. 8) $$c_2 = \pm 1.$$

Consider $c_2 = 1$. If we choose the unit of length properly, $c_3 = c_1$ and $e^{\lambda} = 1$. Then (5. 5) becomes

(5. 9) $$e^{\nu} = c_1 u.$$

Differentiating (5. 9) and on using (5. 3) we find

(5. 10) $$e^{\nu}(d\nu/dz) = c_1 e^{\mu}.$$

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* A. S. Eddington, *loc. cit.*, p. 93.

Here we have again an infinite number of solutions of ν and μ. To avoid this indeterminateness we use Newton's theory. By setting

$$g_{44} \equiv e^{2\nu} = 1 + 4\pi\sigma z \tag{5.11}$$

and identifying c_1 as $2\pi\sigma$ where σ is the surface density of matter on the given plane, we get $\mu + \nu = 0$ and the final form of (5.1) is

$$ds^2 = -(1 + 4\pi\sigma z)^{-1}dz^2 - (d\rho^2 + \rho^2 d\phi^2) + (1 + 4\pi\sigma z)\,dt^2. \tag{5.12}$$

The additive constant in (5.11) is chosen to be unity. Here we are dealing with a body whose mass extends to an infinite distance and ds^2 is not Galilean at infinity. The latter condition, however, can be replaced by the one that space surrounding the plane is flat if the density of matter on the plane vanishes. This is satisfied by (5.12).

The solution (5.12) can be regarded as the limiting case of Schwarzschild's solution of a spherical shell when the radius of the shell becomes infinitely great (neglecting the infinite constant obtained in this limiting process). In fact Whittaker * uses this method to obtain his "quasi-uniform" gravitational field which, as we see in the present discussion, is the field outside an infinite material plane. The case $c_2 = -1$ and hence $e^\lambda = c_4 u^2$ has been treated by Levi-Civita † and the result extended to the gravitational field of a charged plane by Kar.‡

b. *The motion of a particle in the field.* Now it is possible to transform away the field represented by (5.12). Let

$$\begin{aligned} x' &= x, \quad y' = y, \\ z' &= (1/2\pi\sigma)\{(1 + 4\pi\sigma z)^{1/2}\cos h\, 2\pi\sigma t - (1 + 4\pi\sigma h)^{1/2}\}, \\ t' &= (1/2\pi\sigma)(1 + 4\pi\sigma z)^{1/2}\sin h\, 2\pi\sigma t, \end{aligned} \tag{5.13}$$

where h is a constant and when $(x, y, z, t) = (0, 0, h, 0)$, we have $(x', y', z', t') = (0, 0, 0, 0)$. Then (5.12) becomes

$$ds^2 = -dx'^2 - dy'^2 - dz'^2 + dt'^2. \tag{5.14}$$

If we expand (5.13) in terms of σ, we get

$$\begin{aligned} x' &= x, \quad y' = y, \\ z' &= z - h + \tfrac{1}{2}gt^2 + (g/2c^2)(h^2 - z^2 + gzt^2) + \cdots, \\ t' &= t + (g/c^2)(zt + \tfrac{1}{6}gt^3) + \cdots. \end{aligned} \tag{5.15}$$

* E. T. Whittaker, *Proceedings of the Royal Society* (A), Vol. 116 (1927), p. 722.
† T. Levi-Civita, *Accademie dei Lincei*, Vol. 27, ii (1918), pp. 240.
‡ S. C. Kar, *Physik Zeit*, Vol. 27 (1926), pp. 208.

where c, the velocity of light in vacuo is restored and $g = 2\pi\sigma$, the gravity. From this we see that (x, y, z, t) are the coördinates used by an observer on the plane and (x', y', z', t') those of one who falls freely toward the plane at an initial height h. In the (x', y', z', t') system gravitation of the plane vanishes.

From (5.13) and (5.14) it is obvious that the differential equations defining the geodesics in the field (5.12) can be integrated rigorously. Analysis shows that the orbit of an infinitesimal particle in the (x, y, z, t) system is a parabola, though in the primed system it is a straight line. Furthermore, this Einsteinian parabola is slightly different from that predicted according to Newton's theory, but when the velocity of light in vacuo becomes infinite, this difference disappears.

III. General Solution of the Problem.

6. *Transformation of the fundamental quadratic differential form.* The foregoing two special cases are solvable from (2.3), (2.4) and (2.6). This is because (2.3) degenerates into an ordinary differential equation in each case. In reality when Φ is a general function of u and v, the problem can be hardly solvable on account of the uncertainty of the boundary conditions of Φ in the (u, v, ϕ, t) manifold as we have pointed out in § 3. In the following section we shall avoid this difficulty by introducing the Newtonian potential into the present problem. As we shall see presently, the problem of the general static gravitational field of a finite body with rotational symmetry can be solved provided we can solve a non-linear partial differential equation of the second order.

We start with the cylindrical coördinates (ρ, z, ϕ), the z-axis being the axis of symmetry of the given body which is finite in extent. Consider the meridianal plane containing the z-axis. Choose in this plane as in ordinary potential theory a more general set (ξ, η) which is conformally mapped upon (ρ, z) by

$$z + i\rho = F(\xi + i\eta) \tag{6.1}$$

where $F(\xi + i\eta)$ is a monogenic function of $\xi + i\eta$ so that

$$dz^2 + d\rho^2 = h^2(d\xi^2 + d\eta^2), \quad h^2 \equiv F'(\xi + i\eta)F'(\xi - i\eta). \tag{6.2}$$

Let $\psi(\xi, \eta) =$ const., $\theta(\xi, \eta) =$ const., be two orthogonal (in the Euclidean sense) families of curves to be determined in the plane. Denote partial differentiations by subscripts as in § 1. Then

(6.3) $$d\psi = \psi_\xi d\xi + \psi_\eta d\eta, \quad d\theta = \theta_\xi d\xi + \theta_\eta d\eta \quad \text{where}$$

(6.4) $$\psi_\xi \theta_\xi + \psi_\eta \theta_\eta = 0.$$

Choose the Jacobian of transformation of (6.3) to be

(6.5) $$J \equiv \frac{\partial(\psi, \theta)}{\partial(\xi, \eta)} \equiv \psi_\xi \theta_\eta - \psi_\eta \theta_\xi = e^f (\psi_\xi^2 + \psi_\eta^2) \quad \text{where}$$

(6.6) $$e^f = \rho.$$

Solving θ_ξ, θ_η from (6.4) and (6.5) simultaneously we obtain

(6.7) $$\theta_\xi = - e^f \psi_\eta, \quad \theta_\eta = e^f \psi_\xi.$$

Since $d\theta$ is an exact differential, (6.7) must satisfy the necessary and sufficient condition,

(6.8) $$\frac{\partial}{\partial \xi} \theta_\eta = \frac{\partial}{\partial \eta} \theta_\xi \quad \text{giving}$$

(6.9) $$\psi_{\xi\xi} + \psi_{\eta\eta} + f_\xi \psi_\xi + f_\eta \psi_\eta = 0.$$

By (6.1) and (6.6), f is a known function of ξ and η. Simple verification shows that (6.9) is Laplace's equation in the (ξ, η, ϕ) coördinates independent of ϕ.

Now by (6.3) we obtain (6.2) in the form

(6.10) $$dz^2 + d\rho^2 = h^2 (\psi_\xi^2 + \psi_\eta^2)^{-1} (d\psi^2 + \rho^{-2} d\theta^2).$$

Consequently the fundamental quadratic form for a flat space-time continuum in the present (ψ, θ, ϕ, t) variables is

(6.11) $$ds^2 = - h^2 (\psi_\xi^2 + \psi_\eta^2)^{-1} (d\psi^2 + \rho^{-2} d\theta^2) - \rho^2 d\phi^2 + dt^2.$$

When matter is present, ds^2 is no more Galilean. We suppose that in such cases (6.11) is replaced by

(6.12) $$ds^2 = - e^{-2H} (e^{2\lambda} d\psi^2 + \rho^{-2} e^{2\mu} d\theta^2) - \rho^2 e^{2\gamma} d\phi^2 + e^{2\nu} dt^2,$$

(6.13) $$e^{-2H} = h^2 (\psi_\xi^2 + \psi_\eta^2)^{-1},$$

where $\lambda, \mu, \gamma, \nu$ are functions of ψ and θ to be determined according to Einstein's law of gravitation, with the condition that at infinite distances from the body all four approach zero as a limit.

7. *Introduction of the Newtonian potential.* Next transform (2.3), (2.4) and (2.6) into the (ψ, θ, ϕ, t) system. Consider the following expression from (6.12),

(7.1) $$d\psi^2 + \rho^{-2} e^{-2\lambda + 2\mu} d\theta^2 \equiv d\psi^2 + e^{-2g} d\theta^2 = (d\psi + i e^{-g} d\theta)(d\psi - i e^{-g} d\theta)$$

where we put $e^g = \rho e^{\lambda-\mu}$. Let $(\alpha + i\beta)^{-1} \neq 0$, where both α and β are real, be an integrating factor of $d\psi + ie^{-g}d\theta$ so that

(7.2) $\quad d\psi + ie^{-g}d\theta = (\alpha + i\beta)(du + idv)$, and (6.2) becomes

(7.3) $$ds^2 = -e^{-2H}e^{2\lambda}(\alpha^2+\beta^2)(du^2+dv^2) - \rho^2 e^{2\gamma}d\phi^2 + e^{2\nu}dt^2.$$

Comparing (7.3) and (1.4) we obtain

(7.4) $$e^{2m} = e^{-2H}e^{2\lambda}(\alpha^2+\beta^2).$$

Equating real and imaginary parts in (7.2) we get

(7.5) $$d\psi = \alpha du - \beta dv, \quad d\theta = e^g(\beta du + \alpha dv),$$

from which the conditions of integrability for $d\psi$ and $d\theta$ give

(7.6) $\quad \alpha_v + \beta_u = 0, \quad \alpha_u - \beta_v = \beta g_v - \alpha g_u$, and furthermore

(7.7) $$\psi_{uu} + \psi_{vv} = -(\alpha^2+\beta^2)\frac{\partial g}{\partial\psi}, \quad \theta_{uu} + \theta_{vv} = (\alpha^2+\beta^2)e^{2g}\frac{\partial g}{\partial\theta}.$$

By (7.5) and (7.7), equation (2.3) becomes

(7.8) $$\frac{\partial^2\Phi}{\partial\psi^2} + e^{2g}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{\partial g}{\partial\psi}\ \frac{\partial\Phi}{\partial\psi} + e^{2g}\frac{\partial g}{\partial\theta}\ \frac{\partial\Phi}{\partial\theta} = 0.$$

In the like manner we get (2.4) in the form,

(7.9) $$\frac{\partial^2\nu}{\partial\psi^2} + e^{2g}\frac{\partial^2\nu}{\partial\theta^2} - \frac{\partial g}{\partial\psi}\ \frac{\partial\nu}{\partial\psi} + e^{2g}\frac{\partial g}{\partial\theta}\ \frac{\partial\nu}{\partial\theta} + \frac{\partial\nu}{\partial\psi}\ \frac{\partial\chi}{\partial\psi} + e^{2g}\frac{\partial\nu}{\partial\theta}\ \frac{\partial\chi}{\partial\theta} = 0.$$

In (6.12) we have four functions λ, μ, γ, ν to determine by means of the three independent equations (7.8), (7.9) and (2.7) which will be rendered into the (ψ, θ) coördinates by the foregoing analysis presently. In order to avoid the one degree of arbitrariness existing in this problem we assume the solution of ν to be

(7.10) $$g_{44} \equiv e^{2\nu} = 1 - 2M\psi$$

where M is the mass of the body and ψ according to (6.9) is the Newtonian potential per unit mass. This general assumption includes apparently Schwarzschild's solution as a special case. Equation (7.9) then becomes

(7.11) $$\partial n/\partial\psi - \partial g/\partial\psi - \partial\nu/\partial\psi = 0,$$

giving

(7.12) $$e^{\gamma} = e^{\nu+\lambda-\mu}\Theta(\theta)$$

where $\Theta(\theta)$ is an arbitrary function. At infinity where $\psi = 0$, $\lambda = \mu = \gamma$

$=\nu=0$ for all values of θ. Hence we have $\Theta(\theta)\equiv 1$ and (7.12) can be rewritten in the form,

$$\lambda+\nu=\gamma+\mu. \tag{7.13}$$

This condition is evidently satisfied by Schwarzschild's solution and even by the plane (5.12).

By (7.10) and (7.11) and since $\Phi=e^{n+\nu}$, (7.8) becomes

$$2\frac{\partial}{\partial\psi}\left[(1-2M\psi)\frac{\partial n}{\partial\psi}\right]+\frac{\partial^2}{\partial\theta^2}e^{2n}=0. \tag{7.14}$$

Consider (2.6). By (2.2), the exact differential

$$dm=m_u du+m_v dv \tag{7.15}$$

can be integrated into the form,

$$2m=\log(\chi_u^2+\chi_v^2)+\chi+2\int Pdu+\theta dv, \quad \text{where} \tag{7.16}$$

$$P=\chi_v C+\chi_u D,\quad Q=\chi_u C-\chi_v D, \tag{7.17}$$

$$\begin{aligned} C&=(\chi_u^2+\chi_v^2)^{-1}(n_u n_v+\nu_u\nu_v),\\ D&=\tfrac{1}{2}(\chi_u^2+\chi_v^2)^{-1}(n_u^2+\nu_u^2-n_v^2-\nu_v^2). \end{aligned} \tag{7.18}$$

Expression (7.16) contains only first partial derivatives and is simpler than (2.6). By (7.5) and the inverse relations, (7.16) becomes

$$2m=\log(\alpha^2+\beta^2)+\log(\chi_\psi^2+e^{2g}\chi_\theta^2)+\chi+2\int P'd\psi+Q'e^{-g}d\theta \tag{7.19}$$

$$\begin{aligned} &P'=e^g\chi_\theta C'+\chi_\psi D',\quad Q'=\chi_\psi C''-e^g\chi_\theta D\ ;\\ &C'=(\chi_\psi^2+e^{2g}\chi_\theta^2)^{-1}e^g(n_\psi n_\theta+\nu_\psi\nu_\theta),\\ &D'=\tfrac{1}{2}(\chi_\psi^2+e^{2g}\chi_\theta^2)^{-1}(n_\psi^2+\nu_\psi^2-e^{2g}[n_\theta^2+\nu_\theta^2]). \end{aligned} \tag{7.20}$$

Between (7.4) and (7.19) we can eliminate the auxiliary functions m and $\alpha^2+\beta^2$. A recapitulation of results gives

$$2\lambda=\log(\chi_\psi^2+e^{2g}\chi_\theta^2)+2H+\chi+2\int P'd\psi+Q'e^{-g}d\theta, \tag{7.21}$$

$$g_{44}\equiv e^{2\nu}=1-2M\psi, \tag{7.10}$$

$$\nabla^2\psi=0, \tag{6.9}$$

$$\lambda+\nu=\gamma+\mu, \tag{7.13}$$

$$2\frac{\partial}{\partial\psi}\left[(1-2M\psi)\frac{\partial n}{\partial\psi}\right]+\frac{\partial^2}{\partial\theta^2}e^{2n}=0, \tag{7.14}$$

$$e^n=\rho e^\gamma,\quad \chi=n+\nu,\quad e^g=\rho e^{\lambda-\mu}=\rho e^{\gamma-\nu},$$

(6.12) $ds^2 = e^{-2H}\,[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2] - \rho^2 e^{2\gamma}d\phi^2 + e^{2\nu}dt^2.$

(6.13) $e^{-2H} = h^2(\psi\ ^2 + \psi\ ^2)^{-1}.$

From the above list we see immediately that the solution of the whole problem depends upon the solution of the well known Laplace's equation, (6.9), the non-linear equation, (7.14) and a quadrature, (7.21).

IV. Fields of Spheroidal Homoeoids.

8. *Oblate spheroidal homoeoid.* Let the equation of the homoeoid be

$$\rho^2/a^2 + z^2/c^2 = 1, \qquad (\rho^2 = x^2 + y^2,\quad a^2 > c^2). \tag{8.1}$$

Use spheroidal coördinates, ξ, η, defined by

$$\rho + iz = \kappa \cos(\xi + i\eta) \qquad (\kappa^2 = a^2 - c^2). \tag{8.2}$$

Then $\eta =$ const. represents a family of oblate spheroids confocal with (8.1), which is $\kappa \cosh \eta = a$ in the family and $\xi =$ const. a family of hyperboloids of one sheet confocal with and orthogonal to the spheroids.

The Newtonian potential for an oblate spheroidal homoeoid with unit mass is

$$\psi = \kappa^{-1} \cot^{-1}(\sinh \eta). \tag{8.3}$$

The function, θ, defined by (6.7) may be taken as

$$\theta = \sin \xi. \tag{8.4}$$

From (8.2), (8.3) and (8.4),

$$\rho^2 = \kappa^2(1 - \theta^2)\csc^2 \kappa\psi. \tag{8.5}$$

In the present case (6.12) is

$$ds^2 = -\, e^{-2H}\,[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2] - \rho^2 e^{2\gamma}d\phi^2 + e^{2\nu}dt^2, \tag{8.6}$$

where
$$e^{-2H} = \kappa^4 \cosh^2 \eta(\sinh^2 \eta + \sin^2 \xi),$$

and λ, μ, γ are to be determined, ν being given by (7.10).

The equation (7.14) that γ must satisfy becomes

$$\frac{d}{d\psi}\left[(1 - 2M\psi)\left(\frac{d\gamma}{d\psi} - \kappa \cot \kappa\psi\right)\right] - \kappa^2 e^{2\gamma} \csc^2 \kappa\psi = 0, \tag{8.7}$$

in which we assume that γ is a function of ψ alone. Equation (8.7) is solvable by the following changes of variables,

(8.8) $R = e^{\gamma}(1 - 2M\psi)^{1/2} \csc \kappa\psi, \quad du = -\csc \kappa\psi \cdot e^{\gamma}(1 - 2M\psi)^{-1/2}\kappa\, d\psi,$

where the negative sign in du is chosen to make u tend toward positive infinity as ψ approaches zero. (8. 7) then becomes

(8. 9) $$d^2R/du^2 - R = 0, \quad \text{giving}$$

(8. 10) $$R = c_1 \sinh u.$$

By (8. 8) and (8. 10), we have

(8. 11) $$c_1 \sinh u = e^{\gamma}(1 - 2M\psi)^{1/2} \csc \kappa\psi$$

from which, and the choice of du in (8. 8), we see that c_1 must be a positive constant. Eliminating e^{γ} between du in (8. 8) and (8. 11) we get

(8. 12) $$-c_1\kappa(1 - 2M\psi)^{-1}\, d\psi = \operatorname{csch} u\; du, \quad \text{which gives}$$

(8. 13) $$(c_1\kappa/2M)\log(1 - 2M\psi) = \log(\coth u - \operatorname{csch} u) + c_2.$$

Eliminating u between (8. 11) and (8. 13) we find

(8. 14) $$\gamma = -\tfrac{1}{2}(c_1\kappa/M + 1)\log(1 - 2M\psi) + \log\{[e^{2\gamma}(1 - 2M\psi) + c_1^2 \sin^2 \kappa\psi]^{1/2} - c_1 \sin \kappa\psi\} + c_2,$$

the general solution of (8. 7) involving the two constants, c_1 and c_2. To determine c_2 let ψ approach zero. Then γ tends toward zero and $c_2 = 0$. The constant c_1 can be identified with M/k, which is also a constant of integration in Newton's theory. (8. 14) now becomes

(8. 15) $$e^{\gamma} = k^{-1}\psi^{-1} \sin k\psi.$$

Obviously e^{γ} approaches unity as ψ tends toward zero.

Last, we must obtain λ. Knowing λ we can get μ by (7. 13) and (8. 15). In order to avoid cumbersome differentiations in integrating (7. 21) directly we use the transformation of ψ in (8. 8), and furthermore put $dv = d\xi$. By (7. 13), (8. 3), (8. 5), (8. 8) and (8. 15), (8. 6) can be written in the form,

(8. 16) $$ds^2 = -\kappa^{-2}e^{-2H}e^{2\mu}\sin^2 \kappa\psi\,[du^2 + dv^2] - \psi^{-2}\cos^2 v\, d\phi^2 + e^{2\nu}dt^2,$$

which has the same form as (1. 4), provided

(8. 17) $$e^{2m} = \kappa^2 e^{2\mu}(\cot^2 \kappa\psi + \sin^2 v).$$

By (8. 8) and (8. 15), we see that ψ can be expressed as an explicit function of u, and (2. 6), that must be satisfied by μ, can be computed with the aid of R in (8. 10). The quadrature in terms of the u, v variables is quite simple. Coupled with the condition that at infinite distances from the body μ must vanish, the function, μ, is found to be

(8. 18) $$e^{2\mu} = \kappa^{-2}\psi^{-2}(\sinh^2 \eta + \sin^2 \xi)^{-1}.$$

From (7.13) and (8.18), λ is given by

$$(8.19)\qquad e^{2\lambda} = \kappa^{-4}\psi^{-4}\operatorname{sech}^2\eta(1-2M\psi)^{-1}(\sinh^2\eta+\sin^2\xi)^{-1}.$$

gain, ds^2 in (8.6) becomes

$$(8.20)\qquad \begin{aligned} ds^2 = -\kappa^{-2}\psi^{-2}\,[\psi^{-2}(1-2M\psi)^{-1}\operatorname{sech}^2\eta d\eta^2 \\ +\kappa^2 d\xi^2+\kappa^2\cos^2\xi d\phi^2]+(1-2M\psi)\,dt^2. \end{aligned}$$

Solving for $\sinh\eta$ and $\sin\xi$ from (8.2) we get

$$(8.21)\qquad \begin{aligned} 2\kappa^2\sinh^2\eta &= r^2-\kappa^2+[r^4-2\kappa^2(\rho^2-z^2)+\kappa^4]^{1/2}, \\ 2\kappa^2\sin^2\xi &= -(r^2-\kappa^2)+[r^4-2\kappa^2(\rho^2-z^2)+\kappa^4]^{1/2}. \end{aligned} \qquad (r^2=\rho^2+z^2)$$

When κ is small compared with r, these expressions can be expanded in the following forms:

$$(8.22)\qquad \begin{aligned} \kappa^2\sinh^2\eta &= r^2\left[1-\frac{\rho^2}{r^4}\kappa^2+\frac{(1-\omega^2)}{4r^4}\kappa^4+\cdots\right], \\ \sin^2\xi &= \frac{z^2}{r^2}+\frac{1-\omega^2}{4r^2}\kappa^2+\frac{\omega(1-\omega^2)}{4r^4}\kappa^4+\cdots, \\ \omega &= (\rho^2-z^2)/(\rho^2+z^2). \end{aligned}$$

It is interesting to observe from (8.22) that when κ approaches zero, namely, when the spheroidal homoeoid tends toward a spherical shell as a limit, the line element (8.20) becomes Schwarzschild's solution. Furthermore, (8.20) is also the solution of an infinitely thin material disc with mass M and radius κ.

9. *Prolate spheroidal homoeoid.* The treatment of the prolate spheroidal homoeoid is analogous to the preceding problem. Here the equation of the surfaces of the body is given by

$$(9.1)\qquad \rho^2/a^2+z^2/c^2=1 \quad\text{with}\quad c^2>a^2.$$

The spheroidal coördinates ξ, η, used are defined by

$$(9.2)\qquad z+i\rho=\kappa\cos(\xi+i\eta), \qquad (\kappa^2=c^2-a^2).$$

The Newtonian potential for (9.1) with unit mass is

$$(9.3)\qquad \psi=\frac{1}{2\kappa}\log\frac{\cosh\eta+1}{\cosh\eta-1}.$$

The function, θ, defined by (6.7) becomes

$$(9.4)\qquad \theta=-\cos\xi,$$

and ds^2 given in (6.12) is then

$$(9.5)\qquad \begin{aligned} ds^2 &= - e^{-2H}\left[e^{2\lambda}d\psi^2 + \rho^{-2}e^{2\mu}d\theta^2\right] - \rho^2 e^{2\gamma}d\phi^2 + e^{2\nu}dt^2, \\ e^{-2H} &= \kappa^4 \sinh^2\eta\,(\sinh^2\eta + \sin^2\xi), \quad e^{2\nu} = 1 - 2M\psi, \end{aligned}$$

where γ is assumed to be a function of ψ alone while both λ and μ are functions of ψ and θ. Between λ, μ, γ, ν we have the relation (7.13), namely $\lambda + \nu = \gamma + \mu$. The equation (7.14) for γ in the present case is similar to (8.7), so the remaining analysis will be similar. The result is

$$(9.6)\qquad ds^2 = -\kappa^2(\sinh^2\eta + \sin^2\xi)\left[e^{2\lambda}d\eta^2 + e^{2\mu}d\xi^2\right] - \rho^2 e^{2\gamma}d\phi^2 + e^{2\nu}dt^2,$$

where

$$\begin{aligned} e^{2\lambda} &= \kappa^{-4}\psi^{-4}\operatorname{csch}^2\eta\,(1-2M\psi)^{-1}(\sinh^2\eta + \sin^2\xi)^{-1}, \\ e^{2\mu} &= \kappa^{-2}\psi^{-2}\left[\sinh^2\eta + \sin^2\xi\right]^{-1}, \\ e^{2\gamma} &= \kappa^{-2}\psi^{-2}\operatorname{csch}^2\eta. \end{aligned}$$

We notice that (9.6) is also the solution for a rod of length κ and mass M lying on the z-axis. Similarly when the prolate spheroidal homoeoid approaches a spherical shell as a limit, (9.6) degenerates into Schwarzschild's solution.

10. *Motion of a particle in the field of an oblate spheroidal homoeoid.* The fundamental quadratic differential form (8.20) for an oblate spheroidal homoeoid can also be written in the form

$$(10.1)\quad ds^2 = -\psi^{-4}(1-2M\psi)^{-1}d\psi^2 - \psi^{-2}d\xi^2 - \psi^{-2}\cos^2\xi d\phi^2 + (1-2M\psi)dt^2.$$

If for convenience we put

$$(10.2)\qquad \psi = 1/r, \quad \xi = \theta - \pi/2,$$

where it must be remembered that r and θ are not the r and θ used in previous sections, then (10.1) becomes

$$(10.3)\quad ds^2 = -(1-2M/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + (1-2M/r)dt^2,$$

which has the same analytical form as Schwarzschild's solution. The results worked out in the latter case are immediately applicable to the present problem, provided we interpret the symbols in (10.3) appropriately.

The four differential equations,

$$(10.4)\qquad \frac{d^2x_\alpha}{ds^2} + \{\mu\nu, \alpha\}\frac{dx_\mu}{ds}\;\frac{dx_\nu}{ds} = 0,$$

defining the motion of an infinitesimal particle in the four dimensional continuum characterized by (10.3) are *

* A. S. Eddington, *loc. cit.*, pp. 85.

$$(10.5)\qquad \frac{d^2r}{ds^2}+\lambda'\left(\frac{dr}{ds}\right)^2-re^{-2\lambda}\left(\frac{d\theta}{ds}\right)^2-r\sin^2\theta e^{-2\lambda}\left(\frac{d\phi}{ds}\right)^2 + e^{2\nu-2\lambda}\nu'\left(\frac{dt}{ds}\right)^2=0,$$

$$(10.6)\qquad \frac{d^2\theta}{ds^2}+\frac{2}{r}\,\frac{dr}{ds}\,\frac{d\theta}{ds}-\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2=0,$$

$$(10.7)\qquad \frac{d^2\phi}{ds^2}+\frac{2}{r}\,\frac{dr}{ds}\,\frac{d\phi}{ds}+2\cot\theta\,\frac{d\theta}{ds}\,\frac{d\phi}{ds}=0,$$

$$(10.8)\qquad \frac{d^2t}{ds^2}+2\nu'\,\frac{dr}{ds}\,\frac{dt}{ds}=0,$$

where $\qquad e^{2\nu}=1-2M/r,\quad \lambda+\nu=0,\quad \nu'=d\nu/dr.$

Instead of using (10.5) we can take (10.3), which can be written as

$$(10.9)\qquad e^{-2\nu}\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\theta}{ds}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2-e^{2\nu}\left(\frac{dt}{ds}\right)^2=-1.$$

Equations (10.7) and (10.8) are immediately integrable, giving respectively

$$(10.10)\qquad r^2\sin^2\theta(d\phi/ds)=c_2,$$

$$(10.11)\qquad dt/ds=c_1e^{-2\nu}.$$

Let the constants of integration c_1 and c_2 be positive.

Eliminating $d\phi/ds$ between (10.6) and (10.10), we find

$$(10.12)\qquad \frac{d^2\theta}{ds^2}+\frac{2}{r}\,\frac{dr}{ds}\,\frac{d\theta}{ds}-\frac{c_2^2}{r^4}\cos\theta\csc^3\theta=0,\quad\text{giving}$$

$$(10.13)\qquad r^4(d\theta/ds)^2+c_2^2\csc^2\theta=c_3^2\qquad(\text{Take } c_3>0).$$

Eliminating ds from (10.10) and (10.13), we get

$$(10.14)\qquad d\phi=-c_2\left[(c_3^2-c_2^2)-c_2^2\cot^2\theta\right]^{-1/2}\csc^2\theta\, d\theta,$$

in which we choose the negative sign to make θ decrease when ϕ increases. Let

$$(10.15)\qquad p=(c_3^2-c_2^2)^{1/2}/c_2.$$

By (10.13) since r, θ, s are all real we see that $c_3^2\geqq c_2^2$ and consequently p is real. Integrating (10.14), we obtain

$$(10.16)\qquad \cot\theta\doteq p\sin(\phi-\Omega),$$

where Ω is the node, and θ is taken to be $\pi/2$ when $\phi=\Omega$. The geo-

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metrical meaning of $\theta = \pi/2$ is that $z = 0$ where the particle crosses the equatorial plane of the oblate homoeoid [cf. (10.2) and (8.2)].

By using (10.9), (10.10), (10.11), (10.13) and (10.16), we obtain the following relation between ψ and ϕ,

(10.17) $$c_2[2Mf(\psi)]^{-1/2}d\psi = -c_3[1+p^2\sin^2(\phi - \Omega)]^{-1}d\phi, \quad \text{where}$$

(10.18) $$f(\psi) \equiv \psi^3 - \frac{1}{2M}\psi^2 + \frac{1}{c_3^2}\psi + \frac{c_1^2-1}{2Mc_3^2};$$

the negative sign in (10.17) will be explained presently.

The right hand side of (10.17) is immediately integrable in terms of circular functions. The rigorous integration of the left hand side in terms of elliptic functions has been discussed by Forsyth.* Let α, β, γ $(\alpha > \beta > \gamma)$ be the three roots of $f(\psi)$. Then ψ can lie only with the interval $\beta \geqq \psi \geqq \gamma$. When $\psi = \beta$, we have the analogous "perihelion" and when $\psi = \gamma$, the "aphelion." Let $\phi = \phi_0$, when $\psi = \beta$. Integrating, we have

(10.19) $$c_2\int_\beta^\psi [2Mf(\psi)]^{-1/2}d\psi = -c_3\int_{\phi_0}^\phi [1+p^2\sin^2(\phi - \Omega)]^{-1}d\phi.$$

Here we see that since ψ decreases after $\psi = \beta$, but that ϕ continues to increase after $\phi = \phi_0$, the negative sign in (10.17) must be taken.

From (10.19) we obtain

(10.20) $$\psi = \gamma + (\beta - \gamma)(1 - cn2\mu)/(1 + dn2\mu),$$

where μ is defined by the equation,

(12.21) $$2\mu = 2K - (1/P)\{\tan^{-1}[\sigma\tan(\phi - \Omega)] - \tan^{-1}[\sigma\tan(\phi_0 - \Omega)]\}$$

in which $$\sigma = c_3/c_2, \quad P = [2M(\alpha - \gamma)]^{-1/2},$$

and K is the complete elliptic integral of the first kind with modulus, k, given by

(10.22) $$k^2 = (\beta - \gamma)/(\alpha - \gamma).$$

From (8.2), (8.3), (10.2), (10.16), (10.20) and (10.21), we obtain the equations of the orbit of the particle in the following forms,

(10.23) $$\begin{aligned} \rho &= \kappa[1+p^2\sin^2(\phi - \Omega)]^{-1/2}\csc\kappa\psi, \\ z &= \kappa p\sin(\phi - \Omega)[1+p^2\sin^2(\phi - \Omega)]^{-1/2}\cot\kappa\psi. \end{aligned}$$

The equation $\theta = $ const. [cf. (10.2) and (8.2)] represents the family of hyperboloids of one sheet orthogonal to the family of spheroids $\psi = $ const.

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* A. R. Forsyth, *Proceedings of the Royal Society* (A), Vol. 97 (1920), pp. 145.

Then (10.16) shows that the maximum and minimum latitudes of the particle in its orbit are invariable for given initial conditions.

The function, ψ, in (10.20) is a Jacobian elliptic function of ϕ. Hence the analogous "line of apsides" of the orbit precesses about the z axis. The amount of this precession for the particle to describe the orbit once can be calculated in the following manner.* In (10.20) we have so chosen ψ, ϕ that at perihelion $\psi = \beta$, $\phi = \phi_0$. Then at aphelion $\psi = \gamma$, let $\phi = \phi_1$. From (10.20) and (10.21),

$$(10.24) \quad \tan^{-1}[\sigma \tan(\phi_1 - \Omega)] - \tan^{-1}[\sigma \tan(\phi_0 - \Omega)] = 2PK.$$

At the next perihelion let $\phi = \phi_2$. The relation analogous to (10.21) is

$$(10.25) \quad \tan^{-1}[\sigma \tan(\phi_2 - \Omega)] - \tan^{-1}[\sigma \tan(\phi_1 - \Omega)] = 2PK.$$

Adding (10.24) and (10.25), we get

$$(10.26) \quad \tan^{-1}[\sigma \tan(\phi_2 - \Omega)] - \tan^{-1}[\sigma \tan(\phi_0 - \Omega)] = 4PK.$$

The precession is given by

$$(10.27) \qquad \Delta = \phi_2 - \phi_0 - 2\pi.$$

Solving ϕ_2 from (10.26), we get

$$(10.28) \quad \Delta = \tan^{-1} \frac{1}{\sigma} \left\{ \frac{\tan 4PK + \sigma \tan(\phi_0 - \Omega)}{1 - \sigma \tan(\phi_0 - \Omega) \tan 4PK} \right\} - (\phi_0 - \Omega) - 2\pi.$$

It is interesting to observe from (10.13), (10.15) and (10.23) that if the particle lies initially in the equatorial plane of the homoeoid, i.e. $d\theta/ds = 0$ when $\theta = \pi/2$, then subsequently $\theta = \pi/2$ and the particle will continually lie there. The approximate formula for Δ in this case can be calculated as follows: Regard (r, ϕ) as configurational polar coördinates of the particle. Then (10.3) shows that the motion of the particle in these coördinates is the same as the motion of a corresponding particle in Schwarzschild's solution. Hence the constants, c_1 and c_2, in (10.11) and (10.10) are given by †

$$(10.29) \quad c_2^2 = r_0(1 - e^2)M, \quad c_1^2 - 1 = -M/r_0, \quad e^2 = (r_0^2 - r_1^2)/r_0^2,$$

where M is the mass of the homoeoid, r_0 the semi-major axis, r_1 the semi-minor axis, and e the eccentricity of the orbit in the configurational coördinate system. The advance of the perihelion is given approximately by

$$(10.30) \qquad \Delta = 2\pi \cdot 3M/r_0(1 - e^2).$$

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* A. R. Forsyth, *loc. cit.*, p. 148.

† A. R. Forsyth, *loc. cit.*, p. 145.

From (8.3) and (8.21) with $z=0$, and (10.2), we obtain

$$\frac{1}{r}=\psi=\frac{1}{\kappa}\cot^{-1}\left[\frac{1}{\kappa}(\rho^2-\kappa^2)^{1/2}\right] \tag{10.31}$$

When $\rho^2-\kappa^2>\kappa^2$ which is obviously satisfied by large values of ρ, we can expand (10.31) in ascending powers of κ/ρ in the form,

$$\frac{1}{r}=\frac{1}{\rho}\Big[[1+\frac{1}{6}\frac{\kappa^2}{\rho^2}+\frac{3}{40}\frac{\kappa^4}{\rho^4}+\cdots]. \tag{10.32}$$

Equation (10.31) shows that ρ is a monotonic function of r, and consequently the value of Δ in (10.30), which is primarily for the orbit in the (r,ϕ) configurational coördinates will hold also in the (ρ,ϕ) system. Knowing the "semi-major" and "semi-minor" axes, ρ_0 and ρ_1 of the particle's orbit in the latter system we can compute the corresponding values of r_0 and r_1 by (10.32). Then (10.30) shows that the oblateness of the central body causes a small increase in the advance of the perihelion of the orbit predicted from Schwarzschild's solution. This increase vanishes when $\kappa=0$, namely, when the oblate spheroidal homoeoid degenerates into a spherical shell.

In conclusion I wish to thank Prof. E. T. Bell heartily, whose interest in this problem and encouragement in the course of work have made this investigation possible.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIFORNIA.

ON THE NON-EXISTENCE OF CURVES OF ORDER 8 WITH 16 CUSPS.

By Oscar Zariski.

1. In a paper, which will be published in the "Annals of Mathematics", I prove incidentally by means of several examples, that the answer to the well-known question,* as to whether there always exist plane algebraic curves with assigned Plückerian characters, is negative. It is understood, of course, that the assigned characters satisfy the Plücker relations, and are essentially non-negative integers. For instance, I prove that a curve of order 7 cannot possess 11 (or more) cusps.† Thus, with the usual notations for the Plückerian characters of a curve (n-order, m-class, d-number of nodes, k-number of cusps, τ-number of double tangents, i-number of flexes), we have then that the following sets of Plückerian characters do not correspond to any effectively existent curve:

$$(1) \quad n=7,\ d=0,\ k=11;\ m=9,\ \tau=7,\ i=17;$$
$$(2) \quad n=m=7,\ d=\tau=1,\ k=i=11.$$

Similarly, I prove that a curve of order 8 cannot possess 16 (or more) cusps. For instance, there do not exist (self-dual) curves of order 8 with 16 cusps and no nodes.‡

The proof of the non-existence of the above curves is based on the following general theorem, proved in my paper as a consequence of certain results concerning the determination of irregular cyclic multiple planes: *the linear*

* See the interesting paper by S. Lefschetz, "On the Existence of Loci with Given Singularities," *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 23-41, which constitutes perhaps the first attempt to throw light on questions as yet unsolved in all their generality, relative to the *dimension* and *existence* of continuous systems of cuspidal curves. The examples given below solve some of the questions explicitly raised by Lefschetz in his paper.

† On the other hand, seventics with 10 cusps exist, as is shown by the example of the seventic with 10 cusps and 3 nodes, which is the dual of a sextic curve with 7 cusps and one node.

‡ In contradiction with a statement made by B. Segre in his paper "Esistenza e dimensione di sistemi continui di curve piane algebriche con dati caratteri," *Rendiconti della R. Accademia Nazionale dei Lincei*, (July, 1929), p. 38. The above examples correspond to the lowest possible values of n. In fact, it is easily shown that when $n \leqq 6$, any set of (non-negative) Plückerian characters belong to some effectively existent curve. For instance, the existence of a curve of order 6 with 7 cusps and one node, mentioned in the previous footnote, can be ascertained as follows: The dual of a quartic curve with two cusps is a curve of order 6 possessing 8 cusps and one node. Since the complete continuous system of curves, having the same characters, has its characteristic series non-special, one of the 8 cusps can be considered as virtually non-existent.

system of curves of order $n-3-j$ *determined by simple basis points at the cusps of a plane irreducible algebraic curve of order* n, *is regular* (effective dimension = virtual dimension) *for any value of the integer* j, *such that* $6j < n$. The non-existence of the above mentioned cuspidal curves follows at once from this theorem, if we observe that in each of these examples the corresponding curves, if they existed, would be certainly irreducible and that in each case one of the linear systems, mentioned in the above theorem, would have a virtual dimension equal to -2.

In my paper the quoted theorem is based on considerations, which involve the elements of the theory of algebraic surfaces. Since the question of the Plückerian characters of a curve, so elementary as far as its formulation is concerned, is essentially a problem of plane geometry, it seems desirable to consider one of the examples quoted above and to arrive at the desired conclusion of the non-existence in a more elementary way, by making use only of the elements of plane geometry and of the geometry on an algebraic one-dimensional variety. Accordingly, we propose to give in this paper a direct proof of the non-existence of curves of order 8 with 16 cusps.

2. For the proof, let us suppose that a curve,

$$f(x, y) = 0,$$

of order 8 with 16 cusps and no nodes exists. This curve is self-dual, i. e., $m = 8$, $\tau = 0$, $i = 16$. We shall have to consider the curve f from its dual aspect, and in doing this we shall face the question whether the dual singularities of f are exactly of the same type as the point singularities of f, namely ordinary flexes and no double tangents. This is by no means obvious. The general question involved is the following: given a curve f possessing cusps and nodes only, what can be said about the nature of the dual singularities of f? Obviously, it would not be correct to say, that the dual singularities of f are necessarily double tangents and ordinary flexes. Thus a curve without point singularities at all can possess flexes of order $s > 1$ (where the flex tangent has $s + 2$ coincident intersections with the curve), triple tangents, etc. The question raised must refer, in its correct form, not to the individual curve f, but to the most general curve of the same order and having the same singularities as f, or better, to the generic curve of a complete continuous system of such curves. We can ask namely, whether the dual singularities of the *generic* curve of a complete continuous system of curves, having cusps and nodes only, are exactly double tangents and ordinary flexes. Obvious as the answer may seem, it requires a proof. However, it is not our intention to attempt such a proof in this paper. For our purpose it will be sufficient to show that in one particular case the answer to the above question is in the

affirmative, and not only for the generic curve of the system but for *every* individual curve of the system possessing nodes and cusps only. This is the case in which $\tau = 0$. We propose then to prove the following

LEMMA. *Given an irreducible curve f of order n, possessing d nodes and k cusps. If the number τ of the double tangents, evaluated according to the Plücker relations, has the value* 0, *then the only dual singularities of f are exactly ordinary flexes.*

Denote, as usual, by i the number of flexes, as evaluated according to the Plücker formulas, and by p the genus of f. We shall have

$$p = (n-1)(n-2)/2 - d - k = (m-1)(m-2)/2 - i,$$

where m is the class of f. Denote by $\bar{f}$ the transform of f by duality. Let $q_1, q_2, \cdots, q_l$ be the singular tangents of f, and let $Q_1, Q_2, \cdots, Q_l$ be the corresponding singular points of $\bar{f}$. The line q_j will touch f at certain points, having at each point a contact of a certain order with f. In addition q_j may pass through some of the nodes and cusps of f. To each contact of order s will correspond by duality a branch of $\bar{f}$ of order s and of origin Q_j, in the neighborhood of which there will be only simple points infinitely near to Q_j. To distinct points of contact there will correspond branches with distinct tangents. If q_j passes through a node, we shall have an additional branch of origin Q_j, if and only if q_j coincides with one of the principal tangents of the double point. Similarly, if q_j passes through a cusp, we shall have an additional linear branch of origin Q_j, if and only if q_j coincides with the cusp tangent.

Let $s_{j1}, s_{j2}, \cdots, s_{jr_j}$ be the orders of contact of q_j at the r_j points at which it touches f, including possibly the nodes and cusps, through which q_j happens to pass and at which it coincides with a principal tangent.*

Let

$$s_j = s_{j1} + s_{j2} + \cdots + s_{jr_j}.$$

Then Q_j is an s_j-fold point of $\bar{f}$, and since in its neighborhood there are only simple points it follows that Q_j absorbs $s_j(s_j - 1)/2$ ordinary double points of $\bar{f}$. Hence

$$p = (m-1)(m-2)/2 - \sum_{j=1}^{l} s_j(s_j - 1)/2,$$

and consequently

$$\sum_{j=1}^{l} s_j(s_j - 1)/2 = i. \tag{1}$$

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* At any of these cusps we must put $s = 1$. At a node the order of contact of q_j with one of the branches through the double point is to be considered.

We also have,

$$n = m(m-1) - 3i.$$

On the other hand it is obvious that the multiple point Q_j of $\bar{f}$ diminishes the class of $\bar{f}$ by

$$s_j(s_j-1) + (s_{j1}-1) + (s_{j2}-1) + \cdots + (s_{jr_j}-1) = s_j^2 - r_j.$$

Hence

$$n = m(m-1) - \sum_{j=1}^{l} (s_j^2 - r_j),$$

and consequently

$$(2) \qquad \sum_{j=1}^{l} (s_j^2 - r_j) = 3i.$$

From (1) and (2) we deduce:

$$3\sum_{j=1}^{l} s_j(s_j-1)/2 = \sum_{j=1}^{l} (s_j^2 - r_j),$$

or

$$(3) \qquad \sum_{j=1}^{l} [s_j(s_j-3)/2 + r_j] = 0.$$

Since for any j, $s_j \geqq 2$ and r_j is positive, it follows that each term of the written sum is $\geqq 0$, where the sign $=$ holds, if and only if $s_j = 2$ and $r_j = 1$. Hence the relation (3) implies that $s_j = 2$, $r_j = 1$ for $j = 1, 2, \cdots, l$, i. e., that the only singularities of $\bar{f}$ are ordinary cusps, which proves our Lemma.

COROLLARY. *A curve f of order* 8 *with* 16 *cusps and no nodes, if it exists, possesses exactly* 16 *ordinary flexes, i. e., the transform $\bar{f}$ of f will also possess* 16 *ordinary cusps.*

3. Let $f(x, y) = 0$ be a curve (necessarily irreducible) of order 8 with 16 cusps and no nodes. Its genus is 5. Denote by Γ_8 a set of 8 points cut out on f by a line of the plane, by $\bar{\Gamma}_8$ a set of 8 points, outside the cusps, cut out on f by a first polar of f, and by $|\Gamma_8|$ and $|\bar{\Gamma}_8|$ the corresponding complete series on f. Let moreover H_8 be a canonical set on f, i. e., a set of the canonical series g_8^4. Furthermore denote by K_{16} the set of 16 cusps of f. We then have the following relations:

$$(4) \qquad 2\,\Gamma_8 + H_8 \equiv \bar{\Gamma}_8 + K_{16};$$

$$(4') \qquad 5\,\Gamma_8 \equiv H_8 + 2\,K_{16}.$$

The relation (4) expresses the known fact that the Jacobian set $\Gamma_{24} \equiv \bar{\Gamma}_8 + K_{16}$ of the g_8^1 cut out on f by a pencil of lines is equivalent to the sum of a canonical set and of the two-fold of a set of the g_8^1. The relation (4′) expresses the fact that the canonical series is cut out on f by the adjoint curves of order 5. Eliminating the set K_{16} between (4) and (4′) we obtain

(5) $$\Gamma_8 + 2\,\bar{\Gamma}_8 \equiv 3\,H_8.$$

Since the curve f is self-dual, and since by the Lemma proved above, the curve $\bar{f}$ dual to f is exactly of the same type as f, the relation (5) can be dualized, by interchanging Γ_8 and $\bar{\Gamma}_8$ and by leaving H_8 unaltered. Hence

(5′) $$\bar{\Gamma}_8 + 2\,\Gamma_8 \equiv 3\,H_8.$$

From (5) and (5′) we deduce $\Gamma_8 \equiv \bar{\Gamma}_8$, i. e., the two series $|\Gamma_8|$ and $|\bar{\Gamma}_8|$ coincide.

There are now two possible cases to consider: (1) the series $|\Gamma_8|$ is non-special, and hence is a $g_8{}^3$; (2) the series $|\Gamma_8|$ is special, and then it necessarily coincides with the canonical series $g_8{}^4 = |H_8|$. We investigate the two cases separately.

4. Let us first suppose that $|\Gamma_8| = |\bar{\Gamma}_8|$ is a non-special series $g_8{}^3$. Let $g_8{}^2$ and $\bar{g}_8{}^2$ denote the (incomplete) series cut out on f by the lines of the plane and by the first polars of f respectively. Since the two series, of dimension 2, are both contained in the series $|\Gamma_8|$ of dimension 3, they have a $g_8{}^1$ in common. This $g_8{}^1$ is cut out on f by a pencil of lines, say of center A, and on the other hand the same $g_8{}^1$ is cut out on f by a pencil of first polars; the pole B of the variable polar of the pencil will describe a line b. We have then the following situation: there exsts a point A and line b in the plane of f, such that for any point B of the line b the points of contact with f of the 8 tangents drawn from B are the intersections with f of a line a on A. It is easily seen that this situation is impossible. For, let us consider a point B_1 at which the line b meets the curve f. If A happens to be on b and on f, we may suppose that B_1 is distinct from A, since f possesses only ordinary flexes. Let a_1 be the line on A which corresponds to B_1. Since, as the variable point B of b approaches B_1, two or more of the points of contact with f of the tangents drawn from B approach B_1, it follows that the line a_1 necessarily coincides with the lino AB_1. It follows that the line a_1 must absorb all the 8 tangents, which can be drawn through B_1, which is impossible, since f possesses only ordinary flexes.

5. We now consider the case, in which the series $|\Gamma_8| = |\bar{\Gamma}_8|$ coincides with the canonical series $g_8{}^4$. In this case there exist adjoint curves of order 5, which cut out on f sets of the $g_8{}^2$ cut out by the lines of the plane. These adjoint curves necessarily degenerate into a line and into a fixed quartic curve. Hence the 16 cusps of f lie on a quartic curve, which obviously does not meet f outside the cusps. It can also be shown that there exists a sextic curve, which passes through the cusps of f and touches at each cusp the cusp tangent (and which therefore does not meet f outside the cusps). In fact, the relations (4) and (4′) yield by subtraction:

$$K_{16} \equiv 3\,\Gamma_8 + \bar{\Gamma}_8 - 2\,H_8,$$

or, in view of (5′),

$$K_{16} \equiv \Gamma_8 + H_8.$$

This relation shows that the set K_{16} of the cusps of f belongs to the series $|\,\Gamma_8 + H_8\,|$, which is cut out on f by the adjoint curves of order 6. It follows that there exists an adjoint sextic curve, whose 16 intersections with f (generally outside the cusps) fall at the cusps of f. Such a sextic must touch at each cusp the cusp tangent.

Let $\psi_6(x, y) = 0$ and $\phi_4(x, y) = 0$ be the equations of this sextic curve and of the above quartic curve respectively. Let us consider the pencil

$$[\psi_6(x, y)]^2 - t[\phi_4(x, y)]^3 = 0.$$

The curves of this pencil do not have variable intersections with the curve f, since all the intersections fall at the cusps of f. Hence, for a proper value of t, which we may suppose to be $t = 1$, the corresponding curve of the pencil will contain the curve f as a component. We have then

$$(6) \qquad [\psi_6(x, y)]^2 - [\phi_4(x, y)]^3 = A_4(x, y) \cdot f(x, y),$$

where $A_4(x, y)$ is a polynominal of order 4 in x and y.

To prove that the relation (6) cannot hold we first observe that the curves $\psi_6 = 0$ and $\phi_4 = 0$, which we shall denote in the sequel by C_6 and C_4, satisfy the following conditions: (1) the cusps of the curve f are *simple* points of the two curves, and at each cusp the two curves have distinct tangents; (2) if the two curves are reducible, they do not have common components; (3) each curve possesses only a finite number of multiple points. In fact, (1) holds, because the two curves C_6 and C_4 have at each cusp of f exactly 3 and 2 coincident intersections with f respectively, otherwise the total number of intersections of one of these curves with f would be greater than the product of its order and the order of f. The condition (2) holds, because a common component of the curves would have to meet f at the cusps only, and the two curves would have at some cusp a common tangent. Finally neither one of the curves can possess a curve of multiple points, because such a curve would have to meet f at the cusps only, which contradicts the condition (1).

It follows that C_6 and C_4 have exactly 8, distinct or coincident, intersections outside of the cusps of f. Each of these intersections is at least a double point of the curve $A_4 = 0$. Thus, if O is a common simple point of the curves C_6 and C_4, then in general O will be a cusp of the curve $A_4 = 0$. If, however, the point O absorbs two intersections of the curves C_6 and C_4, then O is either a tacnode of the second kind or a triple point for the curve $A_4 = 0$, according as C_6 possesses at O a simple point or a double point. In

both cases O will absorb 3 double points of the curve $A_4 = 0$. At any rate the meet points of C_6 and C_4 constitute a set of multiple points of the curve $A_4 = 0$, which will absorb at least 8 double points of the curve. It follows that the curve $A_4 = 0$ possesses *infinite* multiple points, since the maximum number of double points (or of equivalent singularities) which a curve of order n can possess, without possessing infinite multiple points, is $n\,(n-1)/2$. Hence the curve $A_4 = 0$ contains a line of multiple points or is a conic counted twice. We consider in the next two sections these two cases. The reader should bear in mind that the quartic $A_4 = 0$ cannot pass through a cusp of f, and that any point common to the curve $A_4 = 0$ and C_4 (or C_6) is also on C_6 (or C_4), but is not on the curve f.

6. Let the curve $A_4 = 0$ contain a line of multiple points, say the line $x = 0$. Let

$$\psi_6(x, y) = a_6(y) + a_5(y)x + a_4(y)x^2 + a_3(y)x^3 + \cdots,$$
$$\psi_4(x, y) = b_4(y) + b_3(y)x + b_2(y)x^2 + \cdots,$$

where all the missing terms contain higher powers of x. The coefficients $a_i(y), b_j(y)$ are polynominals in y of degrees indicated by the indices. Since, by hypothesis, x^2 is a factor of the polynominal $[\psi_6(x, y)]^2 - [\phi_4(x, y)]^3$, we must have

$$[a_6(y)]^2 = [b_4(y)]^3; \tag{7}$$
$$2a_6(y)a_5(y) = 3[b_4(y)]^2 b_3(y). \tag{7'}$$

From (7) we deduce that $a_6(y)$ and $b_4(y)$ are the cube and the square respectively of a polynomial of second degree.* Let the roots of this polynominal be assumed to be $y = 0$ and $y = \eta$. Then

$$a_6(y) = y^3(y-\eta)^3, \quad b_4(y) = y^2(y-\eta)^2; \tag{8}$$
$$2a_5(y) = 3y(y-\eta)b_3(y). \tag{8'}$$

We find then

$$\{[\psi_6(x, y)]^2 - [\phi_4(x, y)]^3\}/x^2 = y^2(y-\eta)^2\{-3/4[b_3(y)]^2 + 2y(y-\eta)a_4(y) - 3y^2(y-\eta)^2 b_2(y)\} + \cdots, \tag{9}$$

where all the missing terms involve the variable x. The points $(0, 0)$ and $(0, \eta)$ are on C_6, C_4 and on the curve $A_4 = 0$, which is made up of the line of double points $x = 0$ and of a residual conic. If $\eta \neq 0$, then, by (9), this residual conic passes through the points $(0, 0)$ and $(0, \eta)$, touching there the

* None of the polynomials $a_6(y)$, $b_4(y)$ can vanish identically, because the identical vanishing of one would imply the identical vanishing of the other, and the line $x = 0$ would be a common component of the two curves C_6 and C_4, which, as we observed above, is impossible.

line $x=0$. If $\eta=0$, the conic has at the point $(0,0)$ at least a 4 point contact with the line $x=0$. In both cases we deduce that this conic must degenerate into two lines, one of which is the line $x=0$, the points of which are therefore at least triple points of the curve $A_4=0$. Hence the left-hand member of (9) must be divisible by x, and we must have

$$(10)\qquad 3/4\,[b_3(y)]^2 = 2y(y-\eta)a_4(y) - 3y^2(y-\eta)^2b_2(y).$$

If $\eta\neq 0$, then it follows that $b_3(y)$ is divisible by $y(y-\eta)$ and that consequently also $a_4(y)$ is divisible by $y(y-\eta)$. Putting

$$b_3(y) = y(y-\eta)\,c_1(y),\quad a_4(y) = y(y-\eta)\,c_2(y),$$

we have, by (8) and (8′),

$$\begin{aligned}\psi_6(x,y) &= y^3(y-\eta)^3 + \tfrac{3}{2}y^2(y-\eta)^2c_1(y)x\\ &\qquad + y(y-\eta)c_2(y)x^2 + a_3(y)x^3 + \cdots,\\ \phi_4(x,y) &= y^2(y-\eta)^2 + y(y-\eta)c_1(y)x + b_2(y)x^2 + \cdots.\end{aligned}$$

We see that the curve C_6 possesses at the origin a triple point (at least) and that C_4 possesses at the origin a double point (at least). The origin is therefore at least a 6-fold point of the curve $[\psi_6(x,y)]^2 - [\phi_4(x,y)]^3 = 0$, which is impossible since this would imply that the curve f has at the origin a double point (at least), whereas we know that it does not pass through the origin at all.

Let now $\eta=0$. Then, by (10), $b_3(y)$ is divisible by y,

$$(11)\qquad b_3(y) = yc_2(y),$$

and (10) becomes

$$(10')\qquad 3/4\,[c_2(y)]^2 = 2a_4(y) - 3y^2b_2(y).$$

We find

$$\begin{aligned}(12)\quad \{[\psi_6(x,y)]^2 - [\phi_4(x,y)]^3\}/x^3 &= y^3\,\{3c_2(y)a_4(y)\\ &+ 2y^3a_3(y) - [c_2(y)]^3 - 6y^2c_2(y)b_2(y) - 3y^5b_1(y)\} + \cdots,\end{aligned}$$

where all the missing terms involve the variable x. The curve $A_4=0$ is made up of the line of triple points $x=0$, and of a residual line. The presence of the factor y^3 in the right-hand member of the relation (12) shows that this residual line must coincide with the line $x=0$, so that the quartic $A_4=0$ is merely the line $x=0$ counted 4 times. Expressing the fact that the term independent of x in the right-hand member of (12) vanishes identically, we find that $c_2(y)\{3a_4(y) - [c_2(y)]^2\}$ must be divisible by y^2. Since, by (10′), also $3/4\,[c_2(y)]^2 - 2a_4(y)$ is divisible by y^2, we deduce that $a_4(y)$ is divisible by y^2. Taking in account (8), (8′) and (11), we then deduce, as in the previous case $(\eta\neq 0)$, that the curves C_6 and C_4 have at the origin at least a triple point and a double point respectively, which is impossible.

7. There remains to consider the case in which the quartic curve $A_4 = 0$ is a conic C_2 counted twice. Let O be a point at which C_2 meets the curve C_4 and which is therefore also on the curve C_6, *but not on the curve f*. Since O is a double point of the curve $[\psi_6(x,y)]^2 - [\phi_4(x,y)]^3 = 0$, it is necessarily a simple point of the curve C_6. It is obvious that C_2 and C_6 have the same tangent at O. We suppose for simplicity that the point O is at the origin, and that the common tangent of C_6 and C_2 at O is the axis $y = 0$. We consider the expansions of y

$$\text{on } C_6\colon\ y = b_2x^2 + b_3x^3 + \cdots + b_kx^k + \cdots;$$
$$\text{on } C_2\colon\ y = c_2x^2 + c_3x^3 + \cdots + c_kx^k + \cdots,$$

which represent the curve C_6 and the conic C_2 respectively in the neighborhood of the origin. Let $A(x,y) = 0$ be the equation of the conic C_2. We may write then

$$(13)\quad \begin{cases} \psi_6(x,y) = (y - b_2x^2 - b_3x^3 - \cdots - b_kx^k - \cdots)\bar{\psi}(x,y); \\ A(x,y) = (y - c_2x^2 - c_3x^3 - \cdots - c_kx^k - \cdots)\bar{A}(x,y), \end{cases}$$

where $\bar{\psi}(x,y)$ and $\bar{A}(x,y)$ are polynomials in y, whose coefficients are functions of x, which are regular in the neighborhood of the value $x = 0$. Moreover, $\bar{\psi}(0,0) \neq 0$ and $\bar{A}(0,0) \neq 0$.

We now have the following relation:

$$(14)\quad \begin{aligned} &(y - b_2x^2 - b_3x^3 - \cdots - b_kx^k - \cdots)^2[\bar{\psi}(x,y)]^2 - [\phi_4(x,y)]^3 \\ &= (y - c_2x^2 - c_3x^3 - \cdots - c_kx^k - \cdots)^2[\bar{A}(x,y)]^2 f(x,y) \end{aligned}$$

It should be noticed that $f(0,0) \neq 0$. Our proof will consist in showing that (14) cannot hold unless $a_k = b_k$ for every value of k, which is impossible, since this would mean that the curve C_6, and hence also C_4, contain the conic C_2 as a component, which contradicts the fact that C_6 and C_4 have no common components. Let us suppose that the first $k - 2$ coefficients ($k \geqq 2$) of the above expansions are alike:

$$(15)\qquad b_2 = c_2,\ b_3 = c_3, \cdots,\ b_{k-1} = c_{k-1}.$$

We propose to prove that $b_k = c_k$. Let

$$[\bar{\psi}(0,0)]^2 = \bar{\psi}_0^2 \neq 0,\ [\bar{A}(0,0)]^2 f(0,0) = a_0 \neq 0.$$

Recalling that $\phi_4(0,0) = 0$, we deduce immediately from (14)

$$(16)\qquad \bar{\psi}_0^2 = a_0.$$

If we put in (14)

$$(17)\qquad y = y_1 = b_2x^2 + b_3x^3 + \cdots + b_{k-1}x^{k-1},$$

the functions $\bar{\psi}(x,y)$ and $\bar{A}(x,y)$ become integral power series in x, and we easily conclude, in view of (15), that the polynominal in x, $\phi_4(x,y_1)$, cannot

contain terms of degrees less than $2k/3$. If k is not divisible by 3, we put in (14)

$$y = y_2 = b_2x^2 + b_3x^3 + \cdots + b_{k-1}x^{k-1} + b_kx^k. \tag{17'}$$

We then observe that after this substitution is made the left-hand member of (14) does not involve terms of degree $\leqq 2k$ in x, and that in the right-hand member the coefficient of x^{2k} is $a_0(b_k - c_k)^2$. We deduce that $c_k = b_k$, which proves our assertion for the case $k \not\equiv 0 \pmod{3}$.

Let now k be divisible by 3, $k = 3k_1$. If we again use the substitution (17) and if we denote by β_{2k_1} the coefficient of x^{2k_1} in $\phi_4(x, y_1)$, we obtain from (14) by equating the coefficients of x^{2k_1},

$$\bar{\psi}_0{}^2b_k{}^2 - \beta^3_{2k_1} = a_0c_k{}^2. \tag{18}$$

If $\beta_{2k_1} = 0$, then the reasoning employed above, in the case $k \not\equiv 0 \pmod{3}$, can be used again in order to prove that $b_k = c_k$. We may then suppose that $\beta_{2k_1} \not\equiv 0$. We use the substitution (17′) and we first prove that as in $\phi_4(x, y_1)$ so also in $\phi_4(x, y_2)$ the term of lowest degree in x is $\beta_{2k_1}x^{2k_1}$. In fact, let $y = y_1'$, $y = y_2'$, etc., be the Puiseux expansions in the neighborhood of the origin of the different branches of the function y defined by the equation $\phi_4(x, y) = 0$ (y_1', y_2', etc., denoting fractional power series in x). Then we can write

$$\phi_4(x, y) = \rho(y - y_1')(y - y_2') \cdots,$$

where ρ is a constant or a polynominal in x. By the hypothesis made on the function $\phi_4(x, y_1)$ it follows that each of the series $y_1 - y_1'$, $y_1 - y_2' \cdots$ contains terms of lowest degree $\leqq 2k_1$. Since $y_2 = y_1 + b_kx^{3k_1}$, it is obvious that the terms of lowest degree of the series $y_1 - y_1'$, $y_1 - y_2', \cdots$, coincide with the terms of lowest degree of the corresponding series $y_2 - y_1'$, $y_2 - y_2', \cdots$, which proves that the terms of lowest degree of $\phi_4(x, y_1)$ and of $\phi_4(x, y_2)$ are the same.

If we now put in (14) $y = y_2$ and if we equate the coefficients of x^{2k_1}, we obtain

$$-\beta^3_{2k_1} = a_0(b_k - c_k)^2,$$

which combined with (16) and (18) yields

$$(b_k - c_k)^2 = c_k{}^2 - b_k{}^2. \tag{19}$$

In a similar way we obtain

$$(c_k - b_k)^2 = b_k{}^2 - c_k{}^2, \tag{19'}$$

by using the substitution $y = b_2x^2 + b_3x^3 + \cdots + b_{k-1}x^{k-1} + c_kx^k$.

From (19) and (19′) it follows that $b_k = c_k$. Q. E. D.

THE JOHNS HOPKINS UNIVERSITY.

CONSTRUCTION OF PENCILS OF EQUIANHARMONIC CUBICS.

By JACOB YERUSHALMY.

1. *Introduction.* By Salmon's theorem, the cross-ratio of the four tangents, which may be drawn to a plane cubic curve from a point on it, is constant for the cubic. A certain function of the cross-ratio, which is rational in the coefficients of the cubic, constitutes the only absolute rational invariant of the cubic. In fact, if α is one of the values which this cross-ratio assumes, then

$$J=4(\alpha^2-\alpha+1)^3/(\alpha+1)^2(1-2\alpha)^2(2-\alpha)^2$$

is the absolute rational invariant of the cubic. This invariant is known as the modulus of the cubic.

In terms of the two relative invariants S and T of the cubic, which are of degrees 4 and 6 respectively in the coefficients of the cubic,

$$J=S^3/T^2$$

and hence involves the coefficients to the 12th degree.

If $\alpha=-1$, then $J=\infty$, $T=0$ and the cubic is said to be *harmonic.*

If $\alpha=-\epsilon(\epsilon^3=1)$, then $J=S=0$ and the cubic is said to be *equianharmonic.*

If $\alpha=1$, then $S^3-T^2=0$ and the cubic has a double point.

If α is indetermined, $S=T=0$ and the cubic has a cusp.

Since J involves the coefficients of the cubic to the 12th degree, there are in an arbitrary pencil of cubics 12 cubics of an assigned generic modulus, but only six harmonic and four equianharmonic cubics.

2. *Pencils of Cubics of Equal Modulus.* O. Chisini * proposes to determine all the pencils having the property that all the cubics of one pencil have the same modulus. He succeeds in determining them in the following manner.

He observes that to a pencil of cubics all having the same modulus corresponds a pencil of lines cutting a quartic curve in quadruples of points all having the same cross-ratio, and conversely. He then determines all the

* "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo,* Vol. 41 (1916).

quartic curves admitting the above property and studying their singularities he arrives at the construction of the pencils of cubics.

In the equianharmonic case Chisini proves that every quartic that is cut by all the lines of a pencil in equianharmonic quadruples is represented by an equation of the type

$$f_4 = \frac{\partial^2 \phi_3(x_1x_2x_3)}{\partial x_3{}^2}\, \phi_3(x_1x_2x_3) - \tfrac{1}{2}\left[\frac{\partial \phi_3(x_1x_2x_3)}{\partial x_3}\right]^2 = 0,$$

where $\phi_3 = 0$ is an arbitrary cubic not passing thru the center of the pencil of lines which is taken to be the point $C \equiv (0, 0, 1)$. It is easily verified that the cubic $\phi_3 = 0$ is the polar of the point C with respect to the quartic $f_4 = 0$. He also shows that the six points C_1, C_2, C_3, C_4, C_5, C_6 of intersection of $\phi_3 = 0$ with the polar conic of C with respect to $\phi_3 = 0$ are flex-points for $f_4 = 0$ whose flex-tangents pass thru C. Obviously the cubic $\phi_3 = 0$ touches $f_4 = 0$ at the six points C_i.

Chisini, however, does not give the actual construction of the pencils of equianharmonic cubics. He only points out that such pencils are characterized by possessing six cuspidal cubics (corresponding to the six flex-tangents CC_i of the above quartic). It is the object of this paper to make a closer investigation of these pencils of cubics with special reference to their actual construction. We show that every such pencil is contained in a net of equianharmonic cubics thru six base-points. These base-points form the vertices of two in-circumscribed triangles of an arbitrary cubic curve, which are three-fold perspective from the vertices of a third in-circumscribed triangle of the same cubic.

3. *Cremona Transformations Leaving a Pencil of Equianharmonic Cubics Invariant.* It is well known that a general elliptic cubic is transformed into itself by 8 cyclic collineations of period three whose fixed points do not lie on the cubic. These collineations are given in terms of the abelian parameter u in the form

$$u' = u + \omega/3 \qquad (\omega \text{ is a period}).$$

In addition to these general transformations, an equianharmonic cubic is transformed into itself by singular homographies and homologies of period three with three fixed points on the cubic. In terms of the abelian parameter u these singular transformations are given by

$$u' = +\epsilon u + b, \qquad (\epsilon^3 = 1).$$

The existence of such homographies and homologies characterizes the equianharmonic cubics.

If, then, we have a linear system of equianharmonic cubics, say a pencil, the homology may be fixed or variable, i. e., there may exist one homology leaving invariant all the cubics of the pencil, or the homology will vary from cubic to cubic. However Chisini proves (*loc. cit.*, p. 90) that if the homology is fixed the pencil is special, and on a general pencil the homology is variable. It is natural, therefore, to look for Cremona Transformations of the plane into itself leaving the cubics of a pencil of equianharmonic cubics invariant. For this purpose we must investigate more closely the quartic curve $f_4 = 0$ which is cut by a pencil of lines $\{C\}$ in equianharmonic quadruples of points.

As is known, a cubic surface F_3 may be mapped on a double plane with a quartic branch-curve by projection from a point O on the surface. Consider, therefore, the plane π of $f_4 = 0$ as the projection of a cubic surface F_3, and $f_4 = 0$ as the branch-curve. The equation of $f_4 = 0$ is, as we noted,

$$f_4 \equiv \frac{\partial^2 \phi_3}{\partial x_3{}^2}\, \phi_3 - \tfrac{1}{2}\left(\frac{\partial \phi_3}{\partial x_3}\right)^2 = 0$$

and $\phi_3 = 0$ is the polar of C (the center of $\{C\}$) with respect to $f_4 = 0$. Take an arbitrary line a of $\{C\}$. It cuts $f_4 = 0$ in an equianharmonic quadruple of points E_1, E_2, E_3, E_4 and $\phi_3 = 0$ in three points P_1, P_2, P_3 which constitute the polar group of C with respect to $E_1, \cdots E_4$. To a will correspond on F_3 a plane equianharmonic cubic curve Ψ_3 cut out by the plane thru a and O (O is the center of projection). To P_1, P_2, P_3 will correspond on Ψ_3 two triples of points. We proceed to prove that each triple of points is a group of three fixed points of a singular birational transformation T_3, cyclic of order 3, of the cubic Ψ_3 into itself. What we have to prove is, in fact, the following:

THEOREM. *The lines joining any point O on an equianharmonic cubic to the ∞^1 groups of three fixed points, on it, of the ∞^1 singular birational transformations cyclic of period 3, of the cubic into itself form a $g_3{}^1$ in the pencil $\{O\}$, and this $g_3{}^1$ is precisely the $g_3{}^1$ obtained by taking the polar group of a variable line of the pencil with respect to the four tangents to the cubic from O.*

Proof. Let Ψ_3 be an equianharmonic cubic and O a point on it. Let γ be a transformation of Ψ_3 into itself having three fixed points A_1', A_2', A_3' on the cubic. Let ω be the transformation determined by the $g_2{}^1$ cut out on Ψ_3 by the pencil $\{O\}$, i. e. the transformation which interchanges the points of each pair of the $g_2{}^1$.

Evidently $\omega^{-1}\gamma\omega$ is a transformation of Ψ_3 into itself having as fixed

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points the three further intersections A_1'', A_2'', A_3'' of the lines OA_1', OA_2', OA_3' with Ψ_3. Hence given any line of the pencil, each of the two points in which it cuts Ψ_3 outside of O will define, by a known result, two other points which together with it form a group of three fixed points for some birational transformation of Ψ_3 into itself, and by the previous remark these two triples will form three groups of the g_2^1 mentioned above. It follows that the series ∞^1 of triples of lines mentioned in the theorem is such that each line belongs to one and only one triple, and therefore this series is a g_3^1.

To prove that this g_3^1 coincides with the g_3^1 of the polar groups we take the trace of $\{O\}$ on a line l.

Consider the transformation γ of Ψ_3 into itself having one of its fixed points at the point of contact C of one of the tangents thru O. γ will leave invariant the g_2^1 cut out by $\{O\}$ since it leaves one group of it (the point C counted twice), invariant, and will permute cyclically the three points of contact of the other 3 tangents to Ψ_3 from O. The other two fixed points of γ are on a line with O. We have in $\{O\}$ a cyclic projectively of order 3. The trace of it on l may be given by $x' = \epsilon x$ ($\epsilon^3 = 1$), having the points 0 and ∞ as invariant points. The equianharmonic quadruple of points on l can be taken to be 0, 1, ϵ, ϵ^2 and the polar group of any point (a_1, a_2) with respect to it is given by

$$(1) \qquad 4a_1x_1^3 - 3a_2x_1x_2^2 - a_1x_2^3 = 0 \quad \text{or} \quad 4ax^3 - 3x - a = 0.$$

The polar group of the trace of OC on l is the polar group of 0 with respect to 0, 1, ϵ, ϵ^2. Putting in (1) $a = a_1/a_2 = 0$ we obtain the point O and the point ∞ counted twice, and this triple of points obviously coincides with the traces on l of the lines joining O to the three fixed point of γ. The same will hold for any of the 4 groups having one of the 4 points of contact as a fixed point, hence the $2g_3^1$'s coincide. q. e. d.

From this theorem we conclude that the two triples of points on F_3 are each a group of fixed points of some birational transformation of Ψ_3 into itself.

To ϕ_3 will correspond on F_3 a curve of order 9, C_9. It will be seen later from the analytical expression for F_3 that C_9 is degenerate, but it can also be seen from the following consideration. Since $\phi_3 = 0$ touches the branch curve $f_4 = 0$ whenever they meet, the doubly-covered cubic $\phi_3 = 0$, i. e. C_9 does not possess branch-points. Moreover the six points of intersection of $\phi_3 = 0$ and $f_4 = 0$ lie on a conic ($\Psi_2 = \partial\phi_3/\partial x_3 = 0$) and by a known theorem * C_9 is reducible. It will be seen later that C_9 breaks up into a plane

* See for instance F. Enriques and O. Chisini, *Courbes et Fonctions Algébriques d'une Variable*, Chap. IV, p. 444.

cubic curve which may be supposed to coincide with the cubic $\phi_3=0$ itself and another sextic curve. The two triples of points on F_3 corresponding to P_1, P_2, P_3 are, hence, separable. One group of 3 points will be on $\phi_3=0$ and the other group will be on the sextic.

Consider only the cubic $\phi_3=0$. It is traced out by the groups of fixed points of the singular birational transformations, cyclic of order 3, sending into themselves the cubics of the pencil which we obtain on F_3 as the line a varies in the pencil $\{C\}$, i. e., the cubics cut out by the planes thru OC.

These singular transformations of the cubics of the pencil are, in fact, homologies because the three fixed points on each cubic are on a line (the line of the pencil $\{C\}$ corresponding to the cubic).

Since thru each point of F_3 there passes only one cubic of the pencil we have a Cremona transformation Γ of the space sending F_3 into itself defined in the following way: For any point P take the plane of the pencil containing it. In this plane we have an homology sending P into P' in the same plane. P' is the homologous point of P under Γ. Evidently Γ possesses on F_3 an entire curve of invariant points, the curve $\phi_3=0$ of fixed-points.

4. *The Equation of F_3.* Consider the equation

$$\frac{\partial^2\phi_3(x_1x_2x_3)}{\partial x_3{}^2}x_4{}^2+(2)^{1/2}\frac{\partial\phi_3(x_1x_2x_3)}{\partial x_3}x_4+\phi_3(x_1x_2x_3)=0.$$

It is an equation of a cubic surface F. If we project F from the point $O\equiv(0,0,0,1)$ on it upon the plane $x_4=0$ we get as branch-curve

$$(\partial^2\phi_3/\partial x_3{}^2)\phi_3-\tfrac{1}{2}(\partial\phi_3/\partial x_3)^2=0$$

which coincides with our $f_4=0$.

The following consideration shows that we can really take the above equation to represent our F_3.

The plane $x_4=0$ is a double plane both for F and the plane α of the pencil of equianharmonic cubics (see Chisini, *loc. cit.*, p. 62), the branch-curve being the same in both cases. The net of cubics on F cut out by the bundle of planes thru O will be mapped into the net of cubics thru the seven base-points in α. The system ∞^3 of the plane sections of F will go by this mapping into the web of cubics thru 6 of the 7 base-points, the seventh base-point will correspond to O. This web will therefore contain the pencil of equianharmonic cubics and we can take as the equation of F_3:

$$F_3\equiv(\partial^2\phi_3/\partial x_3{}^2)x_4{}^2+(2)^{1/2}(\partial\phi_3/\partial x_3)x_4+\phi_3=0.$$

From this equation it is evident that the plane $x_4=0$ cuts out on F_3

the cubic $\phi_3 = 0$, and that the C_9 on F_3 corresponding to $\phi_3 = 0$ breaks up into $\phi_3 = 0$ itself and another sextic curve. $\phi_3 = 0$ is, therefore, the curve of invariant points for the Cremona Transformation Γ sending F_3 into itself.

5. *Every Pencil of Equianharmonic Cubics is Contained in a Net of Such Cubics.* The only points on F_3 which may be fundamental points for Γ are the three points O, O_1, O_2, in which OC meets F_3. We proceed to prove, however, that they are ordinary points and consequently Γ possesses no fundamental points on F_3.

The singular birational transformations sending the cubics of the pencil, cut out on F_3 by the planes on OC, into themselves are, as we have noted, homologies. Therefore the three fixed points are flexes for the cubic, and it will be shown later that the line joining them is a side of the Hessian triangle of the cubic. The pencil $\{C\}$ is, hence, composed of the flex lines for the corresponding cubics. Each line forms the axis of the homology; the center being the point common to the three flex tangents at the invariant points.

Since O, O_1, O_2 are not on $\phi_3 = 0$, one of the following two cases may happen: either these points go by each homology into different points on the different cubics and will, therefore, be fundamental points for Γ; or they form for each homology a cycle. This will happen if for each cubic the three flex tangents at the invariant points meet on OC. In this case O, O_1, O_2 are ordinary points for Γ which possesses, then, no fundamental points on F_3. To prove that this latter case takes place, it is sufficient, remembering that on a P_1, P_2, P_3 are the polar group of C with respect to E_1, E_2, E_3, E_4, to prove the following

THEOREM. *The lines joining any point O of an equianharmonic cubic to any three flexes P_1, P_2, P_3 on a side of the Hessian triangle form the polar group of the line joining O to the point K common to the three flex-tangents at P_1, P_2, P_3 with respect to the 4 tangents drawn from O to the cubic.*

Proof. Let the equation of the equianharmonic cubic be

$$\Psi_3 \equiv x_1^3 + x_2^3 + x_3^3 = 0.$$

The Hessian of $\Psi_3 = 0$ is the cubic $x_1x_2x_3 = 0$ composed of the three lines $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

Let $O \equiv (y_1, y_2, y_3)$ be any point on $\Psi_3 = 0$.

The four tangents from O to $\Psi_3 = 0$ are given by

$$3(y_1x_1^2 + y_2x_2^2 + y_3x_3^2)^2 - 4(x_1^3 + x_2^3 + x_3^3)(y_1^2x_1 + y_2^2x_2 + y_3^2x_3) = 0.$$

Let P_1, P_2, P_3 be the three flexes on the line $x_2 = 0$. The point K in this case will be the point $x_1 = x_2 = 0$ which is the opposite vertex of the flex triangle. Let the pencil of lines $\{O\}$ be projected upon $x_2 = 0$. The three flexes P_1, P_2, P_3 are given by

$$x_1^3 + x_3^3 = 0.$$

The trace of the line OK is the point (y_1, y_3). The equianharmonic quadruple is given by

$$\phi_4 = (y_1x_1^2 + y_3x_3^2)^2 - 4(x_1^3 + x_3^3)(y_1^2x_1 + y_3^2x_3) = 0.$$

The polar triple of (y_1, y_3) with respect to $\phi_4 = 0$ is given by

$$y_1\partial\phi_4/\partial x_1 + y_3\partial\phi_4/\partial x_3 = 0$$

or

$$\begin{aligned}
&y_1\{12x_1y_1(y_1x_1^2 + y_3x_3^2) - 4y_1^2(x_1^3 + x_3^3) - 12x_1^2(y_1^2x_1 + y_3^2x_3)\}\\
&\quad + y_3\{12x_3y_3(y_1x_1^2 + y_3x_3^2) - 4y_3^2(x_1^3 + x_3^3) - 12x_3^2(y_1^2x_1 + y_3^2x_3)\} = 0\\
&= 12(y_1x_1^2 + y_3x_3^2)(y_1^2x_1 + y_3^2x_3) - 4(x_1^3 + x_3^3)(y_1^3 + y_3^3)\\
&\quad - 12(y_1x_1^2 + y_3x_3^2)(y_1^2x_1 + y_3^2x_3) = 0\\
&= x_1^3 + x_3^3 = 0,
\end{aligned}$$

which coincides with the three flexes. q. e. d.

We conclude that the three points O, O_1, O_2 form a cycle for each homology, and they are ordinary points for Γ. Moreover, since the point K on each cubic must form with the three points O, O_1, O_2 an equianharmonic quadruple, it is the same point for all the cubics of the pencil. The transformation Γ has therefore the point K as an invariant point. Also every point of the plane $x_4 = 0$ is invariant for Γ, since we have in this plane the pencil of lines $\{C\}$ each of which is the axis of an homology. We also see that the projectivity involved on the line OC by the homology on a generic plane on OC does not vary as the plane varies in the pencil. Hence the transformation Γ does not possess fundamental points, and consequently is a collineation. More precisely, Γ is an homology in space, cyclic of order three, with the point K as center and $x_4 = 0$ as plane of invariant points.

The net of cubics cut out on F_3 by the bundle of planes on K is a net of invariant cubics for Γ since the planes of the bundle are invariant. Each cubic of the net has three invariant points on it (the three points in which C cuts the plane $x_4 = 0$) and is therefore equianharmonic.

By mapping the cubic surface F_3 on the plane α of the original pencil of equianharmonic cubics we conclude that every pencil of equianharmonic cubics is contained in a net of such cubics thru 6 of the 9 base points of the pencil.

6. *A Canonical Form for F_3 and the Equation of the Homology.* Let us determine the coördinates of the point K. If we write $\phi_3 = 0$ in the canonical form

$$\phi_3 \equiv x_1^3 + x_2^3 + x_3^3 + 6\lambda x_1 x_2 x_3 = 0,$$

F_3 will take the form

$$F_3 \equiv x_1^3 + x_2^3 + x_3^3 + 6x_3x_4^2 + 3(2)^{1/2}x_3^2x_4 + 6(2)^{1/2}\lambda x_1x_2x_4 + 6\lambda x_1x_2x_3 = 0.$$

Consider the cubic Ψ_3 cut out on F_3 by the plane $x_1 = \mu x_2$:

$$\Psi_3 \equiv (1+\mu^3)x_2^3 + x_3^3 + 6x_3x_4^2 + 3(2)^{1/2}x_3^2x_4 + 6(2)^{1/2}\lambda\mu x_2^2x_4 + 6\lambda\mu x_2^2x_3 = 0.$$

This is an equianharmonic cubic and its Hessian is composed of a flex triangle. The equation of the Hessian is found to be

$$H \equiv x_4\{[(1+\mu^3)x_2 + 2\lambda\mu(x_3 + (2)^{1/2}x_4)](x_3 + (2)^{1/2}x_4) - 2\lambda^2\mu^2x_2^2\} = 0.$$

The second factor breaks up into the product of two linear factors by completing the square and taking the difference of the two squares, and we obtain the Hessian triangle. Since $x_4 = 0$ is one of the sides we conclude that each of the lines of the pencil $\{C\}$ is a side of the Hessian triangle of the cubic corresponding to it on F_3.

To determine the coördinates of K, we consider the cubic cut out on F_3 by the particular plane thru $x_1 = 0$ and O. The Hessian triangle of this cubic is found to be

$$x_4x_2[(2)^{1/2}x_3 + 2x_4] = 0.$$

The sides of the triangle are the lines

$$x_2 = 0; \quad x_4 = 0; \quad (2)^{1/2}x_3 + 2x_4 = 0.$$

The three tangents at the flexes on $x_4 = 0$ meet on the opposite vertex, which is the point of intersection of $x_2 = 0$ and $(2)^{1/2}x_3 + 2x_4 = 0$, or in the point $K \equiv [0, 0, 1, -(2)^{1/2}/2]$.

For simplicity send the point K to $(0, 0, 1, 0)$ and interchange the planes $x_3 = 0$ and $x_4 = 0$.

The transformation sending the points

$$(0,0,0,1),\ (0,0,-(2)^{1/2}/2, 1),\ (0,0,1,0) \text{ into } (0,0,0,1),\ (0,0,1,0),\ (0,0,\epsilon,1)$$

is found to be $x_1 = x_1'$; $x_2 = x_2'$; $x_3 = \epsilon x_3'$; $x_4 = (2)^{1/2}(x_4' - \epsilon x_3')$.

Applying this transformation to F_3 after having interchanged x_3 and x_4 the equation of F_3 turns out to be

$$F_3 \equiv x_1^3 + x_2^3 - 2(2)^{1/2}x_3^3 + 2(2)^{1/2}x_4^3 + 6(2)^{1/2}\lambda x_1x_2x_4 = 0,$$

and the homology in space sending F_3 into itself is evidently

$$\Gamma; \qquad x_1' = x_1, \quad x_2' = x_2, \quad x_3' = \epsilon x_3, \quad x_4' = x_4.$$

7. *A Net of Equianharmonic Cubics thru Six Base-Points.* In order to obtain the equianharmonic net of cubics in the plane we map F_3 on a plane α.

As is known, a cubic surface may be mapped on a plane by means of a system ∞^3 of cubics thru six base-points in the plane, so that the cubics of this system correspond to the plane sections of the cubic surface. The correspondence between the points of the plane and the points of the surface is $(1, 1)$ except for the six base-points i $(i = 1, 2, \cdots 6)$ to which correspond on the surface six lines denoted by a_i. Also to the six conics thru 5 of the base-points correspond on the cubic surface six lines b_i (the line b_i corresponding to the conic leaving out the point i). Finally to the 15 lines joining two of the base points i and j correspond on the surface the lines c_{ij}. In all we have 27 lines on the cubic surface.

F_3 is a particular cubic surface since there are collineations sending it into itself, and when mapped on α the six base points will be in particular position which we proceed to determine.

Evidently to Γ will correspond in α a transformation T. For a point P on F_3 is mapped into a point Q in α. Γ sends P into a point P' which is mapped into a point Q' in α. The correspondence between Q and Q' is $(1, 1)$ and algebraic, and is therefore a Cremona transformation in the plane.

Since Γ has no fundamental points on F_3, the Cremona transformation T will have no fundamental points in α outside the six base-points. Let us determine the order of T, i. e., the order of the curves into which T transforms the lines of α.

Consider an arbitrary line a of α. It is mapped into a twisted cubic C on F_3. Γ sends C into another twisted cubic $\bar{C}$ which is mapped into a curve S of α. As is known there are seventy two systems ∞^2 of twisted cubics on F_3. One system is mapped into the lines of α. Twenty systems are mapped into conics thru three of the base-points, thirty systems into cubics with a double point at one of the base-points and passing simply thru four of the remaining five base-paints. Twenty systems go into quartics with double points in three of the base-points and simple points at the other three base-points, and finally one system is mapped into quintics having double-points at the six base-points. To determine into what system Γ transforms the system of cubics corresponding to the lines of α we must find the number of intersections of C and $\bar{C}$.

C cuts $x_3=0$ in three points which are invariant for Γ and belong, hence, also to $\bar{C}$. If C and $\bar{C}$ have any more common points they must be such points on C which are the transform of points on C. A point and its transform by Γ must be on a line with K. However, there is only one double-secant of C thru K, and since Γ is cyclic of order three, one of the two points in which the double-secant cuts C must go by it into the other point. Hence, one of these two points is common to C and $\bar{C}$. These two curves intersect therefore in four points, and so do S and a in α. Hence S is a quartic curve passing simply thru three of the base-points, say 1, 2, 3 and doubly thru the other base-points 4, 5, 6. The transformation is biquadratic and the fundamental points of the inverse transformation (T^{-1}) coincide with those of the direct transformation. The homoloidal net Σ' of the inverse transformation being known

$$\Sigma': \quad (1^1, 2^1, 3^1, 4^2, 5^2, 6^2)_4,$$

it is easy to determine the homoloidal net Σ of T. T being cyclic of order three must transform the system Σ' into Σ. A quartic C_4 of Σ must therefore have four variable intersections with a quartic C_4' of Σ'. Let Σ have multiplicities r_1, r_2, r_3 at 1, 2, 3 and s_1, s_2, s_3 at 4, 5, 6. We have

$$r_1+r_2+r_3+2(s_1+s_2+s_3)=16-4=12$$

and since three of them must have the value two and the other three must have the value one, we have the result

$$r_1=r_2=r_3=2, \qquad s_1=s_2=s_3=1,$$

and hence

$$\Sigma: \quad (1^2, 2^2, 3^2, 4^1, 5^1, 6^1)_4.$$

Corresponding to the cubic $\phi_3=0$ on F_3, we have in the system ∞^3 of cubics in α a cubic ϕ each point of which is invariant for T. Consider the base-point 4. It is a simple fundamental point and goes by T into a line l which has no variable intersections with the quartics of Σ' and is therefore on two of the three points 4, 5, 6. This says that on F_3, a_4 goes into c_{ij} $(ij=4, 5, 6)$ by Γ. But c_{ij} must pass thru the point of intersection of a_4 and $\phi_3=0$ and therefore the line l in α passes thru 4. It may be either $(45)_1$ or $(46)_1$. Let $T(4)=(45)_1$, then $T(5)=(56)_1$, $T(6)=(64)_1$. Moreover, since $\phi_3=0$ and c_{45} have a point in common on a_4, the line $(45)_1$ and the cubic ϕ have at 4 the same direction. $(45)_1$ is, hence, tangent to ϕ at 4. In the same way it is seen that $(56)_1$ and $(64)_1$ are tangent to ϕ at 5 and 6 respectively. The three points 4, 5, 6 form therefore an in-circum-

scribed triangle in the cubic ϕ. Applying the same reasoning for T^{-1} we see that the other three fundamental points 1, 2, 3 form the vertices of another in-circumscribed triangle in the same cubic.

The point 1 goes by T into a conic passing thru 1. This conic goes thru 4, 5, 6 since it has no variable intersections with the quartics of Σ'. 1 is transformed, hence, into either $(12456)_2$ or $(13456)_2$. Let $T(1)=(12456)_2$ then $T(2)=(23456)_2$ and $T(3)=(31456)_2$. We have for T

$$\begin{array}{llll} 1 \rightarrow (12456)_2 \rightarrow & (13)_1 & \rightarrow 1 \\ 2 \rightarrow (23456)_2 \rightarrow & (12)_1 & \rightarrow 2 \\ 3 \rightarrow (31456)_2 \rightarrow & (23)_1 & \rightarrow 3 \\ 4 \rightarrow \quad (45)_1 & \rightarrow (12346)_2 & \rightarrow 4 \\ 5 \rightarrow \quad (56)_1 & \rightarrow (12345)_2 & \rightarrow 5 \\ 6 \rightarrow \quad (64)_1 & \rightarrow (12356)_2 & \rightarrow 6. \end{array}$$

In order to obtain the complete configuration of the six base-points, consider for example the point 1. We have by T

$$1 \rightarrow (12456)_2 \rightarrow (13)_1 \rightarrow 1.$$

On F_3 we have the cycle $(a_1\, b_3\, c_{31})$. These three lines must concur at a point on $\phi_3=0$ (the invariant point on a_1), and hence both the conic (12456) and the line (31) are tangent to ϕ at 1. The same is true for the other five conics. The complete configuration of the six base-points is therefore as follows: $(1, 2, 3)$ $(4, 5, 6)$ are the vertices of two in-circumscribed triangles on ϕ, such that the lines (13), (12), (23), (45), (56), (64) and the conics (12456), (23456), (31456), (12346), (12345), (12356) are tangent to ϕ at the points 1, 2, 3, 4, 5, 6 respectively.

As is known there are twenty-four in-circumscribed triangles in an arbitrary cubic ϕ. For, if $(4, 5, 6)$ be such a triangle, and u the abelian parameter of the cubic chosen so that for three points on a line the sum of the values of this parameter should be equal to zero (mod. periods), then we have

$$\begin{aligned} 2u_4 + u_5 &\equiv 0 \ (\text{mod.}\ \omega, \omega') \\ 2u_5 + u_6 &\equiv 0 \ (\text{mod.}\ \omega, \omega') \\ 2u_6 + u_4 &\equiv 0 \ (\text{mod.}\ \omega, \omega'). \end{aligned}$$

From which

$$u_4 \equiv (\alpha\omega + \alpha'\omega')/9.$$

Giving to α and α' all the values from 0 to 8 we get 81 points from which we have to exclude the 9 flexes, leaving 72 points forming 24 triangles.

If we start with any ninth of a period for u_4, then u_5 and u_6, and hence

the triangle (4, 5, 6), are determined. In order to determine the second triangle $(1, 2, 3)$ we must put the condition that $(12346)_2$ should touch ϕ at 4. Hence

$$(2) \qquad 2u_4 + u_5 + u_1 + u_2 + u_3 \equiv 0 \pmod{\omega, \omega'}.$$

But since $(1, 2, 3)$ is also an in-circumscribed triangle, the following relations exist

$$\begin{aligned} u_3 &\equiv -2u_1 \pmod{\omega, \omega'} \\ u_2 &\equiv 4u_1 \pmod{\omega, \omega'}. \end{aligned}$$

Substituting in (2) we have

$$2u_4 + u_1 \equiv (\beta_\omega + \beta'_{\omega'})/3,$$
$$u_1 \equiv -2u_4 + 1/3 \text{ of a period} \equiv u_4 + 1/3 \text{ of a period}.$$

We have 9 thirds of a period, and it would seem that associated with any triangle there are 8 more each of which together with it will give the required six points. However, finding the values of u_2 and u_3

$$\begin{aligned} u_2 &\equiv 4u_4 + 1/3 \text{ of a period} \equiv u_4 + 1/3 \text{ of a period} \\ u_3 &\equiv -8u_4 + 1/3 \text{ of a period} \equiv u_4 + 1/3 \text{ of a period} \end{aligned}$$

we see that these also are obtained by adding to u_4 a third of a period, and hence the nine thirds of a period give rise to only 3 triangles. One triangle is to be excluded; this is the one obtained by adding to u_4 the third periods $0, 3u_4, 6u_4$, which give the triangle (4, 5, 6) over again. Hence, starting with any one point (a ninth of a period), the triangle containing it is determined and with it there are determined two associated triangles, each of which can be taken with the original one to form the base of the net.

In fact, start with a ninth of a period $u_1 \equiv \omega/9$. The three triangles are

$$\begin{array}{lll} u_1, & u_2 \equiv u_1 + \omega/3, & u_3 \equiv u_1 + 2\omega/3; \\ u_4 \equiv u_1 + \omega'/3, & u_5 \equiv u_1 + 2\omega/3 + \omega'/3, & u_6 \equiv u_1 + \omega/3 + \omega'/3; \\ u_7 \equiv u_1 + 2\omega'/3, & u_8 \equiv u_1 + 2\omega/3 + 2\omega'/3, & u_9 \equiv u_1 + \omega/3 + 2\omega'/3. \end{array}$$

It is easily verified from the above that any two of the three triangles satisfy our conditions, and any two of them can be taken to form the base.

It is also easily seen that any two of the above three triangles are threefold perspective from the vertices of the third. For

$$\begin{aligned} u_1 + u_4 + u_8 &\equiv u_2 + u_5 + u_8 \equiv u_3 + u_6 + u_8 \equiv 0; \\ u_1 + u_5 + u_7 &\equiv u_2 + u_6 + u_7 \equiv u_3 + u_4 + u_7 \equiv 0; \\ u_1 + u_6 + u_9 &\equiv u_2 + u_4 + u_9 \equiv u_3 + u_5 + u_9 \equiv 0. \end{aligned}$$

Finally we want to show that if we take on an arbitrary cubic ϕ six oints forming the vertices of two in-circumscribed triangles and satisfying he above conditions, then in the system ∞^3 of cubics thru these points there contained a net of equianharmonic cubics.

Consider a biquadratic transformation T having these six points as fundaiental points $(1^2, 2^2, 3^2, 4^1, 5^1, 6^1)$. The fundamental points of the inverse ransformation will generally be some other six points $(1_0{}^1, 2_0{}^1, 3_0{}^1, 4_0{}^2, 5_0{}^2, {}_0{}^2)$ of the plane. T sends the cubics thru $(1, 2, \cdots 6)$ into the cubics thru $1_0, 2_0, \cdots 6_0)$. It will send ϕ into some cubic ϕ_0.

The quartics of Σ' having double-points at 1, 2, 3 and passing simply hru 4, 5, 6 cut the cubic ϕ in ∞^2 groups of 3 points. But since

$$2(u_1 + u_2 + u_3) \equiv 2\omega/3,$$
$$u_4 + u_5 + u_6 \equiv \omega/3,$$

nd hence

$$2(u_1 + u_2 + u_3) + u_4 + u_5 + u_6 \equiv 0,$$

t follows that the sum of the Abelian parameters at each group of three points s equal to zero, i. e. the three points of each group are on a line. Σ' cuts, herefore, out on the cubic ϕ the same $g_3{}^2$ cut out on it by the lines of the lane. The transformation T involves, therefore, a collineation γ sending ϕ nto ϕ_0. γ sends the points $1, 2, \cdots 6$ into the points $1_0, 2_0 \cdots 6_0$. To prove his take for example the line (13). T sends it into one of the points 1_0, 2_0, 3_0 ay into 1_0. Consider the pencil of quartics in Σ' degenerating into the fixed ine (13) and the pencil of cubics $(2^2, 4^1, 5^1, 6^1, 1^1, 3^1)$. Then $g_3{}^1$ cut out y it on ϕ has a fixed point which falls at 1 since (13) is tangent to the ubic ϕ at this point. To it will correspond the $g_3{}^1$ cut out on ϕ_0 by the ines on 1_0, and the fixed point 1_0 corresponds to the fixed point 1. And by he same reasoning γ send 2 into 2_0 etc.

Multiplying T by γ^{-1} we get a Cremona transformation R having the undamental points of both the direct and inverse transformations coincident. This transformation will leave invariant the linear system ∞^3 of cubics on $1, \cdots 6$. Moreover, each point of the cubic ϕ is an invariant point for R.

The cube of this transformation is a collineation in the plane. Because he lines of the plane go by it into the quartics of Σ': $(1^1, 2^1, 3^1, 4^2, 5^2, 6^2)$, which go by the same transformation into quartics having four variable interections with those of Σ'. R sends, therefore, Σ' into Σ. Applying R to Σ we get again the lines of the plane. But a collineation having a cubic of nvariant points is the identity, hence $R^3 = I$ and the transformation is yclic of period 3.

If we consider then the linear space S_3 whose elements are the cubics of our system ∞^3, our transformation R will be a cyclic projectivity because it is an algebraic $(1, 1)$ correspondence between the elements. This projectivity has one invariant point (the cubic ϕ) and also every line on this point is invariant: in fact every pencil of cubics obtained by combining linearly any cubic of the system with ϕ goes into itself since the three base-points of the pencil outside the fundamental points are invariant. By duality, since the projectivity has a bundle of invariant lines, it must have a plane of invariant points. The transformation R in the plane has, therefore, a net of invariant cubics. Each of the cubics of this net has three fixed points on it (the three points in which it cuts the cubic ϕ) and is therefore equianharmonic.

It is of interest to note that the cubic ϕ is the locus of the cusps of all the cuspidal cubics of the net, because the cusp must be an invariant element and is therefore on the invariant cubic.

THE PLANAR IMPRIMITIVE GROUP OF ORDER 216.

By John Rogers Musselman.

Introduction.

In a paper,[*] which discussed the imprimitive group of order 5184 in S_3, the existence of an imprimitive group of order 216 in S_2 connected with the Hesse configurations was indicated. The purpose of this paper is to discuss this planar group and the geometry associated with it. Of special interest is a configuration of twelve triangles which possesses important properties. These are dealt with in Section II. In Section I is a study of four-fold perspective triangles. A new canonical form is used which leads to a ruler construction for them and to a compass construction for a Clebsch six-point.

I. Four-fold Perspective Triangles.

Four-fold perspective triangles have been discussed by Schröter,[†] Hess,[‡] Valyi [§] and others—all using the same canonical form. As a knowledge of these triangles is necessary for the development of the next section of this paper it has seemed best to study them using a new canonical form, which has certain advantages over the former one. Among others it leads to a simple ruler construction for ordinary four-fold perspection, and to a compass construction for a Clebsch six-point.

Let us call the vertices of one triangle A, B, C; those of the second triangle a, b, c. We shall choose for our canonical form the coördinates of the six vertices to be

$$\begin{array}{lll} A:\ (1,0,0) & B:\ (0,1,0) & C:\ (0,0,1), \\ a:\ (1,1,1) & b:\ (r_1,r_2,r_3) & c:\ (s_1,s_2,s_3). \end{array}$$

Let us indicate the four-fold perspections as $A_aB_bC_c$, $A_bB_cC_a$, $A_cB_aC_b$, $A_aB_cC_b$. To be perspective in these four ways requires the following conditions on the six points

$$(1) \qquad r_1s_2 = r_3s_1, \quad r_2s_3 = r_1s_2, \quad r_3s_1 = r_2s_3, \quad r_1s_3 = r_2s_1.$$

[*] Musselman, *American Journal of Mathematics*, Vol. 49 (1927), pp. 355-366.

[†] Schröter, *Mathematische Annalen*, Vol. 2 (1869), p. 553.

[‡] Hess, *Mathematische Annalen*, Vol. 28 (1887), p. 167.

[§] Valyi, *Archiv der Mathematik und Physik*, Vol. 70 (1884); Vol. A1 (1885), p. 320; *Monatshefte für Mathematik und Physik* (1898), p. 169.

As is well known the third condition above is merely a consequence of the first two and it is only three conditions on two triangles to be four-fold perspective. If we eliminate r from the three actual conditions we have $s_1(s_1s_2 - s_3^2) = 0$; if we eliminate s from the conditions we have $r_1(r_1r_3 - r_2^2) = 0$. Now $s_1 = 0$ together with conditions (1) would compel one of the points of the second triangle to coincide with one of the first triangle and we should not have two distinct triangles. Similarly if $r_1 = 0$. Hence we see that for four-fold perspective triangles point c must lie on the conic $x_1x_2 - x_3^2 = 0$ and point b on the conic $x_1x_3 - x_2^2 = 0$. The first conic is tangent to $\overline{AC}$ at A, tangent to $\overline{BC}$ at B and passes through a. The second conic is tangent to $\overline{AB}$ at A, tangent to $\overline{BC}$ at C and passes through a.

The equation of the line joining points b and c is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0,$$

which can be written, due to (1), as

$$r_1/s_1(s_2 - s_3)[-(s_2 + s_3)x_1 + s_1(x_2 + x_3)] = 0.$$

Now $s_2 \neq s_3$ else point c would coincide with point a, so the equation of the line is simply

$$(2) \qquad (s_2 + s_3)x_1 - s_1x_2 - s_1x_3 = 0.$$

If we let points b and c run over the conics on which they lie and ask for the envelope of the line $\overline{bc}$ we find that all these lines pass through the point $(0, -1, 1)$. To locate this point p, note that if we call the point of intersection of $\overline{BC}$ and $\overline{Aa}$ by p', then p lies on $\overline{BC}$ and is the fourth harmonic of p' as to B and C. Moreover point p and line $\overline{Aa}$ set up the reflexion $\rho x_1' = x_1$, $\rho x_2' = x_3$, $\rho x_3' = x_2$ which interchanges the two conics $x_1x_2 - x_3^2 = 0$ and $x_1x_3 - x_2^2 = 0$.*

The above furnishes a construction for four-fold perspective triangles given one triangle and one vertex of the second triangle. Call the given triangle ABC and the fourth point a. Produce $\overline{Aa}$ to cut $\overline{BC}$ at p'. Construct p as the fourth harmonic of p' as to B and C. Construct any point on the conic which is tangent to $\overline{AC}$ at A, tangent to $\overline{BC}$ at B, and passes through a. Call this point c, then b is the reflexion of c in the line $\overline{Aa}$ with p as center.

* It is of interest to note that the two imaginary intersections of these conics form with the four points A, B, C, a six-fold perspective triangles.

Since the line $\overline{pc}$ ordinarily cuts the conic $x_1x_2 - x_3^2 = 0$ in two distinct points we find then on every line through p there are two sets of two points each which together with point a form a triangle four-fold perspective to ABC. There are two lines, however, through p which contain only one set of points. One is $\overline{BC}$ *; the other will be mentioned later.

Let us write the coördinates of the six points in terms of a parameter as follows:

$$\begin{array}{lll} A\colon\ (1,0,0), & B\colon\ (0,1,0), & C\colon\ (0,0,1), \\ a\colon\ (1,1,1), & b\colon\ (\mu^2,\mu,1), & c\colon\ (\mu^2,1,\mu). \end{array}$$

The coördinates of the four centers of perspection are $(\mu^2,1,1)$, $(\mu,1,\mu)$, $(\mu,\mu,1)$ and $(\mu,1,1)$. The condition that the first three centers lie on a line is

$$\begin{vmatrix} \mu^2 & 1 & 1 \\ \mu & 1 & \mu \\ \mu & \mu & 1 \end{vmatrix} = 0,$$

which reduces to $\mu(\mu-1)^2(\mu+2)=0$. If μ equals 0 or 1, two points will coincide; hence *the condition that three centers of perspection lie on a line is* $\mu = -2$.

The condition that a conic can be put on the six vertices of the two triangles is

$$\begin{vmatrix} 1 & 1 & 1 \\ \mu^2 & 1 & \mu \\ \mu & 1 & \mu^2 \end{vmatrix} = 0,$$

which reduces to $\mu(\mu-1)^2(\mu+2)=0$. The equation of the conic on the six points when $\mu=-2$ is $x_1x_2 - 2x_2x_3 + x_3x_1 = 0$; the equation of the line of centers is $x_1 - 2x_2 - 2x_3 = 0$; the odd center has coördinates $(-2, 1, 1)$. Hence the theorem *when* $\mu = -2$ *the six vertices of the two triangles lie on a conic, that three centers of perspection lie on a line and the fourth center of perspection is the pole of this line as to the conic.*

To construct the line of centers produce $\overline{Ba}$ to cut $\overline{AC}$ at D. On AC construct the fourth harmonic of C as to A and D; call this point q. Then $\overline{pq}$ is the required line.

Of the four intersections of the conics $x_1x_2 - 2x_2x_3 + x_3x_1 = 0$ and $x_1x_2 - x_3^2 = 0$ three are at A, B, a; the other is the point $(4, 1, -2)$. But this latter is the point of tangency of the tangent (other than $\overline{BC}$) from

* Obviously the set on BC coincides with B and C and the six points are not distinct.

p to $x_1x_2 - x_3^2 = 0$. This is the second exceptional line through p which contains only one set of two points which with the given point a makes a four-fold perspective triangle with ABC. From this we can readily deduce a construction for four-fold projective triangles whose vertices are on a conic. But a simpler method is available. We express the conic $x_1x_2 - 2x_2x_3 + x_3x_1 = 0$ in parametric form as

$$(5) \qquad x_1 = t(2-t), \qquad x_2 = t, \qquad x_3 = 2 - t,$$

then the parameters of the points A, B, C are $t = \infty, 2, 0$ respectively while those of a, b, c are $t = 1, 4, -2$. Hence we have the theorem that *the six vertices of the two triangles on the conic are a cubic and its jacobian.* So the well known construction* on a conic for a cubic and its jacobian will furnish us four-fold perspective triangles whose vertices lie on a conic.

The Clebsch † six-point has the property that the 15 joins of the points meet by threes at ten points. The points can be arranged into two four-fold perspective triangles. The usual canonical form for the six-point is $1, 0, 0$; $1, 2\epsilon^4, 2\epsilon$; $1, 2\epsilon, 2\epsilon^4$; $1, 2, 2$; $1, 2\epsilon^2, 2\epsilon^3$ and $1, 2\epsilon^3, 2\epsilon^2$. The linear transformation which sends the first four of these points into the reference triangle and the unit point carries points 5 and 6 into $1+\nu, 1, \nu$ and $1+\nu, \nu, 1$ where $\nu = \frac{1}{2}(-1+5^{\frac{1}{2}})$. These points are on the line $-x_1 + x_2 + x_3 = 0$ which line is on p, the center of the reflexion and on $(1, 0, 1)$, the point where $\overline{Ba}$ cuts $\overline{AC}$. Points 5 and 6 are pairs in the reflexion with p as center and $\overline{Aa}$ as axis. To construct a Clebsch six-point with the apparatus available for four-fold perspective triangles means we must find the points where the conic cuts a definite line. This is a compass construction and the line can be constructed as indicated above. Since the conic cuts the line in two points there is another pair of points $5'$ and $6'$ which together with A, B, C, a will form a Clebsch six-point. The coördinates of this second set of points are $\nu, -1, 1+\nu$ and $\nu, 1+\nu, -1$ where $\nu = \frac{1}{2}(-1+5^{\frac{1}{2}})$.

For further study of four-fold perspective triangles let us choose a second canonical form. Let ABC be the reference triangle; in the previous form we let point a be the unit point and saw that it was on the conic $x_1x_2 - x_3^2 = 0$ so now we shall let point a run over this conic $x_1x_2 - x_3^2 = 0$ by giving it the coördinates $1, t^2, t$. Then point b will be on the conic $t^3x_1x_3 - x_2^2 = 0$ while point c lies on $x_1x_2 - x_3^2 = 0$. These two conics are reflected into each other by the reflexion with center $0, -t, 1$ and axis

* See for example Winger, *Projective Geometry,* p. 257.

† Clebsch, *Mathematische Annalen,* Vol. 4 (1871), p. 336.

$tx_3 - x_2 = 0$. The equations of the reflexion are $\rho x_1 = tx_1'$, $\rho x_2 = t^2 x_3'$, $\rho x_3 = x_2'$. The coördinates of the six points are now

$$A:\ (1, 0, 0), \qquad B:\ (0, 1, 0), \qquad C:\ (0, 0, 1),$$
$$a:\ (1, t^2, t), \qquad b:\ (t\mu^2, t^2\mu, 1), \qquad c:\ (\mu^2, 1, \mu).$$

The four centers of perspection are $t\mu^2, t, 1$; $\mu, t, t\mu$; $\mu, t^2\mu, 1$; and $\mu, t, 1$. The condition that the first three centers lie on a line is $t\mu + 2 = 0$. The equation of the line is then $t^2x_1 - 2x_2 - 2tx_3 = 0$, whose envelope is $x_3^2 + 2x_1x_2 = 0$, a conic having double contact with $x_1x_2 - x_3^2 = 0$. The six vertices of the triangles now lie on the conic $tx_1x_2 - 2x_2x_3 + t^2x_3x_1 = 0$ and the odd center $-2, t^2, t$ has for its polar line as to this conic, the line on which lie the first three centers of perspection. In the general case, the odd center $\mu, t, 1$ and the center $t\mu^2, t, 1$ lie on the axis of reflexion $tx_3 - x_2 = 0$; the two remaining centers lie on a line through the center of reflexion $0, -t, 1$.

The axes of perspection have the following coördinates:

$$(6) \qquad \begin{array}{llll} A_1 \equiv A_aB_bC_c: & -t^2, & 1 + t\mu, & t(1 + t\mu), \\ A_2 \equiv A_bB_cC_a: & 1 + t\mu, & -\mu^2, & \mu(1 + t\mu), \\ A_3 \equiv A_cB_aC_b: & t(1 + t\mu), & \mu(1 + t\mu), & -t^2\mu^2, \\ A_4 \equiv A_aB_cC_b: & t, & \mu, & t\mu. \end{array}$$

The condition that the three axes A_1, A_2 and A_3 be on a point reduces to $(2t\mu + 1)(t\mu + 2) = 0$. If now $2t\mu + 1 = 0$, the three axes coincide and the vertices of the triangle abc are on a line. The second triangle has degenerated, but the centers of perspection are distinct and not on a line. Hence $t\mu + 2 = 0$ is the condition that the three axes meet in a point; it is likewise the condition that three centers of perspection be on a line and that the six vertices of the two triangles be on a conic. If, however, $2t\mu + 1 = 0$ the equation of the line on which the points a, b, and c lie is $2t^2x_1 - x_2 - tx_3 = 0$. The envelope of this line is $x_3^2 + 8x_1x_2 = 0$, another conic having double contact with $x_1x_2 - x_3^2 = 0$.

When $t\mu + 2 = 0$ the axes A_1, A_2 and A_3 meet at the point $2, -t^2, -t$; as t varies this point describes the conic $2x_3^2 + x_1x_2 = 0$. Likewise when $t\mu + 2 = 0$, the six sides of the two triangles touch a conic whose equation in line coördinates is $2tu_1u_2 - t^2u_2u_3 + 2u_3u_1 = 0$. The odd axis has coördinates $-t^2, 2, 2t$ and the three other axes meet on the point whose equation is $2u_1 - t^2u_2 - tu_3 = 0$. The pole of the odd axis as to the line conic is the point where the axes A_1, A_2 and A_3 meet. In general, the odd axis of perspection $tx_1 + \mu x_2 + t\mu x_3 = 0$ and the A_1 axis are on the center of re-

flexion $0, -t, 1$; while the axes A_2 and A_3 are on a point on the axis of reflexion $tx_3 - x_2 = 0$.

To summarize, when $t\mu + 2 = 0$, the six vertices of the two triangles lie on a conic; three centers of perspection are on the odd axis; three axes of perspection are on the odd center and this odd center and odd axis are pole and polar as to the conic. The three axes of perspection cut the odd axis in three points q_i which are the jacobian points of the three centers of perspection p_i on the line. The hessian points are the intersections of the line and conic. The three lines joining the points p_i to the odd center cut the conic in the six vertices of the two triangles; moreover the six tangents to the conic from the points q_i touch the conic at the same six vertices.

II. The Planar Group of Order 216.

On any two sides of the reference triangle in the plane choose a set of points such that the vertices of the triangle on those sides are the Hessian points of the set. Construct on both sides the cubicovariant points of each given set. Join now the six points on one side with the six points on the second side in all possible ways. These 36 lines will cut the third side of the triangle in six points having the same relation to it as the originally chosen sets have to the sides on which they are located. We thus have a set of 18 points and 36 lines such that 3 points are on each line and 6 lines on each point. The 36 lines fall naturally into four sets of nine each, such that each set together with the reference triangle form the flex triangles of a pencil of cubic curves.

If we designate by a_i the points $0, \omega^i, 1$; by b_i the points $\omega^i, 0, 1$; by c_i the points $\omega^i, 1, 0$; $(i = 0, 1, 2;\ \omega^3 = 1)$; and if we call the cubicovariant sets respectively by a_i', b_i' and c_i' one can easily verify that

$$
(7)\qquad
\begin{array}{ll}
a_i',\ b_i',\ c_i' \text{ are the flexes of} & x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0,\\
a_i',\ b_i,\ c_i \text{ are the flexes of} & -x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0,\\
a_i,\ b_i',\ c_i \text{ are the flexes of} & x_1^3 - x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0,\\
a_i,\ b_i,\ c_i' \text{ are the flexes of} & x_1^3 + x_2^3 - x_3^3 - 6mx_1x_2x_3 = 0.
\end{array}
$$

The configuration consists of four Hesse configurations with a common triangle—the reference triangle— which is invariant under all the operations of the group. The group is the product of that subgroup G_{54} of a Hesse G_{216} which leaves one flex triangle invariant, and the G_4 which sends any one of the four Hesse configurations into each other. Hence we have an imprimitive G_{216}.

Now these 36 lines intersect in 360 points outside of the points on t

sides of the reference triangle. These 360 points divide into three sets containing 36, 108 and 216 respectively. For each line is conjugate under the G_{216} and hence is unaltered by a G_6. If two of the 36 lines should be unaltered by the same G_3, then the G_2 which interchanges them will leave their join unaltered by a G_6. Hence a set of 36 conjugate points. This will occur if the two lines belong to a set of flex lines of a cubic. This is evident also from the duality existing. For the 18 lines $x_1 \pm \omega^i x_2 = 0$, $x_2 \pm \omega^i x_3 = 0$, $x_3 \pm \omega^i x_1 = 0$ $(i = 0, 1, 2)$ which go by six through the vertices of the reference triangle meet in the 36 points $1, \pm\omega^i, \pm\omega^j$; $(i = j = 0, 1, 2)$. These are the coördinates of the set of 36 conjugate points mentioned above. These 36 points are on 360 lines, outside of those passing through the vertices of the reference triangle. These 360 lines fall into three sets containing 36, 108 and 216 respectively. The set of 36 lines are the flex lines of the four cubics; the 36 points are the flexes.

Furthermore, if two of the original 36 lines are not invariant under the same G_3, they may be interchanged by a G_2 or not. In the former case we shall get a set of 108 conjugate points, in the latter case a set of 216 points conjugate under the group. These will be discussed later, the dual sets of 108 and 216 lines will also appear.

Now the 36 points $1, \pm\omega^i, \pm\omega^j$ $(i = j = 0, 1, 2)$ fall into twelve triangles. Let

$$
(8)\qquad
\begin{array}{llll}
A_{11} \text{ be} & 1, 1, 1; & 1, \omega, \omega^2; & 1, \omega^2, \omega; \\
A_{12} \text{ be} & 1, 1, \omega^2; & 1, \omega^2, 1; & 1, \omega, \omega; \\
A_{13} \text{ be} & 1, 1, \omega; & 1, \omega, 1; & 1, \omega^2, \omega^2; \\
A_{21} \text{ be} & 1, -1, 1; & 1, -\omega, \omega^2; & 1, -\omega^2, \omega; \\
A_{22} \text{ be} & 1, -1, \omega^2; & 1, -\omega^2, 1; & 1, -\omega, \omega; \\
A_{23} \text{ be} & 1, -1, \omega; & 1, -\omega, 1; & 1, -\omega^2, \omega^2; \\
A_{31} \text{ be} & 1, 1, -1; & 1, \omega, -\omega^2; & 1, \omega^2, -\omega; \\
A_{32} \text{ be} & 1, 1, -\omega^2; & 1, \omega^2, -1; & 1, \omega, -\omega; \\
A_{33} \text{ be} & 1, 1, -\omega; & 1, \omega, -1; & 1, \omega^2, -\omega^2; \\
A_{41} \text{ be} & -1, 1, 1; & -1, \omega, \omega^2; & -1, \omega^2, \omega; \\
A_{42} \text{ be} & -1, 1, \omega^2; & -1, \omega^2, 1; & -1, \omega, \omega; \\
A_{43} \text{ be} & -1, 1, \omega; & -1, \omega, 1; & -1, \omega^2, \omega^2.
\end{array}
$$

We shall arrange these twelve triangles in four rows of three each as follows:

$$
\begin{array}{ccc}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} \\
A_{41} & A_{42} & A_{43}.
\end{array}
$$

Now this set of twelve triangles has the following remarkable property—*any two triangles in the same row are six-fold perspective; any two triangles in the same column are four-fold perspective with three centers of perspection on a line;* finally *any two triangles in different rows and columns are four-fold perspective.* Since the triangles in any row together with the reference triangle are the flex triangles of a cubic we know they are six-fold perspective by pairs, with the six centers of perspection being the other six points.

Any two triangles in the same column are four-fold perspective with three centers of perspection on a line. For A_{11} and A_{21} the centers of perspection are $1, 0, \omega^i$; $(i = 0, 1, 2)$ and $0, 1, 0$; the three flexes on one side of the reference triangle and the opposite vertex. Since the G_{216} is doubly transitive on the triangles in a column, this proves the theorem for any two triangles in any column. The triangles can be paired thus in 18 ways. In each case the three centers of perspection on a line, lie on the sides of the reference triangle. The three axes of perspection on a point for A_{11} and A_{21} are $x_1 + \omega^i x_3 = 0$ $(i = 0, 1, 2)$ on the point $0, 1, 0$; the odd axis is the opposite side of the reference triangle $x_2 = 0$. Moreover $A_{11}A_{21}, A_{12}A_{22}$, and $A_{13}A_{33}$ have the same centers and same axes of perspection. If two triangles are four-fold perspective with three centers on a line, the six vertices lie on a conic. These conics are

$$
(9) \qquad
\begin{aligned}
A_{1i}A_{2i}:&\quad x_2^2 - \omega^{i-1}x_1x_3 = 0\\
A_{1i}A_{3i}:&\quad x_3^2 - \omega^{i-1}x_1x_2 = 0\\
A_{1i}A_{4i}:&\quad x_1^2 - \omega^{i-1}x_2x_3 = 0\\
A_{2i}A_{3i}:&\quad x_1^2 + \omega^{i-1}x_2x_3 = 0\\
A_{2i}A_{4i}:&\quad x_3^2 + \omega^{i-1}x_1x_2 = 0\\
A_{3i}A_{4i}:&\quad x_2^2 + \omega^{i-1}x_1x_3 = 0 \qquad\qquad (i = 1, 2, 3).
\end{aligned}
$$

These 18 conics belong to three pencils of double contact conics, the parameters of the six conics in each pencil being the sixth roots of unity. The chord of contact of each pencil is one side of the reference triangle. Each conic is unaltered by a G_{12} of the group.

Two triangles in a different row and column are four-fold perspe
Thus the four centers of perspection for the triangles A_{13} and A_{22} are 0,
$1, 1 - \omega^2, \omega^2$; $1, \omega - 1, \omega$; and $1, \omega^2 - \omega, 1$. Two such triangles can be j
in 72 ordered pairs or 36 non-ordered pairs. The G_{18} which leaves
invariant is generated by

$$
(10) \qquad
\begin{aligned}
\rho x_1' &= x_1, & &= x_3, & &= x_1\\
\rho x_2' &= x_2, & &= x_1, & &= x_3\\
\rho x_3' &= \omega^2 x_3, & &= x_2, & &= x_2.
\end{aligned}
$$

This sends triangle A_{22} into itself, or A_{32} or A_{42}. The G_{216} sends A_{13} into each of the twelve triangles, and carries its partner A_{22} along into three, consequently we get 36 non-ordered pairs. Hence any pair of triangles can be sent by the group into any other pair in a certain order. Thus all 36 pairs are four-fold perspective. The G_6 leaving both A_{13} and A_{22} unaltered is generated by

$$(11)\qquad \begin{array}{ll} \rho x_1' = x_1, & = x_3 \\ \rho x_2' = \omega x_2, & = x_2 \\ \rho x_3' = \omega x_3, & = x_1. \end{array}$$

This G_6 leaves the center of perspection $(0, 1, 0)$, a vertex of the reference triangle, unaltered; it permutes the other three centers in all possible ways. We have thus $36 \times 3 = 108$ points, conjugate under the group. These points can be identified as that set of 108 conjugate points mentioned earlier in this section. Similarly the 108 axes of perspection of these 36 non-ordered pairs of triangles, outside of the sides of the reference triangle, are the set of 108 conjugate lines previously mentioned.

Under the G_{216} points fall into sets of 216 conjugates points. But this particular set of 216 points which appeared earlier is worth noticing. They lie by 12's on 36 lines. Thus on $-x_1 + x_2 + x_3 = 0$ we have the pair $\omega^2 - \omega, 1, 2\omega^2$; $\omega^2 - \omega, 2\omega^2, 1$. The remaining five pairs of points are given by that G_6 which sends the line into itself.

$$(12)\qquad \begin{array}{llllll} \rho x_1 = x_1', & = x_2', & = x_3', & = x_1', & = x_3', & = x_2' \\ \rho x_2 = x_2', & = - x_3', & = x_1', & = x_3', & = - x_2', & = x_1' \\ \rho x_3 = x_3', & = x_1', & = - x_2', & = x_2', & = x_1', & = - x_3'. \end{array}$$

Let us consider the six-point $1, 0, 0$; $0, 1, 0$; $0, 0, 1$; $1, 1, 1$; $\omega^2 - \omega, 1, 2\omega^2$; $\omega^2 - \omega, 2\omega^2, 1$. Coble* has given, in a paper on Point Sets and Cremona Groups, a method for calculating certain irrational invariants of a six-point which he has called $\bar{a}, \bar{b}, \cdots \bar{f}$ where $\sum^{6} \bar{a} = 0$. If two of these invariants equal, certain lines on the six points meet by threes. For our special t $\bar{a} = \bar{c} = \bar{d} = \bar{e}$. This means geometrically that points 5 and 6 are tc points in the reflexion set up with center at meet of $\overline{23}$ and $\overline{56}$ h axis as $\overline{14}$; also that the pair of points 1 and 4 are apolar to the $\overline{56}$. Consequently the six-point is self associated in the order (2536). the double ratio of the four lines 1-2356 or of 4-2356 has the value ilarly the double ratio of the lines 5-1234, or of 6-1234, or of 2-1456,

B. Coble, *Transactions of the American Mathematical Society*, Vol. 16 (1915).

or of 3-1456 is 4ω. Certain of the lines on the six-point meet by threes in six points whose coördinates are $\omega^2 - \omega, 1, 1$; $1 - \omega^2, 2, 2$; $1, 1, 0$; $0, 2\omega^2, 1$; $0, 1, 2\omega^2$; $1, 0, 1$. It is interesting to note this new six-point is of the same type as the first one, since four of the six irrational invariants of it are equal.

Now on the line $-x_1 + x_2 + x_3 = 0$ are six pairs of points; each pair together with the reference triangle and one other point from $1, 1, 1$; $1, -1, 1$; or $1, 1, -1$ forming a six-point of the type above. Hence the set of 216 points fall into pairs such that each pair with the reference three-point and a definite flex triangle vertex forms a six-point of the above nature. We have thus 216 such six-points, with each of the 36 flex triangle vertices used six times.

A PREPARED SYSTEM FOR TWO QUADRATICS IN SIX VARIABLES.

By J. WILLIAMSON.

Introduction. In a previous paper,* a prepared system was determined, in terms of which every concomitant of two quadratics in n variables could be expressed, if the concomitants were multiplied by suitable invariant factors. In this paper a prepared system, for the case $n = 6$, is determined, in terms of which every concomitant can be expressed, without being multiplied by an invariant factor. It is found that 52 new factors must be added to the $2^6 - 1 = 63$ factors already determined, making a total of 115.

The notation of the previous paper is used throughout except that, for convenience in printing, dashes are used instead of dots to denote determinantal permutations; i. e. the series $(abc)d_x - (abd)c_x - (adc)b_x$ is denoted by one of the three expressions $(ab'c')d_x'$, $(ab''c'')d_x''$, $(ab'''c''')d_x'''$. In addition, for the six sets of cogredient point variables, that are necessary for this discussion, x, y, z, t, w, k are now used, while Q, P, p, and u are written for the compound coördinates π_2, π_3, π_4 and π_5 respectively.†

The first section gives a list of the results while the second is devoted to their determination.

1. *The Prepared System.* The prepared system consists of 115 factors. Of these 63 are simple factors;

$$\left.\begin{array}{l} 6\ x\text{-factors of type } i_x, \\ 6\ u\text{-factors of type } (jkmnt), \end{array}\right\}\text{duals;}$$

$$\left.\begin{array}{l} 15\ Q\text{-factors of type } (ij), \\ 15\ p\text{-factors of type } (kmnt), \end{array}\right\}\text{duals.}$$

20 P-factors of type (ijk), 1 factor (123456).

47 are linear in two sets of variables;

3 ux-factors of type $i_x'(j'kmnt)$,

3 pQ-factors of type $(i'j')(k'mnt)$,

$$\left.\begin{array}{l} 8\ Qu\text{-factors of type } (ij')(imntk') \\ 8\ px\text{-factors of type } (mntk')j_x', \end{array}\right\}\text{duals.}$$

* J. Williamson, "A Special Prepared System for Two Quadratics in n Variables," *American Journal of Mathematics*, Vol. 52 (April, 1930), pp. 399-412.

† *Loc. cit.*, §§ 1 and 2.

$$\left.\begin{array}{l}6\ Pu\text{-factors of type } (ijk')(ijl'nm)\\ 6\ Px\text{-factors of type } (mnt')k_x',\end{array}\right\}\text{ duals;}$$

12 pQ-factors of type $(ij')(k'imn)$

1 pQ-factor $(123'5'')(4'6'')$.

1 is quadratic in the variable P: $(123')(4'56)$.

$$\left.\begin{array}{l}1\text{ is quadratic in } x \text{ and linear in } p:\ (123'5'')4_x'6_x''.\\ 1\text{ is quadratic in } u \text{ and linear in } Q:\ (4''6')(123''56)(12345')\end{array}\right\}\text{ duals.}$$

2 are linear in the three variables P, u and x;

$$\left.\begin{array}{l}(1''6'3)(2''6543)5_x',\\ (5''42')(12346'')1_x',\end{array}\right\}\text{ duals.}$$

These factors illustrate very clearly how the principle of duality* applies to the non-simple bracket factors. Corresponding to every non-simple factor is a dual factor formed by taking the duals of the component factors and permuting the same symbols. For example, $1_x'(2'34)$ yields the dual factor $(2'3456)(1'56)$. A factor may of course be self dual, as is the case with $(123')(4'56)$.

A complete list of these 115 factors is given below. In this list I denotes a product of invariant factors formed from a_ρ, (AR_4), (A_3R_3), (A_4R), r_α, where A, R, α and ρ are written for A_2, R_2, A_5 and R_5 respectively. If in a factor 12 is convolved, I includes a_ρ, if 23, (AR_4) etc. and in any particular case the value of I may be written down immediately. If two factors are similar,† only one has been defined, since the other may be obtained by replacing a, A, A_3, A_4, α by r, R, R_3, R_4, ρ respectively.

List of factors.

$1_x = a_x$, 6_x, $2_x = (A\rho x) = a_\rho' b_x'$, 5_x,

$3_x = (A_3R_4x) = (a'b'R_4)c_x'$, 4_x;

$(12) = a_\rho(AQ)$, (65), $(13) = (aA_3R_4Q) = (a'b'R_4)(ac'Q)$, (64),

$(14) = (aA_4R_3Q) = (a'b'c'R_3)(ad'Q)$, (63),

$(15) = (aR\alpha Q) = r_\alpha'(as'Q)$, (62), $(16) = (arQ)$,

$(23) = (AR_4)(A_3\rho Q) = (AR_4)a_\rho'(b'c'Q)$, (54),

$(34) = (A_3R_3)(A_4R_4Q) = (A_3R_3)(a'b'R_4)(c'd'Q)$,

$(24) = (A\rho A_4R_3Q) = (A_4r''s'')(b't''Q)a_\rho'$, (53),

$(25) = (A\rho R\alpha Q) = a_\rho''r_\alpha'(b''s'Q)$;

$(123) = I(A_3P)$, (654),

$(124) = I(AA_4R_3P) = I(A_4r's')(At'P)$, (653),

$(125) = I(AR\alpha P) = Ir_\alpha'(As'P)$, (652), $(126) = I(ArP)$, (651),

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* *Loc. cit.*, § 5.

† *Loc. cit.*, § 6.

$(134) = I(aA_4R_4P) = I(a'b'R_4)(ac'd'P)$, (643),
$(135) = (aA_3R_4R\alpha P) = (a''b''R_4)r_\alpha'(ac''s'P)$, (642),
$(136) = (aA_3R_4rP) = (a'b'R_4)(ac'rP)$, (641),
$(154) = I(aR_3\alpha P) = Ir_\alpha'(as't'P)$, (623),
$(234) = I(A_4\rho P) = Ia_\rho'(b'c'd'P)$, (543),
$(254) = I(A\rho R_3\alpha P) = Ia_\rho'(b'R_3\alpha P)$, (523);
$(6543) = I(R_4p)$, (1234), $(6542) = I(R_3A\rho p) = I(R_3b'p)a_\rho'$, (1235),
$(6523) = I(RA_3\rho p) = Ia_\rho'(Rb'c'p)$, (1254),
$(6234) = I(rA_4\rho p) = Ia_\rho'(rb'c'd'p)$, (1543),
$(2345) = I(\alpha\rho p) = Ia_\rho'(b'c'd'e'p)$, $(1654) = I(aR_3p)$, (6123),
$(1365) = I(aA_3R_4Rp) = I(a'b'R_4)(ac'Rp)$, (6412),
$(1346) = I(aA_4R_4rp) = I(r's'A_4)(ak't'p)$, $(1265) = I(ARp)$;
$(12345) = I(\alpha u)$, (65432), $(12346) = I(A_4ru)$, (65431),
$(12365) = I(A_3Ru)$, (65412);
$(123456) = a_\rho(AR_4)(A_3R_3)(A_4R)(\alpha r)$;
$(12, 6543) = 1_x'(2'6543) = -6_x'(125'4'3') = I(AR_4ux) = Ia_x'(b'R_4u)$,
$(65, 1234) = 6_x'(5'1234) = -1_x'(652'3'4')$,
$(123, 654) = 1_x'(2'3'654) = 6_x'(1235'4') = I(A_3R_3ux) = Ia_x'(b'c'R_3u)$;
$(123, 1654) = (12')(3'1654) = (16')(1235'4') = I(A_3aR_3Qu) = I(a'b'Q)(c'aR_3u)$.
$(654, 6123) = (65')(4'6123) = (61')(6542'3')$,
$(165, 1234) = (16')(5'1234) = (13')(6514'2') = I(aRA_4Qu) = I(ar'Q)(s'A_4u)$,
$(612, 6543) = (61')(2'6543) = (64')(1263'5')$,
$(265, 1234) = (26')(5'1234) = (3'2)(65124') = I(A\rho RA_4Qu) = I(A\rho r'Q)(s'A_4u)$.
$(512, 6543) = (51')(2'6543) = (4'5)(12653')$,
$(123, 6543) = (1'3)(2'6543) = (6'3)(215'4'3) = I(A_3R_4Qu) = I(a'c'Q)(b'R_4u)$,
$(654, 1234) = (6'4)(5'1234) = (1'4)(562'3'4)$,
$(1265, 6543) = (1'65)(2'6543) = (4'65)(21653')$,
$= I(ARR_4Pu) = I(a'RP)(b'R_4u)$.
$(6512, 1234) = (6'12)(5'1234) = (3'12)(56124')$.
$236, 6543) = (1'36)(2'6543) = (5'36)(1264'3)$,
$= I(A_3rR_4Pu) = I(a'c'rP)(b'R_4u)$,
$541, 1234) = (6'41)(5'1234) = (2'41)(6513'4)$,
$234, 6543) = (1'34)(2'6543) = (5'34)(126'43)$,
$= I(A_4R_4Pu) = I(a'c'd'P)(b'R_4u)$,
$236, 6541) = (12'6)(3'6541) = (15'6)(3624'1)$,
$= I(A_3rR_3aPu) = I(a'b'rP)(c'R_3au)$;
$(123, 654) = (1'2')(3'654) = (6'5')(1234')$,
$= I(A_3R_3Qp) = I(a'b'Q)(c'R_3p)$,
$(126, 543) = (1'6)(2'543) = (5'6)(214'3')$,
$= I(ArR_4\alpha Qp) = I(a'rQ)(b'R_4\alpha p)$,
$(651, 234) = (6'1)(5'234) = (2'1)(563'4')$,

$(123,154)=(12')(3'154)=(15')(1234')$,
$\quad =I(A_3aR_3\alpha Qp)=I(a'b'Q)(c'aR_3\alpha p)=I(a'b'Q)(c'as''t''p)r_a''$,
$(654,623)=(65')(4'623)=(62')(6543')$,
$(123,165)=(13')(2'165)=(15')(1236')=I(A_3aRQp)$,
$\quad =I(a'c'Q)(b'aRp)=-I(ar'Q)(A_3s'p)$,
$(654,612)=(64')(5'612)=(62')(6541')$,
$(134,165)=(13')(4'165)=(16')(4135')$,
$\quad =I(aA_4R_4aRQp)=I(ar'Q)(aA_4R_4s'p)$,
$(643,612)=(64')(3'612)=(61')(3642')$,
$(234,265)=(23')(4'265)=(26')(2345')=I(A_4\rho A\rho RQp)$,
$\quad =I(b'c'Q)(d'A\rho Rp)a_\rho'=I(b'c'Q)(d'Ab''p)a_\rho'a_\rho''$,
$(543,512)=(54')(3'512)=(51')(5432')$,
$(123,543)=(1'3)(2'543)=(5'3)(214'3)$,
$\quad =I(A_3R_4\alpha Qp)=I(a'c'Q)(b'R_4\alpha p)$,
$(654,234)=(6'4)(5'234)=(2'4)(563'4)$,
$(123,365)=(1'3)(2'365)=(6'3)(1235')=I(A_3A_3R_4RQp)$,
$\quad =I(a'c'Q)(b'A_4R_3Rp)=I(r'A_3R_4Q)(A_3s'p)$,
$(654,412)=(64')(5'412)=(1'4)(6542')$,
$(12,34,65)=(123'6'')(4'5'')=(1'436'')(2'5'')=(1'53''6)(2'4'')$,
$\quad =I(AA_4R_4RQp)=I(Ac'r''p)(d's''Q)(a'b'R_4)$;
$(12,543)=1_x'(2'543)=4_x'(125'3')=I(AR_4\alpha px)=Ia_x'(b'R_4px)$,
$(65,234)=6_x'(5'234)=3_x'(652'4')$,
$(12,346)=1_x'(2'346)=4_x'(123'6)=I(AA_4R_4rpx)=Ia_x'(b'A_4R_4rp)$,
$(65,431)=6_x'(5'431)=3_x'(654'1)$,
$(12,654)=1_x'(2'654)=(126'4')5_x'=I(AR_3px)=Ia_x'(b'R_3p)$,
$(65,123)=6_x'(5'123)=2_x'(651'3')$,
$(23,654)=2_x'(3'654)=5_x'(236'4')=I(A_3\rho R_3px)=Ia_\rho'b_x'(c'R_3p)$,
$(54,123)=5_x'(4'123)=2_x'(541'3')$;
$(12,34)=1_x'(2'34)=4_x'(123')=I(AA_4R_4Px)=Ia_x'(b'A_4R_4P)$,
$(65,43)=6_x'(5'43)=3_x'(654')$,
$(12,65)=1_x'(2'65)=5_x'(126')=I(ARPx)=Ia_x'(b'RP)$,
$(23,54)=2_x'(3'54)=4_x'(235')=I(A_3\rho R_3\alpha Px)=Ia_\rho'b_x'(c'R_3\alpha P)$,
$(12,54)=1_x'(2'54)=4_x'(125')=I(AR_3\alpha Px)=Ia_x'(b'R_3\alpha P)$,
$(65,23)=6_x'(5'23)=3_x'(652')$;
$(12,34,65)'=(123'6'')4_x'5_x''=(1'436'')2_x'5_x''=(1'4''56)2_x'3_x''$,
$\quad =I(AA_4R_4Rpxx)=I(Ac'r''p)(a'b'R_4)d_x's_x''$;
$(1265,6543,1234)=(1'6'')(2'6543)(5''1234)$,
$\quad =I(AR_4RA_4Quu)=I(a'r''Q)(b'R_4u)(s''A_4u)$;
$(123,6543,65)=(1'6''3')(2'6543)5_x''$,
$\quad =I(A_3R_4RPux)=I(a'r''c'P)(b'R_4u)s_x''$,
$(654,1234,12)=(6'1''4')(5'1234)2_x''$;
$(12,34,65)''=(123')(4'65)=(1'43)(2'65)=(126')(435')$,
$\quad I(AA_4R_4RPP)=I(Ac'P)(d'RP)(a'b'R_4)$.

In finding the values of I in the above list symbols separated by a comma are not to be counted as convolved; thus in $(12, 34)$ the value of I is $a_\rho(A_3R_3)$ and in $(12, 34, 65)$ is $a_\rho(A_3R_3)r_a$. Throughout we have written

$$A = ab, \quad A_3 = abc, \quad A_4 = abcd, \quad \alpha = abcde, \quad R = rs, \quad R = rst, \quad R_4 = rstk.$$

In the definitions of the bracket factors, it is sometimes necessary to use previous definitions; for example,

$$(12, 34) = Ia_x'(b'A_4R_4P) = Ia_x'(a''b''R_4)(b'c''d''P),$$

by the definition of (134).

2. *Determination of the Prepared System.* Since we are now considering two quadratics in n variables for the case $n = 6$, there are seven invariants and six quadratic covariants i_x^2, $(i = 1, 2, 3, 4, 5, 6)$.* By theorem I every concomitant,† multiplied by a suitable invariant factor, can be expressed in terms of the symbolic factors,

$$i_x, \ (ij), \ (ijk), \ (ijkm), \ (ijkmn), \ (123456) \qquad (i, j, k, m, n = 1, 2, 3, 4, 5, 6).$$

We must now determine if ever in forming these bracket factors, we have disturbed any of the invariant factors, which appear when 12, 23, 34, 45, 56 are convolved together. Originally we have six sets of cogredient point variables x, y, z, t, w, k, which are convolved as $\Delta = (xyztwk)$, $u = xyztw$, $p = xyzt$, $P = xyz$, $Q = xy$. Since the only factor involving all six variables is (123456), and in this 12, 23, 34, 45, 56 are convolved, it follows that no invariant factor is disturbed in forming Δ. Let us consider the formation of the factors involving the variable u first of all. For simplicity we call a factor containing m symbols an m-factor. If one of the variables x, y, z, t, w convolved to form u appear in a four-factor, we may take four of these variables as appearing in that factor. For ‡

$$(ijkr \mid xyzt')(mn \mid w'y) = (ijkr \mid x'y'wt)(mn \mid z'y) + (ijkrm' \mid u)n_y',$$

and on the right w and t are both convolved in the same factor. Proceeding in this way we see that we lose nothing by assuming that four of the variables occur in the four-factor. We have then to consider the cases in which the fifth variable occurs in a two-factor, a three-factor or a four-factor. The

* *Loc. cit.*, p. 404.

† *Loc. cit.*, § 3.

‡ Both here and later the symbols $i, j, k, t, m, n, a, b, c, d, e, f$ are used to denote any of the symbols 1, 2, 3, 4, 5, 6.

case with the fifth variable in a one-factor obviously does not require to be considered. If the four-factor is $(ijkr \mid xyzt)$ or more shortly $(ijkr)$, the two-factor cannot contain any of the symbols i, j, k, r; the three-factor can contain at most one and the four-factor at most two of the symbols i, j, k, r, for otherwise in the formation of u no invariant factors would be disturbed. Since there are only six possible values for i, j, k, r, we must consider

$$(ijkrn')\,m', \quad (ijkrn')\,(im'), \quad (ijkrn')\,(ijm'),$$

where i, j, k, m, n are all distinct and we have not written in the variables.

If none of the variables, convolved to form u, occur in a four-factor, one may occur in a three-factor. In this case, as before, we may assume that three of the variables forming u occur in this factor and we have to consider the cases: (a) three variables in one three-factor, two in another; (b) three variables in one three-factor, one in each of two three-factors; (c) three variables in one three-factor, one in another three-factor, one in a two-factor; (d) three variables in one three-factor, one in each of two two-factors. In case (a) the same symbol cannot appear in both three-factors and accordingly we have the sole possibility $(ijkr'm')\,n'$. In case (b) we have

$$(ijkr'a'')\,(m'n')\,(b''c''),$$

where no two of i, j, k are the same as two of r, m, n or of a, b, c nor two of a, b, c are the same as two of r, m, n. For, if $rmn = ijk$, (b) becomes $(ijkna'')\,(ij)\,(b''c'')$ and here ijn are still convolved. The rest follows from the fact that we might have started with the factor (rmn) or (abc) in place of (ijk). In case (c) we have $(ijkr'a'')\,(m'n')\,b''$, where $ijkrmn$ must involve at least five distinct symbols. But neither of a, b can be the same as one of i, j, k or the same as one of r, m, n and hence this type is impossible. In case (d) we have $(ijkr'a'')\,m'b''$, where r, m and a, b contain no symbols in common with i, j, k. If $ab = rm$, this type is obviously reducible * and so we must consider the case

$$(ijkr'r'')\,m'n'' \equiv (ijknr)\,m'r' + (i'j'mnr)\,k'r$$

and each term on the right reduces to simpler bracket factors. The further case, in which only two-factors can occur is easily seen to be impossible.

If the variable w does not occur but the variable t does, that is if the

* We use the phrase "is reducible" to denote that the factor under consideration can be expressed in terms of simpler types or of types that have already been considered. The sign $\equiv$ is used for "equal to, apart from reducible terms."

coördinate p appears but not the coödinate u, and one of the three variables convolved to form p occur in a three-factor, three of them may be considered to occur in that factor. Hence we have the types, $(ijkr')(m'n')$ from two three-factors, $(ijkr')(im')$ from two three-factors, $(ijkr')m'$ from one three-factor and one two-factor. But, if no three-factor occur, we have the type $(ijk'm'')r'n''$ from three two-factors.

If no variables w or t occur, but z occurs, we have the single type $(ijk')r'$. Accordingly we have to consider the following types.

A. $(ijkrm')n'$, B. $(ijkrm')(in')$, C. $(ijkrm')(ijn')$,
D. $(ijkr'm')n'$, E. $(ijkr'n'')(im')(jm'')$, F. $(ijkr')(m'n')$,
G. $(ijkr')(im')$, H. $(ijkr')m'$, I. $(ijk'm'')r'n''$,
J. $(ijk')m'$.

We now consider these types in detail.

Type A. In A m, n must be successive integers and so we have the possibilities,

$$1'(2'3456), \qquad 2'(3'1456) = 6'(1234'5') + 1(32456),$$
$$3'(4'1256) = 1'(2'3456) + 6'(12345'),$$
$$4'(5'1236) = 3'(4561'2') + 6(51234), \qquad 5'(6'1234).$$

We are accordingly left with only two of this type, $1'(2'3456)$ and $5'(6'1234)$, if we include type D. For example the term $1(32456)$ on the right of $2'(3'1456)$ can be neglected, since (32456) has $32, 45, 56$ all convolved.

Type B. In type B m, n must be successive integers and by letting $i = 1, 2, 3$ in turn we have the possibilities;

$$(12')(14563'), \quad (13')(12564') \equiv (15')(12346'),$$
$$(14')(12365') \equiv (12')(14563'), \quad (15')(12346'),$$
$$(23')(21564') \equiv (25')(12346'), \quad (24')(21365') \equiv (26)(21345),$$
$$(25')(21346'), \quad (32')(34561'), \quad (34')(32165') \equiv (32')(34561'),$$
$$(35')(32146') \equiv (32')(34561'),$$

together with similar types. These types reduce as indicated above to the eight,

$$(12')(14563'), \quad (65')(63214'), \quad (15')(12346'), \quad (62')(65431'),$$
$$(25')(21346'), \quad (52')(56431'), \quad (31')(34562'), \quad (46')(43215').$$

Type C. In type C, m, n must be successive integers and so must k, r.

Accordingly m, n and k, r may have the following values: $m, n = 1, 2$; $k, r = 3, 4$ or $4, 5$ or $5, 6$: $m, n = 2, 3$; $k, r = 4, 5$ or $5, 6$: $m, n = 3, 4$; $k, r = 5, 6$. From these values we obtain six types,

$$(1'56)(56342'), \quad (1'63)(63452'), \quad (1'34)(34562'),$$
$$(2'16)(16453'), \quad (2'14)(14563'), \quad (4'12)(12563').$$

Type D. Since r, m, n and i, j, k must be successive integers, there is only one factor of this type, $(1234'5')6'$.

Type E. Since

$$\begin{aligned}(ijkr'n'')(im')(jm'') &\equiv (ijkrm)(in'')(jm'') + (ij'mrn'')(ik')(jm''), \\ &\equiv (ij'mrn'')(ik')(jm''),\end{aligned}$$

for the other term contains the simple factor $(ijkrm)$, we may interchange the rôles of ijk and imr, and similarly the rôles of ijk and jmn. If we consider the factors ijk, imr, jmn in turn as the foundation for the u factor, we see that ijk, imr, jmn must all be sets of three successive integers. They must be chosen from 123, 234, 345, 456 and any three of these sets include two with two symbols the same. Accordingly a factor of type E would simplify.

Type F. In this type at least two of i, j, k and at least two of r, m, n must be successive integers and so we have the possible cases,

$$\begin{aligned}&(1234')(5'6'), \quad (1243')(5'6') \equiv (1245')(36'), \\ &(1253')(4'6') \equiv (1253')(4'6) \equiv (1'345)(62'), \\ &(1263')(4'5') \equiv (1'345)(62'), \\ &(1342')(5'6') \equiv (1345')(26') \equiv (1563')(4'2) \equiv (1562')(3'4'), \\ &(1452')(3'6') \equiv (1452')(3'6) \equiv (1234')(5'6), \\ &\qquad\qquad\qquad \equiv (1234')(5'6') - (1236)(45), \\ &(1562')(3'4') \equiv (2345')(16').\end{aligned}$$

These factors reduce as indicated above to the three,

$$(1234')(5'6'), \quad (3452')(61'), \quad (4325')(16').$$

Type G. In this type m, n must be successive integers and so must j, k. We let $i = 1, 2, 3$ in turn and so get the six factors,

$$(12')(13'45), \quad (12')(13'56), \quad (13')(14'56),$$
$$(23')(24'56), \quad (31')(32'45), \quad (31')(32'56),$$

and six similar factors making twelve in all.

Type H. In this type r, m must be successive integers and so must at least two of i, j, k. We accordingly have

$$\begin{array}{llll} 1'(2'345), & 1'(2'346), & 1'(2'356) \equiv 6'(1235'), & 1'(2'456), \\ 2'(3'145) \equiv (1235')4', & 2'(3'156) \equiv (1236')5', & 2'(3'456), & \\ 3'(4'126) \equiv 1'(2'346), & 3'(4'125) \equiv 1'(2'345), & & \\ 3'(4'156) \equiv 6'(1345'), & 3'(4'256) \equiv 6'(2345'), & & \end{array}$$

and similar factors. But these reduce as indicated above to the four factors,

$$1'(2'345), \quad 1'(2'346), \quad 1'(2'456), \quad 2'(3'456),$$

and four similar factors.

Type I. In this type i, j; k, r; m, n must all be distinct and must all be pairs of successive integers. Accordingly there is only the one possibility $(123'5'')4'6''$.

Type J. In this type i, j and k, m must both be pairs of successive integers and so we have the types.

$$1'(2'34), \quad 1'(2'45), \quad 1'(2'56), \quad 2'(3'45), \quad 2'(3'56), \quad 3'(4'56).$$

Further u-factors. Since types F, G, H, I and J only arise when no variable u is present, we need only consider types A, B, C, and D in the formation of new u-factors. Let us first consider $C = (ijkrm')(ijn' \mid Y)$. If one of the variables of Y is to be convolved to form u and none of the components of u occur in a four-factor, then all three variables in Y may be taken as forming part of u. Thus we have the possibilities, $(ijkrm')(ijn'ab)$, $(ijkrm')(ijn'a''b)$, $(ijkrm')(ijn'a''b'')c''$, $(ijkrm')(ijn'a''b''')$.* In the first case neither of $a, b = i, j, m$ or n and so ab must be kr. By the fundamental identities † this reduces. Similarly the second and the fourth obviously reduce. In the third case, none of a, b, c is i or j; hence two of them are m, n or k, r and if $ab = mn$, $c = k$. We have then the type

$$(ijkrm')(ijn'm''r'')n'',$$

which is reducible. Since k, r and n, m are interchangeable in C, this type reduces in every case. But, if one of the variables of u occur in a four-factor, we have to consider the possibility $(ijkrm')(i'j')(n'abcd)$. This is obviously reducible, if $abcd$ includes ij. Let now $abcd$ involve i but not j, then since

* The symbols $'$ $''$ $'''$ after a letter mean that the letter so marked belongs to a convolution of letters even though the other members of the convolution are omitted.

† *Loc. cit.*, p. 408.

k, r and m, n are interchangeable in C, $(ijkrm')(ij')(n'imnr)$ is typical. But this last is equivalent to $(ijkrm')(in')(jimnr)$, which is a product of factor types already considered. We are left to consider $(ijkrm')(i'j')(n'mnkr)$ $\equiv(ijkrm')(i''n')(j''mnkr)$, and this latter can be obtained from two A factors and appears in the consideration of the Q-factors.

We next consider type $B=(ijkrm')(in' \mid Y)$. If one of the component variables of Y is convolved to from u, we have the types, $(ijkrm')(n'abcd)i'$, $(ijkrm')(i'n'abc)$, $(ijkrm')(i'n'a''f''s)(abcde'')$. Of these, the first two are obviously reducible to simpler types and the last, formed from two B factors is also reducible. For $a \neq i, m$, or n and so $a=j$, k or r and

$$(ijkrm')(in'af''s)(abcde'') \\ \equiv(ijkrf'')(inams)(abcde'')+(ijkrs)(inaf''m)(abcde'').$$

Both of the last two terms have a simple bracket factor as an actual factor and hence this type is also reducible.

Since, in the formation of new u-factors, A and D can only occur with simple factors of the type (ij), (ijk) etc., it is easily seen that A and D do not give rise to any new u-factors. Thus there are no new u-factors.

New p-factors. Type C cannot occur with a p-factor, for*

$$(ijkr'm)(ijn's)\equiv(ijkrs)(ijnm)+(ijkr)(ijnms).$$

Thus C yields only six factors of the type $(ijkrm'u)(ijn'P)$. Further type B cannot occur with a p-factor, for $(ijkrm')(in'a''b''')$ is reducible,* where a and b appear from factors of types A, D, F, G, H or I. Also $(ijkrm')(in'ab)$ is reducible, since a, b must be two of r, j, k. We are left to consider $(ijkrm')(in'a''b'')(c''def)$ arising from a B and an H factor. But this is impossible, since i cannot be one of a, b, c or one of d, e, f, since a, b, c and d, e, f are interchangeable. If however one of the components of u occur in a three-factor, we have the type $(ijkrm')(n'abc)i'$, where abc cannot contain i or both of m, n. If $a=m$, we have $(ijkrm')(n'mkr)i'\equiv(ijkrm)(n'm'kr)i'$, and this latter is reducible, being the product of $(ijkrm)$ and $i'(n'm'kr)$. Hence $abc=jkr$ and $(ijkrm')(n'jkr)i'\equiv(ijkr)(n'm'jkr)i'$, and the latter is a product of two simpler factor types.

Type $F=(ijkr')(m'n')$ cannot occur with a p-factor, for as before $(ijkr')(m'n'ab)$ and $(ijkr')(m'n'a''b)$ are both reducible and

$$(ijkr')(m'n'a''b'')(c''def),$$

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* *Loc. cit.*, p. 408, formula (17).

from two F factors, is impossible, since two of a, b, c cannot be the same as two of r, m, n or two i, j, k. Hence we must consider $(ijkr')(n'abc)m'$. In this a, b, c cannot contain two of r, m, n and must therefore contain two of i, j, k which is impossible.

Type G cannot appear with a p-factor, for $(ijkr')(im's''a''')$ and $(ijkr')(im's''a)$ are reducible.* But in $(ijkr')(im'ae'')(abcd'')$, formed from two G factors, a is distinct from i, r, j, m and k and must therefore be equal to n. Similarly i is not equal to any of a, b, c, d, e. Accordingly we must consider $(ijkr')(im'nm'')(nr''jk) \equiv (ijkr')(imnr)(nm'jk)$, which is reducible, and

$$(ijkr')(im'nr'')(nk''mj) \equiv (ijkn)(imrr'')(nk''mj) + (ijkr'')(imnr)(nk''rj),$$

and this is reducible, unless n, r, k are successive integers. Similarly i, j, k; i, m, r; n, m, j must all be sets of successive integers and on trial this is found to be impossible. We have still to consider the type $(ijkr')(m'abc)i'$, where a, b, c cannot contain i or both of r, m or both of j, k, since r, m and j, k are interchangeable in G. Hence we have $(ijkr')(m'njr)i'$ and this type is reducible, if i, m, n or i, j, k are not successive integers. But

$$(ijkr')(m'njr)i' \equiv (ijkn'')(m'r'jr'')i'$$

and this latter is reducible unless n, r are successive integers. Similarly j, r must be successive integers. Since the only possible type of G factor now is $(3124')(35')$, $n=6$ and accordingly n, j, r cannot be successive integers.

Since factors of the type A, F, H and I can only appear with factors of the types (ij), (ijk) or with factors in which the symbols are not convolved, it is easy to show that no new p-factors arise from considering them. We take type H as an illustration. In the type $(ijkr')(m'abc)$ a, b, c cannot include r or m and must be i, j, k or i, j, n. But $(ijkr')(m'ijk)$ is reducible and $(ijkr')(m'ijn) \equiv (ijkn)(mrij)$, where n is still convolved with i and j. The only other possibilities are $(ijkr')(m'abs'')$ and $(ijkr')(m'ab''c'')$ both of which are obviously reducible. Accordingly there are no new p-factors.

New P-factors. The types F and G cannot occur with a P-factor, for †

$$(ijkr')(im's) \equiv (ijks)(imr) + (ijk)(imsr)$$

and no convolutions of successive symbols have been disturbed. A similar

* *Loc. cit.*, formula (17).

† *Loc. cit.*, formula (17).

proof holds for the type F. Accordingly F yields only three factors of the type $(ijkr'p)(m'n'Q)$ and G twelve of the type $(ijkr'p)(im'Q)$.

If type B occur with a further P-factor, it may occur with a single symbol thus yielding $(ijkrm')(in'a)$, where a is not equal to $i, m,$ or n and so this reduces to type C. But B may occur with a simple two-factor giving the type $(ijkrm')(in'a'')b''$, where neither of a, b is equal to i or one of n, m, since $(ijkrm')(in'a'')b'' \equiv (ij'k')(r'imna'')b''$. Therefore we may take $(ijkrm')(in'j'')k''$ as typical. But this reduces to $(ijkrm')(kn'j)i$, unless i, m, n are successive integers. From the list of B factors we have only four possible types, $(3'1456)(12'4'')5''$, $(3'1456)(12'5'')6''$, $(34562')(31'4'')5''$, $(34562')(31'5'')6''$. Of these the first is equivalent to

$$(14''5'')(1236''4')5'$$
$$\equiv (14''5'')(12345)6'' + (14''5'')(1'''2'''5'6''4')3''' \equiv (145)(1'2'564)3',$$

and this last is a product of two simpler factors; similarily the second is reducible; the third is equivalent to

$$(34'5')(3126'4'')5'' \equiv (345)(4561'2')3',$$

and so is reducible; the fourth is equivalent to

$$(34'5')(3125''6')6'' \equiv (356)(1'2'456)3'$$

but is not reducible, since 3, 4, which was originally convolved, is no longer convolved. Thus we have the new factor type $(34562')(31'5'')6''$ and the factor similar to it. This type need not be considered farther, for, if the extra variable attached to $6''$ appear in a P-factor, the resulting factor type obviously reduces except in the case $(34562')(31'5'')(6''ij)$, where i, j are successive integers. But neither of i, j can be 5 or 3 and so ij must be 12 and $(34562')(31'5'')(6''12) \equiv (34562')(312)(561')$, which is reducible. Further $(315')(6'i) \equiv (31i)(65) + (31)(65i)$, and so, if the extra variable is convolved to form Q, no new factor type is obtained.

If a new P-factor is formed from two B factors, we have the general type $(ijkrm')(abcde'')(in'a'')f''$. This type is reducible, unless i, n, m and a, e, f are both sets of successive integers and also if these two sets coincide. From the list of B factors we see that we must consider

$$(12'6'')(14563')(63214'')5'', \quad (12'5'')(43216'')(14563')4'',$$
$$(31'4'')(34562')(43216'')5''.$$

Of these the first is equivalent to

$$(12'6)(14563')(6''3214'')5''+(12'3''')(14563')(66''2'''1'''4'')5'',$$
$$\equiv(12'6)(14563')(123,456)+(123)(14563')(6542'1')6,$$
$$\equiv(12'6)(14563')(123,456);$$

similarly the second and the third are reducible.

We now consider the possibility of a new factor type formed by a B factor and one other factor of the types A, D etc. in turn.

Types A and B. $B=(ijkrm')(in')$, $A=(abcde')f'$. In such a factor e,f is not the same as m,n nor is one of e,f equal to i. If $e=m$, we have

$$(ijkrm')(ijknm'')(r''in')$$
$$\equiv(r''imnj')(ik'r')(ijknm'')\equiv(rimnj')(irk')(ijknm),$$

and the last of these is reducible. Hence $e,f=j,k$ and we must consider $(ijkrm')(inmrj'')(in'k'')$. But there is only one type of A factor and so, from the list of B factors, we are left with

$$(2'3456)(1'65'')(6''3214'')\equiv(2'3456)(1'4562'')(63''1''),$$
$$\equiv(2''3456)(14562)(63''1'')\equiv(2''3456)(14562)(631''),$$

and

$$(2'3456)(43215'')(46''1')$$
$$\equiv(2'3456)(43''2'')(4651'1'')\equiv(1''3456)(432'')(46512),$$

both of which are reducible.

Types B and D. $B=(ijkrm')(in')$, $D=(abcd'e')f'$. Since abc and def are interchangeable in D, i may be taken equal to e and we have the type $(ijkrm')(in'f'')(abcid'')$, which is the same as a factor arising from one A and one B factor and accordingly has already been considered.

Types B and H. $B=(ijkrm')(in')$, $H=(cdeb')a'$. We have the factor type $(ijkrm')(in'a'')(b''cde)$, which reduces as before, if either of a,b is one of i,m,n. Farther, since $(ina')(b'cde)\equiv(i'ba)(n'cde)+(inab)(cde)$, this type reduces, unless i,m,n are consecutive integers and also if i is equal to one of c,d,e. Accordingly $i=f$, and we have

$$(fabcd')(fe'a'')(b''cde)\equiv(fabcd')(be'a)(fcde)+(fabcd')(fe'ab)(cde),$$

and both terms on the right are products of simpler factor types.

Types B and I. $B=(ijkrm')(in')$, $I=(abc'e'')d'f''$. This type is not possible, since ab, cd, and ef are interchangeable in I and neither of c,d can equal i in B.

Types B and J. $B=(ijkrm')(in')$, $J=(bcd')a'$. As in previous cases i, n, m must be consecutive integers and none of a, b, c, d can belong to i, n, m and this is impossible.

P-factors involving type A but not type B. If A occurs with an ordinary two-factor, the resulting factor is obviously reducible. The only other possibility is

$$(ijkrm')(n'a''b''')\equiv(ijkra'')(mnb''')$$
$$+(ijkrb''')(mna'')+(i'j'k')(r'mna''b''')\equiv(i'j'k')(r'mna''b''').$$

Accordingly, if a, b arise from two other A factors, this type is reducible, since there are only two distinct A factors. Thus there is no new P-factor formed by three A factors. Further, since in A $mn=12$ or 56, the combination of an A factor an F factor and any other factor cannot occur. Moreover, since

$$(ijkr')(m'ts)\equiv(ijk)(rmts)+(ijkt)(rms)+(ijks)(mtr),$$

any factor formed from factors of types A, H, I or J is reducible. Similarly type D cannot appear with types H, I or J. If D occur with an ordinary two-factor, we have $(ijkr'm')(n'mj)$, and this is of type C.

We must now consider the types H, I and J with ordinary simple factors (ab). In type $H=(ijkr')(m'ab)$ neither of a, b is equal to one of r, m and so we have $(ijkr')(m'ab)=(ijk)(rmab)+(ijka'')(rmb'')$ and this is reducible, since at least one of a, b is the same as one of i, j. The factor $I=(ijk'm'')r'n''$ with the simple factor (ab) is impossible, since neither of k, r in I can equal one of a, b and since k, r; m, n; i, j are interchangeable in I. But J and the simple factor (ab) yield the new type $(ijk')(m'ab)$ $=(123')(4'56)$.

The only type, which we have neglected, is $(ijk'm'')(r'n''a)$, and this is reducible, for

$$(ijk'm'')(r'n''a)\equiv(ijkr)(mna)+(ijk'a)(r'mn)+(ijk')(mnr'a),$$

and each term on the right has at least one less broken convolution.

New Q-factors. Type J cannot occur with a new Q-factor, for $(ijk')(m'a)$ $\equiv(ija)(km)+(ij)(mka)$. Two factors of the type A yield the new factor type $(12345')(6'1'')(2''3456)$. The only possibility from one A factor and one D factor is

$$(12345')(6'1'')(2''3''456)\equiv(12345')(6'6'')(1234''5''),$$
$$\equiv(1''2'')(3''4''566')(1234'5')\equiv(1''2'')(3''4''456)(12356),$$
$$\equiv(1'4)(3'2'456)(12356),$$

and this last is reducible. From A and H we have the type

$$(ijkrm')(n'a'')(b''cde)\equiv(ijkrm')(n'abc'')(d''e'')\equiv(ijkra')(mnb'c'')(d''e''),$$

where we have neglected terms, which are reducible. From the first identity we see that neither of n, m can be the same as one of a, b and from the second, that neither of a, b can be the same as one of i, j, k, r. But this is impossible and so the type reduces. Since in I i, j; k, r; m, n are interchangeable, no new factor arises from A and I. From two D factors we have the type

$$(1234'5')(6'6'')(1234''5'')\equiv(1234''5'')(1236''\,|\,x'y'z'w')(456\,|\,xyt')\equiv 0$$

by the convolution of 4, 5, 6 in the first factor. From D and H we have the possibility $(ijkr'm')(n'a'')(b''cde)\equiv(ijkr'm')(n'abc'')(d''e'')$. This latter is reducible, if a and b appear among r, m, n or among i, j, k. Accordingly $a, b=3, 4$ and cde is of the type 126 or 125. Therefore a, b and c, d can be interchanged and since $cd=12$, this new factor reduces. Similarly the combinations of D with I, and H with I are reducible. From two H factors we have the type

$$(ijkr')(m'a'')(b''cde)\equiv(ijkr')(m'abc'')(d''e'')$$
$$\equiv(ijkc'')(rmab)(d''e'')+(ijka')(rmb'c'')(d''e'').$$

From the first identity we see that neither of a, b is the same as one of r, m and that therefore one of a, b must equal one of i, j, k. Accordingly this type reduces by the second identity. No new type arises from two I factors, but the single I factor yields the new type $(123'5'')(4'6'')$.

We have now found all possible cases, in which a convolution of successive symbols has been disturbed. These new factor types, together with the 63 simple factors form the prepared system. In § 1 these factors are listed and defined in terms of the symbols a, r etc. of the two quadratics, and if the factors I are removed from this list, we are left with a prepared system similar to that used by Turnbull in his paper on Two Quaternary Quadratic Forms.*

THE JOHNS HOPKINS UNIVERSITY.

* H. W. Turnbull, "The Simultaneous System of Two Quadratic Quaternary Forms," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 18 (1917), parts 1 and 2, pp. 70-94.

A VARIETY REPRESENTING PAIRS OF POINTS OF SPACE.

By F. R. Sharpe.

1. *Introduction.* If $X \equiv (x_1, x_2, x_3, x_4)$ and $Y \equiv (y_1, y_2, y_3, y_4)$ are any two points in space, the 10 quantities,

(1) $$g_{ij} = x_i y_j + x_j y_i, \qquad (i, j = 1, 2, 3, 4),$$

may be taken as coördinates of a point P in S_9. The locus of P, when X and Y, vary is a six-dimensional variety V_6. It will be shown that V_6 is rational, being mappable on the S_6, $(g_{41}, g_{42}, g_{43}, g_{44}, g_{23}, g_{31}, g_{12})$. When X and Y are corresponding points of an involutional transformation, the locus of P is a three-dimensional variety. The case of the cubic inversion $y_i = 1/x_i$ has been partially discussed by Emch,* but the reasons given for rationality are insufficient. The general cubic involution will be shown to be rational by mapping it on the S_3, $(g_{41}, g_{42}, g_{43}, g_{44})$. The proof differs essentially from and is much simpler than the proof given by Sharpe and Snyder.† It is also shown that the complex of lines joining corresponding points of the general cubic involution can also be mapped on S_3, $(g_{41}, g_{42}, g_{43}, g_{44})$ and its equation is obtained.

2. *The variety* V_6. If $p_{ij} = x_i y_j - x_j y_i$ are line coördinates of XY, they are connected with the g_{ij}, by the identities,

(2) $$p_{ij} p_{kl} = g_{il} g_{jk} - g_{ik} g_{jl}.$$

Hence

(3) $$p_{42}p_{43} = g_{42}g_{43} - g_{44}g_{23} = g_1, \qquad p_{43}p_{41} = g_{43}g_{41} - g_{44}g_{31} = g_2,$$
$$p_{41}p_{42} = g_{41}g_{42} - g_{44}g_{12} = g_3$$

so that

(4) $$p_{41} = (g_2 g_3/g_1)^{1/2}, \quad p_{42} = (g_3 g_1/g_2)^{1/2}, \quad p_{43} = (g_1 g_2/g_3)^{1/2},$$

where the signs of the radicals are all $+$ or all $-$. We have therefore

(5) $$2y_4x_4 = g_{44}, \qquad 2y_4x_1 = g_{41} - (g_2g_3/g_1)^{1/2},$$
$$2y_4x_2 = g_{42} - (g_3g_1/g_2)^{1/2}, \qquad 2y_4x_3 = g_{43} - (g_1g_2/g_3)^{1/2},$$

and hence

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* A. Emch, *American Journal of Mathematics*, Vol. 41 (1926), pp. 21-44.

† F. R. Sharpe and Virgil Snyder, *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 1-12.

$$(6) \qquad g_{11}g_{44} = g^2_{41} - g_2g_3/g_1, \qquad g_{22}g_{44} = g^2_{42} - g_3g_1/g_2,$$
$$g_{33}g_{44} = g^2_{43} - g_1g_2/g_3.$$

It follows from (6) that V_6 is rational, being mapped by (6) on the $S_6(g_{41}, g_{42}, g_{43}, g_{44}, g_{23}, g_{31}, g_{12})$. The equations (5) express the $(1, 2)$ correspondence between the spaces $S(x, y)$ and S_6.

3. *The general cubic involution I between X and Y.* The involution is defined by three equations bilinear in (x) and (y), which, by choosing as vertices of the tetrahedron of reference four of the eight invariant points, can be taken in the form.*

$$(7) \qquad g_{23} = a_1g_{41} + a_2g_{42} + a_3g_{43}, \qquad g_{31} = b_1g_{41} + b_2g_{42} + b_3g_{43},$$
$$g_{12} = c_1g_{41} + c_2g_{42} + c_3g_{43}.$$

When account is taken of (7), the equations (5) express the $(1, 2)$ correspondence between the spaces $S(x)$ and $S_3(g_{41}, g_{42}, g_{43}, g_{44})$.

Hence I is rational. From (5) it can be seen that the image of a plane in $S(x)$

$$(8) \qquad A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0$$

is the surface in S_3

$$(9) \quad (A_1g_{41} + A_2g_{42} + A_3g_{43} + A_4g_{44})^2g_1g_2g_3 = (A_1g_2g_3 + A_2g_3g_1 + A_3g_1g_2)^2,$$

which is apparently of order 8 in the g_{ij}. The terms independent of g_{44} however, vanish identically, so that g_{44} is a factor and the image of a plane in $S(x)$ is a surface F_7 of order 7 in S_3. The image of F_7 is the original plane (8) and the cubic surface

$$(10) \qquad A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 = 0,$$

the y_i being found from (8) by substituting from (1) and solving in terms of the x_i.

The surfaces

$$(11) \qquad C_1g_{41} + C_2g_{42} + C_3g_{43} + C_4g_{44} = 0,$$

images of the planes in S_3 are quartic surfaces through the sextic curve common to the cubic surfaces $y_i = 0$ and through the three straight lines, $x_4 = 0$, and one of $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, in which $x_4 = 0$ meets $y_4 = 0$. Any two of the surfaces (11) meet in a residual curve of order 7, image of a line in S_3. It can be shown from (9) that the images of the three invariant points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ in $S(x)$ are the three quadrics $g_1 = 0$, $g_2 = 0$, $g_3 = 0$ in S_3.

The three quadrics meet in the vertices of the tetrahedron of reference

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* Compare F. R. Sharpe and Virgil Snyder, *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 1-12.

and in four other points. These four points and $(0, 0, 0, 1)$ in S_3 are the images of the four invariant points which are not vertices in $S(x)$ and of $(0, 0, 0, 1)$.

4. *The complex of lines $X Y$.* From (2) we have, in addition to (3), the relations

$$(12)\quad p_{41}p_{23} = g_{43}g_{12} - g_{42}g_{31} = g_4, \qquad p_{42}p_{31} = g_{41}g_{23} - g_{43}g_{12} = g_5,$$
$$p_{43}p_{12} = g_{42}g_{31} - g_{41}g_{23} = g_6,$$

and therefore the quadratic identity

$$p_{41}p_{23} + p_{42}p_{31} + p_{43}p_{12} = 0.$$

From (4) and (12) we find

$$(13)\quad p_{41}/g_2g_3 = p_{42}/g_3g_1 = p_{43}/g_1g_2$$
$$= p_{23}/g_4g_1 = p_{31}/g_5g_2 = p_{12}/g_6g_3 = (g_1g_2g_3)^{-\frac{1}{2}}.$$

The p_{ij} and g_{ij} are connected by the identities

$$(14)\quad g_{ij}p_{kl} + g_{jk}p_{li} + g_{jl}p_{ik} = 0.$$

Hence

$$(15)\quad g_{12}p_{34} + g_{23}p_{41} + g_{42}p_{13} = 0, \quad g_{12}p_{43} + g_{41}p_{32} + g_{31}p_{24} = 0,$$
$$g_{31}p_{42} + g_{43}p_{21} + g_{23}p_{14} = 0, \quad g_{41}p_{23} + g_{42}p_{31} + g_{43}p_{12} = 0.$$

Any three of these equations are linearly independent, but the sum of the four vanishes identically.

From (14) we also have

$$(16)\quad g_{44}p_{23} + g_{42}p_{34} + g_{43}p_{42} = 0, \qquad g_{44}p_{31} + g_{43}p_{14} + g_{41}p_{43} = 0,$$
$$g_{44}p_{12} + g_{41}p_{24} + g_{42}p_{41} = 0.$$

When $g_{41}, g_{42}, g_{43}, g_{44}$ are given, then (7) and (13) determine the p_{ij} which satisfy (15) and (16). If we eliminate the g_{ij} from (7) and (15) we have the equation of the cubic complex.

$$(17)\quad \begin{vmatrix} a_1 & a_2 & a_3 & -1 & 0 & 0 \\ b_1 & b_2 & 3 & 0 & -1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & -1 \\ 0 & p_{13} & 0 & p_{41} & 0 & -p_{43} \\ p_{32} & 0 & 0 & 0 & -p_{42} & p_{43} \\ 0 & 0 & p_{21} & -p_{41} & p_{42} & 0 \end{vmatrix} = 0.$$

Conversely if the p_{ij} satisfy (17) and the quadratic identity, then (7), (15), and (16) determine the g_{ij}. Hence the complex is mappable on S_3. The cubic inversion is the special case when $a_1 = b_2 = c_3 = 1$, the other coefficients in (7) being zero.

CORNELL UNIVERSITY.

ON SEMI-METRIC SPACES.*

By Wallace Alvin Wilson.

1. Let Z be a set of points to each pair of which corresponds a positive real number called the distance between them. If a and b are any two points, we designate this distance by ab, and postulate that the following axioms are satisfied:

I. $ab = ba$.

II. *$ab = 0$ if and only if $a = b$.*

A space which satisfies these conditions and in which limiting points are defined in the usual way is called by Frechet an E-space and by Menger a *semi-metric* space.

As a semi-metric space becomes metric when the so-called triangle axiom is added, it is natural to classify these spaces by the degree to which the triangle axiom is approximated. Hence we are led to the following additional axioms.

III. *For each pair of points a and b there is a positive number r such that for every point c, $ac + bc \geqq r$.*

IV. *For each point a and each positive number k there is a positive number r such that, if b is a point for which $ab \geqq k$ and c is any point, $ac + bc \geqq r$.*

V. *For each positive number k there is a positive number r such that, if a and b are any points for which $ab \geqq k$ and c is any point, $ac + bc \geqq r$.*

If Axiom V is further strengthened by requiring r to equal k, our space becomes metric. Furthermore, E. W. Chittenden † has shown by an equivalent definition that a semi-metric space in which Axiom V is valid is homeomorphic with a metric space. It is the purpose of this article to supplement Chittenden's work by investigating this question for the weaker Axiom IV and also to discuss certain other properties of these spaces.

2. Examples of spaces consisting of enumerable sets of points can be

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* Presented to the American Mathematical Society, February, 1931.

† "On the equivalence of ecart and voisinage," *Transactions of the American Mathematical Society*, Vol. 18, pp. 161-166.

constructed, which show that Axiom V is effectively stronger than Axiom IV, and that Axiom IV is effectively stronger than Axiom III.

In demonstration it is often convenient to make use of the following easily proved properties. *For Axiom III to be valid it is necessary and sufficient that there do not exist two points a and b and a sequence $\{c_i\}$ such that $ac_i + bc_i \to 0$. For Axiom IV to be valid it is necessary and sufficient that there do not exist a point a and two sequences $\{b_i\}$ and $\{c_i\}$ such that $ac_i + b_i c_i \to 0$ but not $ab_i \to 0$. For Axiom V to be valid it is necessary and sufficient that there do not exist three sequences $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, such that $a_i c_i + b_i c_i \to 0$, but not $a_i b_i \to 0$.*

In the case of each of the three axioms it is readily shown that there is a greatest r for which $ac + bc \geqq r$. In future reference to the axioms it will be understood that r represents this greatest value. In Axiom III, r is a function of a and b; in Axiom IV, $r = f(a, k)$; and in Axiom V, $r = f(k)$. In Axioms IV and V, r is a monotone increasing function of k. Since c may coincide with a or b, it is clear that in the last two cases $r \leqq k$ and in the first $r \leqq ab$. Finally, in the last two cases the inequality $ac + bc < r$ implies that $ab < k$.

3. The scope of our work is limited by the following theorems:

I. *For Axiom III to be valid in a semi-metric space it is necessary and sufficient that no sequence converge to more than one limit.*

II. *In a semi-metric space satisfying Axiom IV every derived set is closed.*

The proof of the first of these theorems is self-evident. To prove the second let $A = \{a\}$ be any set, $A' = \{b\}$ be the derived set of A, and c be a limiting point of A'. If c does not lie in A', there is a $k > 0$ such that for no point a in $A - c$ is $ca < k$.

Since Axiom IV is valid, let $r = f(c, k)$; then for every point x, $cx + ax \geqq r$. Take $e < r/2$. Since c is a limiting point of A', there is some point b for which $cb < e$. Since b lies in A', it is a limiting point of A, and also of $A - c$. Hence there is a point a in $A - c$ for which $ba < e$. Then $cb + ab < r$, contrary to the statement at the beginning of the paragraph. Hence c lies in A' and A' is closed.

Corollary. *In a semi-metric space satisfying Axiom IV the set of inner points of any set is a region.*

It can be shown by an example that the converse of Theorem II is not valid, and that the theorem itself is not valid in general if only Axiom III

holds. Consequently it does not seem profitable to give any attention to spaces satisfying Axiom III only.

4. The simpler theorems regarding closed sets and regions in general metric spaces may now be proved valid for semi-metric spaces satisfying Axiom IV in the usual way. In working with these concepts, however, it is necessary to keep in mind two respects in which these spaces differ radically from metric spaces. These are:

In a semi-metric space satisfying Axiom IV the Cauchy criterion for the convergence of a sequence of points to a point is not necessary, and the distance function may have discontinuities.

To see this consider a space Z consisting of a point a and an enumerable set of points $\{a_i\}$, where the distances are defined as follows: $aa_i = 1/i$ and $a_ia_j = 1$ if $i \neq j$. It is easily seen that Axiom IV is valid. Obviously $a_i \to a$, but the Cauchy property is not valid. Finally, as $a_2a = 1/2$ and $a_2a_i = 1$ for every $i \geqq 2$, it is evident that a_2a_i does not converge to a_2a, and so the distance function is not continuous.

It is convenient to call the set of points $\{x\}$ whose distances from a fixed point a are less than some fixed k a sphere, but on account of the possible discontinuity shown above a sphere may fail to be a region; also the sets for which $ax \leqq k$ or $ax \geqq k$ may fail to be closed sets. However, if S is a sphere of center a and radius k, and $r = f(a, k)$, it readily follows from Axiom IV that every point of the sphere of center a and radius r is an inner point of S. Likewise, if T denotes the set $\{x\}$ for which $ax \geqq k$, $\bar{T}$ contains no point y for which $ay < r$.

Furthermore, if s' is a sphere of radius $r' < r$ and center a, $\bar{s}' \subset S$. Hence, if $r' = f(a, r)$ and $r'' < r'$, the sphere s of center a and radius r'' is such that $\bar{s} \cdot \bar{T} = 0$. Such a sphere may be called an *inner sphere* corresponding to a and k.

5. Before proceeding further, we turn to a brief consideration of semi-metric spaces in which Axiom V is valid. We first note that in such a space, *every convergent sequence satisfies the Cauchy convergence criterion.* For otherwise we would have a sequence $\{a_i\}$, where $a_i \to a$, and a constant $k > 0$ such that for every integer i' there would exist an i and a j greater than i' for which $a_ia_j \geqq k$. But, as $i' \to \infty$, $a_ia + a_ja \to 0$, which contradicts Axiom V.

Now let Z be any semi-metric space and a, b, and c be any three points. Also let $g(e)$ be a positive function of the positive real variable e which

converges to zero with e, and let the relations $ac < e$ and $bc < e$ always give $ab < g(e)$. Such a space Frechet calls a space with a *regular ecart.*

THEOREM I. *A semi-metric space in which Axiom V is valid has a regular ecart, and conversely.*

Proof. If Axiom V is valid, there is a monotone increasing function $r = f(k)$ such that $ab \geqq k$ implies that $ac + bc \geqq r$ for any three points of the space. If k has a lower bound $k' > 0$, r has a lower bound $r' > 0$; in this event let us define $r = f(k)$ for $0 \leqq k \leqq k'$ by the relation $r/k = r'/k'$. Now let $h = h(r)$ be 4/3 the lower bound of all values of k for which $r = f(k)$. Then h is one-valued and converges to zero as $r \rightarrow 0$. If $ac < r/3$ and $bc < r/3$, then $ab < h$. Taking $e = r/3$, $h = g(e) = h(r)$ is the required function.

To show the converse, assume that Axiom V fails. Then there would be a $k > 0$ and sequences $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ such that $a_ib_i \geqq k$ for every i, but $a_ic_i + b_ic_i \rightarrow 0$. Taking e so that $g(e) < k$, there is an i such that $a_ic_i < e$ and $b_ic_i < e$, while $a_ib_i \geqq k > g(e)$, contrary to the definition of regular ecart.

The above theorem together with Chittenden's theorem (*loc. cit.* in § 1) gives the following result.

THEOREM II. *For a semi-metric space Z to be uniformly homeomorphic with a metric space it is necessary and sufficient that Axiom V be valid.*

That the condition is sufficient is proved by Chittenden. It remains to prove that it is necessary.

Let $Z = \{x\}$ and let $M = \{y\}$ be metric and uniformly homeomorphic with Z. If Axiom V is not valid in Z, there is a $k > 0$ and three sequences $\{x_i\}$, $\{x_i'\}$, and $\{x_i''\}$ such that $x_i'x_i'' \geqq k$ and $x_ix_i' + x_ix_i'' \rightarrow 0$. Let the corresponding sequences in M be $\{y_i\}$, $\{y_i'\}$, and $\{y_i''\}$. Since the homeomorphism is uniform, $y_iy_i' + y_iy_i'' \rightarrow 0$ and consequently, by the triangle axiom, $y_i'y_i'' \rightarrow 0$. But again this result requires that $x_i'x_i'' \rightarrow 0$, which is a contradiction.

6. THEOREM. *Let Z be a semi-metric space satisfying Axiom IV. Then Z is homeomorphic with a semi-metric space satisfying Axiom V.*

*Proof.** For each point x of Z define two descending sequences $\{r_i(x)\}$ and $\{s_i(x)\}$ as follows:

* The following proof is based upon one by Alexandroff and Urysohn, "Une condition nécessaire et suffisante pour qu'une classe L soit une classe D," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 177, pp. 1274-1276.

$$\begin{array}{ll} s_0(x) = 1, & r_1(x) = \tfrac{1}{2}f(x, 1), \\ r_2(x) = \tfrac{1}{2}f[x, r_1(x)], & s_1(x) = \tfrac{1}{2}r_2(x), \cdots, \\ s_{i-1}(x) = \tfrac{1}{2}r_{2(i-1)}(x), & r_{2i-1}(x) = \tfrac{1}{2}f[x, s_{i-1}(x)], \\ r_{2i}(x) = \tfrac{1}{2}f[x, r_{2i-1}(x)], & s_i(x) = \tfrac{1}{2}r_{2i}(x), \cdots. \end{array}$$

Here f is the function of Axiom IV. It is clear that $s_i(x) \leqq \tfrac{1}{2}^{3i}$ for every i and x. For each i and x let $U_i(x)$ be the sphere of center x and radius $s_i(x)$ and K_i be the set of spheres $\{U_i(x)\}$ as x ranges over Z.

We now prove that, if $D_i = U_i(x) \cdot U_i(y) \neq 0$ for some $i \geqq 1$, then $U_i(x) + U_i(y)$ is contained in either $U_{i-1}(x)$ or $U_{i-1}(y)$. Let u be a point of D_i. There are two cases: $s_i(x) \geqq s_i(y)$ or $s_i(x) < s_i(y)$.

In the first case $xu + uy < 2s_i(x) = r_{2i}(x)$, whence $xy < r_{2i-1}(x)$ by Axiom IV. If z lies in $U_i(y)$, $yz < s_i(y) \leqq s_i(x)$. Hence $xy + yz < 2r_{2i-1}(x) = f[x, s_{i-1}(x)]$ and so $xz < s_{i-1}(x)$ by Axiom IV. Obviously $xz < s_{i-1}(x)$ if z lies in $U_i(x)$. Hence $U_i(x) + U_i(y) \subset U_{i-1}(x)$ in this case.

In the second case $xz < s_i(x) < s_i(y)$ for any z in $U_i(x)$; whence $xy + xz < 2r_{2i-1}(y)$ and $yz < s_{i-1}(y)$. Obviously $yz < s_{i-1}(y)$ if z lies in $U_i(y)$. Thus in this case $U_i(x) + U_i(y) \subset U_{i-1}(y)$.

Now let us define a new distance $d(x, y)$ for each pair of points of Z as follows and call the new space Z'. If no sphere of any K_i contains $x + y$, let $d(x, y) = 1$. If no sphere of K_{i+1} contains $x + y$, but some sphere of K_i contains $x + y$, let $d(x, y) = \tfrac{1}{2}^i$.

We first show that Z' is semi-metric. If $x \neq y$, $d(x, y)$ exists and is positive unless some sphere of every K_i contains $x + y$. Then there would be a sequence $\{U_i(c_i)\}$ of spheres whose respective centers and radii are $\{c_i\}$ and $\{s_i(c_i)\}$, each of which would contain $x + y$. Since $s_i(c_i) \to 0$, this gives the contradiction that $c_i \to x$ and $c_i \to y$ in Z.

If x, y, and z are three points, $d(x, z) = 1/2^i$, and $d(y, z) = 1/2^j$, $j \geqq i$, we have two spheres $U_i(a)$ and $U_i(b)$ of K_i containing $x + z$ and $y + z$, respectively. Then by the above, these spheres are contained in either $U_{i-1}(a)$ or $U_{i-1}(b)$. In both cases $d(x, z) \leqq 1/2^{i-1}$, and so the distance $d(x, y)$ is a regular ecart. Hence Axiom V is valid in Z' by § 5, Theorem I.

Let a be a fixed point, $k > 0$, and S be the set of points $\{x\}$ for which $d(a, x) < k$. Take i so large that $1/2^i < k$. Then $U_i(a) \subset S$. For, if x lies in $U_i(a)$, $a + x$ lies in $U_j(a)$ for $0 \leqq j \leqq i$ and hence $d(a, x) \leqq 1/2^i$. That is, S contains every point x for which $ax < s_i(a)$.

Now let S be the set of points $\{x\}$ for which $ax < k$. Take i so large that $s_{i-1}(a) < k$ and j so large that $2/2^{3j} < s_i(a)$. Let S' be the set of points $\{x\}$ for which $d(a, x) < 1/2^j$. Then for each x in S', $a + x$ lies in

some $U_j(b)$; i. e., $ab + bx < 2s_j(b) < 2/2^{3j} < s_i(a)$, whence $ax < s_{i-1}(a) < k$. That is, S contains every point x for which $d(a, x) < 1/2^j$.

The last two paragraphs prove that Z and Z' are homeomorphic.

Corollary. *Let Z be a semi-metric space satisfying Axiom IV. Then Z is homeomorphic with a metric space.*

This is an immediate consequence of the above theorem and Chittenden's theorem (§ 5, Theorem II). Note, however, that in this case homeomorphism between a semi-metric and a metric space does not imply that Axiom IV is valid in the former space.

7. A semi-metric space Z which satisfies Axiom IV and contains an enumerable set $E = \{a_i\}$ such that every point of Z is the limit of a subsequence chosen from E will be called *separable*, as usual. Following Frechet, we shall call Z perfectly separable or *p-separable*, if for each $k > 0$ each point x of Z lies in an inner sphere corresponding to k and some a_i. (See § 4.)

The author does not know whether separability implies p-separability or not. It is possible to construct a space Z containing an enumerable set E dense in Z and such that for a given $k > 0$, Z is not covered by a set of inner spheres having their respective centers in E and corresponding to k, but this does not show that there is no enumerable set having the desired property. Since by the previous section a separable semi-metric space satisfying Axiom IV is homeomorphic with a separable metric space and a separable metric space is always p-separable, it might appear that separability implies p-separability in semi-metric space also. This does not follow, however, because the homeomorphism between the two spaces may fail to be uniform.

It is easy to show that a p-separable semi-metric space satisfying Axiom IV is homeomorphic with a separable semi-metric space without using Chittenden's theorem. In brief we first prove the theorem that, if $\bar{A} \cdot B + A \cdot \bar{B} = 0$, then there are disjoint regions R and S containing A and B, respectively, in much the same way as for metric spaces. Then Urysohn's proof* that for two disjoint closed sets A and B, there is a continuous function $f(x)$ such that $f(x) = 0$ in A, $f(x) = 1$ in B, and $0 \leqq f(x) \leqq 1$ in $Z - (A + B)$ is applicable. Then, for each k_j of a descending sequence $\{k_j\}$ converging to zero and each point a_i of the set E used in defining p-separability, we define a continuous function $f_{ij}(x)$ such that $f_{ij}(x) = 0$ in a closed set A_{ij} containing a_i as an inner point, $f_{ij}(x) = 1$ in a closed set B_{ij} containing every

* "Zum Metrisationsproblem," *Mathematische Annalen*, Vol. 94, pp. 310-311.

x for which $ax \geqq k_j$, and $0 \leqq f_{ij}(x) \leqq 1$ elsewhere in Z. Then the distance between two points x and y is defined by $\sum_{i,j=1}^{i,j=\infty} | f_{ij}(x) - f_{ij}(y) | / 2^{i+j}$.

8. The result of § 6 would appear to be useful in handling upper semi-continuous decompositions of metric spaces into disjoint closed sets. Under certain broad conditions the space whose elements are the closed sets is known to be metric, but the distance between two such elements has no simple relation to the distance between the closed sets in the original space. The following theorem together with § 6 makes it possible to use the distance in the original space as the distance in the new space.

THEOREM. *Let $Z = \{x\}$ be a metric space and $Z = \sum[X]$ be an upper semi-continuous decomposition of Z into disjoint closed sets. Let Z' be the space whose elements are $\{X\}$ and for any two elements X and Y of Z' let XY be the distance between the sets X and Y as measured in Z. Then Z' is a semi-metric space satisfying Axiom IV and, if Z is connected, so is Z'.*

Proof. By the distance between X and Y in Z we mean the lower bound of xy as the point x ranges over X and the point y ranges over Y. If then $XY = 0$, there are sequences $\{x_i\}$ and $\{y_i\}$, chosen from X and Y, respectively, such that $x_i y_i \to 0$. By the definition of upper semi-continuous decompositions it follows that for any $e > 0$ every point of X has a distance from Y less than e and every point of Y has a distance from X less than e. As X and Y are closed, this makes $X = Y$. Hence Z' is semi-metric.

If Axiom IV were not valid, there would be some element A, a constant $k > 0$, and sequences $\{X_i\}$ and $\{Y_i\}$ such that $AX_i \geqq k$, but $AY_i + X_iY_i \to 0$. Take a positive $e < k/3$. Since $AY_i \to 0$ and the decomposition is upper semi-continuous, there is an i' such that every point of Y_i has a distance from A less than e for every $i > i'$. Since $X_iY_i \to 0$, there is an i'' such that for every $i > i''$ there is a point x_i in X_i and a point y_i in Y_i for which $x_iy_i < e$. But then for i greater than both i' and i'' some point of X_i has a distance less than $2e$ from some point of A, and consequently $AX_i < k$, which is false.

If Z' were not connected, it would be the sum of two disjoint non-void sets H' and K' such that $\bar{H}' \cdot K' + H' \cdot \bar{K}' = 0$. In Z let H be the union of the sets $\{X\}$ which are elements of H' and K have a similar relation to K'. If $\bar{H} \cdot K \neq 0$, there would be a point y in K which is the limit of a sequence $\{x_i\}$ of points of H. Now y lies in some element Y and each x_i in some element X_i, whence $X_iY \to 0$. But X_i is an element of H' and Y is an element of K', and $\bar{H}' \cdot K' = 0$, which is a contradiction. Hence $\bar{H} \cdot K = 0$

and, in like manner, $H \cdot \bar{K} = 0$. This is again a contradiction, because Z was connected. Hence Z' is connected.

In connection with this theorem it should be noted that, in general, Axiom V is not satisfied.

9. We shall now investigate certain relationships between semi-metric and topological spaces, defining the latter by these axioms:

A. *Every point x has at least one vicinity $U(x)$ and x lies in $U(x)$.*

B. *If $U(x)$ and $V(x)$ are vicinities of x, there is some $W(x) \subset U(x) \cdot V(x)$.*

C. *For each $U(x)$ there is a $V(x)$ such that, if y lies in $V(x)$, some $U(y) \subset U(x)$.*

4. *If x and y are two distinct points, some $U(x)$ does not contain y.*

Axioms A and B are taken directly from Hausdorff's *Mengenlehre* (pp. 228, 229). Axiom C is the weaker form of Hausdorff's Axiom C suggested by Frechet. This is more convenient, since in semi-metric spaces satisfying Axioms IV or V, a sphere is not necessarily a region, but merely contains a region containing in turn the center. In Axiom 4 "vicinity" has been used instead of "region," since this is more consistent in forming a set of vicinity axioms.

Before proceeding further it will be as well to call attention to two known points which are sometimes overlooked or insufficiently stressed. The first is the fact that it must not be understood that, if some $U(x)$ contains a point y different from x, then $U(x)$ is a vicinity of y. Such an assumption in certain cases vitiates the work. The second is the difference between equivalence and homeomorphism, the former being in some cases an effectively stronger property than the latter. It may be remarked here that the axioms given above are *equivalent* to the corresponding axioms of Hausdorff.

It is clear that relations between semi-metric and topological spaces will involve enumerability axioms of some kind. Consider the following.

9. *There is an equivalent set of vicinities such that every point x has an at most enumerable set of vicinities.*

9′. *If x is a point and $\{U(x)\}$ the set of its vicinities, there is an enumerable sub-set $\{V_i(x)\}$ of these vicinities such that x is the divisor of the set $\{V_i(x)\}$ and each $U(x)$ contains some $V_i(x)$.*

10. *There is an equivalent set of vicinities which is enumerable.*

Axioms 9 and 10 are essentially the same as Hausdorff's axioms with these numbers. It is shown below by an example that Axiom 9′ is effectively weaker than Axiom 9. On the other hand it is readily seen that the restriction of the class of vicinities in Axiom 9′ does not affect the definition of limiting points. This is a case where homeomorphism and equivalence are not the same and it is well illustrated by the following example.

Let Z be the sum of two disjoint sets A and C, where A consists of the points of the interval $0 \leqq x < 1$ and C is a set $\{y\}$ of cardinal number c. For each y let $U(y) = y$; for $x \neq 0$ let $U(x)$ be any open interval of center x contained in A; for $x = 0$ let $U(x)$ be any half-open sub-interval of A whose left end-point is 0 *or* any such sub-interval plus any point of C. Here Axiom 9 is not satisfied. If we restrict the lengths of the sub-intervals to rational numbers and require $U(0)$ to be either such a sub-interval or such a sub-interval plus the point of C corresponding to its length in some correspondence between the rational numbers and an enumerable sub-set of C, Axiom 9 is valid. On the other hand, Axiom 9′ is valid in the original space, since the vicinities of points of C and the vicinities of points of A consisting of sub-intervals of rational length satisfy the requirements. The two spaces are homeomorphic, but not equivalent.

THEOREM I. *Let the topological space Z satisfy Axiom 9′. Then we can take the partial set of vicinities so that for each point* $V_1(x) \supseteqq V_2(x) \supseteqq V_3(x) \supseteqq \cdots$.

Proof. Let $\{U(x)\}$ be the original set and $\{W_i(x)\}$ be any partial set satisfying the requirements of Axiom 9′. Take $V_1(x) = W_1(x)$. Now $W_1(x) \cdot W_2(x) \supset$ some $U(x) \supset$ some $W_{i_2}(x)$. Set $V_2(x) = W_{i_2}(x)$. Likewise the divisor of the first $i_2 + 1$ vicinities $\{W_i(x)\}$ contains some $U(x)$, which in turn contains some $W_{i_3}(x)$; this we take for $V_3(x)$. Continue this process indefinitely. Clearly every $U(x)$ contains some $V_n(x)$ and x is the divisor of the monotone descending sequence $\{V_n(x)\}$.

THEOREM II. *Let the topological space Z satisfy Axiom 9. Then there is an equivalent set of vicinities so that for each point* $V_1(x) \supseteqq V_2(x) \supseteqq \cdots$.

This is proved in the same way as Theorem I.

10. It is apparent that there is an intimate connection between Axioms III, IV, and V of semi-metric spaces and Tietze's separation axioms * for topological spaces. But in studying this connection we meet the following

* See Hausdorff, *Mengenlehre*, p. 229, Axioms 5-8.

difficulty. If in a semi-metric space the sequence of points $\{a_i\}$ converges to a point a, there is for every $r > 0$ an i' such that a sphere of center a and radius r contains every a_i for which $i > i'$ and for such values of i the spheres of radius r and centers a_i all contain a. But this does not hold for topological spaces. There, although each vicinity of a contains every a_i for i greater than some i', it may be that *no* vicinity of any a_i contains a. Therefore it seems well to the author to propose the following axioms in place of Hausdorff's Axioms 5-8 for use in topological spaces satisfying Axioms 9 or 9':

5'. *For every pair of points a and b and every integer n there is an integer $m = g(a, b, n)$ such that m increases indefinitely with n and the relation $V_n(a) \cdot V_n(b) \neq 0$ implies that b lies in $V_m(a)$ and a lies in $V_m(b)$.*

6'. *For each point a and each integer n there is an integer $m = g(a, n)$ such that m increases indefinitely with n and the relation $V_n(a) \cdot V_n(b) \neq 0$ implies that b lies in $V_m(a)$ and a lies in $V_m(b)$.*

7'. *For each integer n there is an integer $m = g(n)$ such that m increases indefinitely with n and the relation $V_n(a) \cdot V_n(b) \neq 0$ implies that b lies in $V_m(a)$ and a lies in $V_m(b)$.*

It is a simple matter to show that Axiom 5' implies Hausdorff's Axiom 5. For Axiom 6' we get the following theorem, which is analogous to a theorem of Tychonoff.*

THEOREM I. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 9 or 9'. If it also satisfies Axiom 6', it satisfies Hausdorff's Axioms 6, 7, and 8.*

Proof. Let our vicinities be monotone descending as in the theorems of § 9. Let A and B be two point-sets such that $\bar{A} \cdot B + A \cdot \bar{B} = 0$. Let a be a fixed point not in the closed set $\bar{B}$ and b be any point of $\bar{B}$. Then there is an n such that $V_n(a) \cdot V_n(b) = 0$. If n is unbounded as b ranges over $\bar{B}$, there is for each n a point b_n in $\bar{B}$ such that $V_n(a) \cdot V_n(b_n) \neq 0$. If $m = g(a, n)$ as in Axiom 6', $V_m(a)$ contains b_n. But m increases indefinitely with n; hence every $V_m(a)$ contains points of $\bar{B}$, a contradiction. Therefore for some n we have $V_n(a) \cdot V_n(b) = 0$ for every point b in $\bar{B}$.

In consequence of this result there is for each integer i a sub-set A_i of A such that for each point a in A_i and each point b in $\bar{B}$, $V_i(a) \cdot V_i(b) = 0$. The set A_i may be void for a particular value of i, but $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and A is the union of the sets $\{A_i\}$. Likewise, B is the union of a monotone

* "Über einen Metrisationssatz" von P. Urysohn, *Mathematische Annalen*, Vol. 95, pp. 139-141.

increasing sequence of sets $\{B_i\}$, such that for each b in B_i and each a in $\bar{A}$, $V_i(a) \cdot V_i(b) = 0$.

Each $V_i(a)$ contains a region $v_i(a)$ which contains a, by Axiom C, and likewise for $V_i(b)$. Let U_i be the union of the regions $\{v_i(a)\}$ as a ranges over A_i, and W_i be the union of the regions $\{v_i(b)\}$ as b ranges over B_i. Let i and j be any two integers and $i \leqq j$. If a lies in A_i, $V_i(a) \cdot V_i(b) = 0$ for every b in $\bar{B}$, and consequently in B_j. Since $V_j(b) \subseteq V_i(b)$, we have $V_i(a) \cdot V_j(b) = 0$ for every a in A_i and b in B_j. Consequently $U_i \cdot W_j = 0$ for this case, and similar reasoning establishes the same fact for the case that $i > j$. If then R and S are the unions of the sets $\{U_i\}$ and $\{W_i\}$, respectively, we have $A \subset R$, $B \subset S$, and $R \cdot S = 0$. As R and S are obviously regions, we have shown that A and B are contained in disjoint regions, which is the requirement in Hausdorff's Axiom 8. A fortiori, Axioms 6 and 7 are also valid.

THEOREM II. *Let Z be a topological space satisfying Axioms A, B, C, 4, 6′, and 9 or 9′, and $\{a_i\}$ a sequence of points converging to a. Then for each m there is an i_m such that a_i lies in $V_m(a)$ and a lies in $V_m(a_i)$ for every $i > i_m$.*

Proof. As usual we assume that our vicinities are monotone descending in accordance with the theorems of § 9. Since by Axiom 6′, $m = g(a, n)$ increases indefinitely with n, there is for each integer m an integer n such that $V_n(a) \cdot V_n(b) \neq 0$ implies that a lies in $V_m(b)$ and b lies in $V_m(a)$.

Since $a_i \to a$, there is an i_n such that each a_i lies in $V_n(a)$ for $i > i_n$. But then $V_n(a) \cdot V_n(a_i) \neq 0$. Consequently the previous paragraph is applicable and we have the theorem on writing i_n as i_m.

11. THEOREM. *Let Z be a semi-metric space satisfying Axiom IV(V). Let $r_1 > r_2 > r_3 \cdots$ and $r_i \to 0$. For each point a of Z let $U_i(a)$ denote a sphere of center a and radius r_i. If these spheres are taken as vicinities, Z is a topological space satisfying Axioms A, B, C, 4, and 9, and also 6′(7′).*

Proof. It is clear that Axioms A, B, 4, and 9 are satisfied. Now take a fixed $U_i(a)$. Then for some $j > i$, it follows from § 4 that $U_j(a)$ contains only inner points of $U_i(a)$ and so for any point x in $U_j(a)$ some $U_k(x) \subset U_i(a)$. Hence Axiom C is also valid.

Now suppose that Axiom 6′ were not valid. Then there would be a fixed integer m and a sequence of points $\{b_n\}$ such that $U_n(a) \cdot U_n(b_n) \neq 0$, but for every n either a would not lie in $U_m(b_n)$ or b_n would not lie in $U_m(a)$. The former statement requires the existence of a sequence $\{c_n\}$ such that

$ac_n \to 0$ and $b_nc_n \to 0$. But then by Axiom IV we have $ab_n \to 0$. Then for some n_0 and every $n > n_0$, $ab_n < r_m$, which contradicts the second statement. Hence Axiom 6′ is valid.

Suppose now that Axiom V is valid, but Axiom 7′ is not. Then there is a fixed m and sequences $\{a_n\}$ and $\{b_n\}$ such that $U_n(a_n) \cdot U_n(b_n) \not\doteq 0$, but either b_n is not in $U_m(a_n)$ or a_n is not in $U_m(b_n)$. Then for a sequence $\{c_n\}$, we have $a_nc_n + b_nc_n \to 0$, whence $a_nb_n \to 0$. This gives a contradiction as above.

Remark. The above theorem is a fortiori true if 9′ is substituted for 9.

12. THEOREM. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 9. If also Axiom 6′(7′) is satisfied, Z is equivalent to a semi-metric space satisfying Axiom IV(V).*

Proof. In accordance with § 9 we assume that the vicinities of each point form a monotone descending sequence of sets. For a pair of points a and b set $f_n(a, b) = 0$ if b lies in $V_n(a)$ and $f_n(a, b) = 1$ if b is not in $V_n(a)$. Likewise define $f_n(b, a)$. Let $d_n(a, b) = d_n(b, a) = [f_n(a, b) + f_n(b, a)]/2^n$ and $ab = ba = \sum_1^\infty d_n(a, b)$. Let Z' be a space having the same points as Z and distances defined in this manner.

It follows at once from Axiom 6′ that there is an integer n' such that $V_n(a) \cdot V_n(b) = 0$ if $n \geqq n'$. Hence $ab \geqq \sum_{n'}^\infty 2/2^n = 1/2^{n'-2} > 0$ and so Z' is semi-metric.

Now let a be any point and $r > 0$. Take m so that $r > 1/2^{m-1}$. In consequence of Axiom 6′ there is an n such that, if b lies in $V_n(a)$, then b lies in $V_m(a)$ and a lies in $V_m(b)$. Hence $ab < \sum_{m+1}^\infty 2/2^i = 1/2^{m-1} < r$. That is, each sphere of radius r contains some $V_n(a)$. On the other hand, let $V_m(a)$ be any vicinity. If $r < 1/2^m$ and $ab < r$, $d_n(a, b) = 0$ for every $n \leqq m$. Hence by the definition of $d_n(a, b)$, b lies in $V_n(a)$ and a in $V_n(b)$ for every $n \leqq m$. Thus every vicinity of a contains some sphere of center a. Hence we have proved that the spaces are *equivalent*.

If Axiom 6′ does not imply Axiom IV, there are two sequences $\{b_i\}$ and $\{c_i\}$ and a constant $k > 0$ such that $ab_i \geqq k$ and $ac_i + b_ic_i \to 0$. These relations show that for a fixed n' there is an i' such that $d_n(a, c_i) = 0$ and $d_n(b_i, c_i) = 0$ for every $i \geqq i'$ and every $n \leqq n'$. Then for such values $V_n(c_i)$ contains a and b_i, and c_i lies in both $V_n(a)$ and $V_n(b_i)$. If $m = g(a, n)$

as defined in Axiom 6′, this means that b_i lies in $V_m(a)$ for $i \geqq i'$. As m increases indefinitely with n, we have b_i approaching a, which is impossible.

If Axiom 7′ does not imply Axiom V, we have three sequences $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, and a $k > 0$, such that $a_ib_i \geqq k$ and $a_ic_i + b_ic_i \to 0$. As above, we have for each n' an i' such that $V_n(a_i) \cdot V_n(b_i) \neq 0$ for $i \geqq i'$ and $n \leqq n'$. Let $m = g(n)$ as defined in Axiom 7′. Then $d_n(a_i, b_i) = 0$ for $n \leqq m$; and, as m increases indefinitely with n, this means that $a_ib_i \to 0$, a contradiction.

Corollary. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 9′. If also Axiom 6′(7′) is satisfied, Z is homeomorphic with a semi-metric space satisfying Axiom IV(V).*

For by § 9 the space Z is homeomorphic with a topological space satisfying Axioms A, B, C, 4, and 9.

Remark. A study of the proof of the theorem of § 6 shows that in the above theorem Axiom 6′ is sufficient for equivalence with a semi-metric space satisfying Axiom V, and an analogous remark is true for the corollary.

13. In the theorem and corollary of the previous section a distinction has been drawn between homeomorphism and equivalence. The same thing is necessary in connection with Urysohn's theorem regarding the equivalence of a topological space satisfying Axioms A, B, C, 6, and 10 to a metric space, which has been referred to in § 7. In his proof it is tacitly assumed that every vicinity containing a point is a vicinity of that point; without that assumption the proof given does not apply and in fact there is homeomorphism and not equivalence. This in part explains the necessity for such axioms as 6′ and 7′.

Since it has been shown that a semi-metric space satisfying Axiom IV is homeomorphic with a metric space, it follows from the previous section that this is also true for a topological space satisfying Axioms A, B, C, 4, 6′, and 9′.

CONCERNING HEREDITARILY LOCALLY CONNECTED CONTINUA.

By G. T. Whyburn.

A continuum every subcontinuum of which is locally connected is said to be *hereditarily locally connected.* The principal contribution of the present paper is the establishing of the proposition that *Every hereditarily locally connected, compact and metric continuum is a rational curve in the Menger-Urysohn* * *sense,* that is, each point of such a continuum M is contained in arbitrarily small neighborhoods having countable boundaries relative to M. This theorem was proved formerly † by the present author for subcontinua of the plane, but the demonstration in the present article is independent of the containing space and is therefore valid for subcontinua of any compact metric space. It is thus demonstrated that in the Menger–Urysohn classification of curves, the hereditarily locally connected continua form a distinct class occupying an intermediate position between the class of all regular curves (= continua each of whose points is contained in arbitrarily small neighborhoods with finite boundaries) and the class of all rational curves. In other words, all regular curves are hereditarily locally connected, but not conversely; and all hereditarily locally connected continua are rational curves, but not conversely.

In the course of the demonstration of the proposition announced above, the author has found and used a number of strong properties of hereditarily locally connected continua, each of which, incidentally, characterizes these continua among the compact metric continua. Proofs for these properties, together with some of their corollaries, form § 2 of the present paper. In § 3 there is given two lemmas of a general character which also are needed in the proof of the main theorem of the paper, given in § 4.

We shall employ the usual terminology and notation of the theory of sets. Our hypotheses ordinarily concern a compact metric continuum M and its subsets, and in such cases we shall consider M as a space and shall speak of the open subsets of M as neighborhoods or open sets. If V is such a set, $F(V)$ will denote the boundary of V, i. e., the point set $\bar{V} - V$. A component of a set K is a connected subset of K which is contained in no other

* See Menger, *Mathematische Annalen*, Vol. 95 (1925), pp. 272-306 and Urysohn, *Verhandelingen der Akademie te Amsterdam*, Vol. 13 (1927), No. 4.

† See *Bulletin of the American Mathematical Society*, Vol. 36 (1930), pp. 522-524.

connected subset of K. The quasi-component of a set K containing the point p of K consists of p together with all points x such that K is not the sum of two mutually separated sets, one containing p and the other x. A collection of sets will be called a *null family* provided that all save a finite number of these sets are of diameter less than any preassigned positive number. A countable sequence of sets whose elements form a null family will be called a *null sequence.*

2. *Properties equivalent to hereditary local connectivity.*

(1). *In order that the compact metric continuum M should be hereditarily locally connected it is necessary and sufficient that if K is any subset of M and G is any collection of open sets covering some subset H of K, the boundary of no one of which contains a point of K, then there exists a null sequence $V_1, V_2, V_3, \cdots$ of mutually exclusive open sets covering H each of which is a subset of some element of G and its boundary is a subset of the sum of the boundaries of a finite number of the sets of G.*

The condition is necessary. For by the Lindelöf Theorem there exists a countable sequence $G_1, G_2, G_3, \cdots$ of the sets of G whose sum covers H. Set

$$G_1 = U_1,\ G_2 - G_2 \cdot \bar{G}_1 = U_2, G_3 - G_3 \cdot (\bar{G}_1 + \bar{G}_2) = U_3, \cdots, \qquad G_n - G_n \cdot \sum_1^{n-1} \bar{G}_i = U_n, \cdots.$$

Then clearly $U_1, U_2, \cdots$ is a sequence of mutually exclusive open sets covering H, and for each n,

$$\text{(i)} \quad F(U_n) = F[G_n - G_n \cdot \sum_1^{n-1} \bar{G}_i] \subset F(G_n) + \sum_1^{n-1} F(G_i) = \sum_1^{n} F(G_i).$$

Now set $U = \sum_1^{\infty} U_n$. Then U is an open subset of M, and hence the components of U may be arranged into a sequence $V_1, V_2, V_3, \cdots$, which clearly must be a null sequence, since M is hereditarily locally connected. Since the sets $[U_n]$ are mutually exclusive, it follows that for each n there exists an i such that $V_n \subset U_i$ and $F(V_n) \subset F(U_i)$. Hence $V_n \subset G_i$, and by (i) we have $F(V_n) \subset \sum_1^{i} F(G_j)$, which completes the proof.

That the condition is sufficient follows at once from the fact* that every continuum M which is not hereditarily locally connected contains an infinite sequence $N_1, N_2, N_3, \cdots$ of mutually exclusive continua, all of diameter greater

* See R. L. Moore, *Bulletin of the American Mathematical Society*, Vol. 29 (1923), p. 296; also C. Zarankiewicz, *Fundamenta Mathematicae*, Vol. 9, p. 134.

than some $\epsilon > 0$, which converges sequentially to a limit continuum N having no point in common with the continua $[N_i]$. For, taking $K = H = \sum_1^\infty N_i$ and, for each i, letting G_i denote the set of all points x in M such that $\rho(x, N_i) < (1/3)\rho(N_i, H - N_i)$, then the collection $[G_i]$ covers H and the boundary of no one of these sets contains a point of K, but clearly no null sequence exists satisfying the terms of our condition.

(2). *In order that the compact metric continuum M be hereditarily locally connected it is necessary and sufficient that if K is any subset of M, and p is any point of a quasi-component C of K, and R is any neighborhood of p, then there exist a neighborhood U of p such that $R \cdot C \subset U \subset R$ and $F(U) \cdot K \subset F(R) \cdot C$.*

The condition is necessary. For let H denote the set $(K - C) \cdot F(R)$. Then since no point of H belongs to C, there exists, for each point x of H, a separation of K into two mutually separated sets one containing x and the other C; and hence there exists an open set G_x containing x but not C and such that $F(G_x) \cdot K = 0$. The collection G of all sets $[G_x]$ for all points x of H covers H, and the boundary of no one of these sets contains a point of K. Therefore, by (1), there exists a null sequence $V_1, V_2, V_3, \cdots$ of open sets covering H each of which contains at least one point of H and is a subset of some G_x and has no boundary point in K. Therefore $\sum_1^\infty V_i$ contains H but contains no point of C, and $K \cdot \sum_1^\infty F(V_i) = 0$. But since $[V_i]$ is a null sequence and since each V_i contains at least one point of $F(R)$, it follows that $F(\sum V_i) \subset \sum F(V_i) + F(R)$, and hence $K \cdot F(\sum V_i) \subset K \cdot F(R)$, because $K \cdot \sum F(V_i) = 0$. Then since H, which is $= F(R) \cdot (K - C)$, is a subset of $\sum V_i$, therefore $F(\sum V_i) \cdot K \subset F(R) \cdot C$.

Now set $V = \sum_1^\infty V_i$. We have just shown that V contains H but no point of C and that $F(V) \cdot K \subset F(R) \cdot C$. Set $U = R \cdot (M - \bar{V})$. Then $R \cdot C \subset U \subset R$ and $F(U) \subset F(V) + [F(R) - H]$, and therefore

$$F(U) \cdot K \subset F(V) \cdot K + F(R) \cdot C \subset F(R) \cdot C.$$

To prove the sufficiency of the condition, take the sets $N, N_1, N_2, \cdots$ as in the proof of the sufficiency part of (1). Let $K = N + \sum_1^\infty N_i$, let p be a point of N, and let R be a neighborhood of p of diameter $< \epsilon/2$. Then although N is a quasi-component of K, there can exist no neighborhood U of

p satisfying the terms of our condition, because any such U would contain a point of some N_i and since $\delta(N_i) > \epsilon$, we would have $N_i \cdot F(U) \neq 0$.

(3). *In order that the compact metric continuum M should be hereditarily locally connected it is necessary and sufficient that the quasi-components and the components of any subset of M be identical.*

The condition is necessary. For let K be any subset of a hereditarily locally connected continuum M. Clearly it is sufficient to prove that every quasi-component of K is connected. Suppose, on the contrary, that some quasi-component C of K is the sum of two mutually separated sets C_1 and C_2. Then there exists an open set R containing C_1 but no point of C_2 and such that $F(R) \cdot C = 0$. Now by (2) there exists an open set U such that $R \cdot C \subset U \subset R$ and $F(U) \cdot K \subset F(R) \cdot C = 0$. Thus U contains C_1 but not C_2, which is impossible since C is a quasi-component of K and $F(U) \cdot K = 0$.

The condition is also sufficient. For consider the sets $N, N_1, N_2, \cdots$ as in the preceding proofs. Let a and b be distinct points of N, and let $K = a + b + \sum_1^\infty N_i$. Then obviously a is a component of K, whereas $a + b$ is the quasi-component of K containing a. Thus our condition is contradicted in any continuum which is not hereditarily locally connected.

(4). *In order that the compact metric continuum M should be hereditarily locally connected it is necessary and sufficient that the components of any subset of M form a null family.*

The condition is necessary. For let M be any hereditarily locally connected continuum and suppose, contrary to our theorem, that there exists an infinite sequence $K_1, K_2, K_3, \cdots$ of distinct components of some subset of M all of which are of diameter greater than some given positive number ϵ. Set $K = \sum_1^\infty K_n$. Then for each n, K_n is a component of K. By (3), K_n is also a quasi-component of K, for each n. Thus there exists a separation of K into two mutually separated sets H_1 and H_2 containing K_1 and K_2 respectively, and hence there exist two mutually exclusive open sets G_1 and Q_1 containing H_1 and H_2 respectively and such that the boundary of neither of these sets contains a point of K. One of these sets, say Q_1, contains infinitely many of the sets $[K_n]$. Likewise there exists a separation of $K \cdot Q_1$ into two non-vacuous mutually separated sets L_1 and L_2, and hence there exist two mutually exclusive open subsets G_2 and Q_2 of Q_1 containing L_1 and L_2 respectively, the boundary of neither of which contains a point of K. One of these sets, say Q_2,

contains infinitely many of the sets $[K_n]$. There exists a separation of $K \cdot Q_2$, and so on. Continuing this process indefinitely, we obtain an infinite sequence of mutually exclusive open sets $G_1, G_2, G_3, \cdots$, such that for each i, G_i contains some component K_{n_i} of K and $F(G_i) \cdot K = 0$. Set $H = \sum_1^\infty K_{n_i}$. Then, by (1), there exists a null sequence $V_1, V_2, \cdots$ of open sets covering H each of which is a subset of some G_i and is such that its boundary contains no point of K. But since the sets G_i are mutually exclusive, no set V_n can contain more than one of the sets K_{n_i}. But then clearly the fact that $\delta(K_{n_i}) > \epsilon$ for all i's contradicts the fact that $V_1, V_2, V_3, \cdots$ is a null sequence.

The sufficiency of the condition is an immediate consequence of the existence of the sets $N, N_1, N_2, N_3, \cdots$, as previously defined, in any continuum which is not hereditarily locally connected, because for each i, N_i is a component of $\sum_1^\infty N_n$ and $\delta(N_i) > \epsilon$, which contradicts our condition.

(5). *In order that the compact metric continuum M should be hereditarily locally connected it is necessary and sufficient that every connected subset of M should be locally connected.* (Theorem of R. L. Wilder.)*

The sufficiency of the condition is obvious. To prove the necessity, let us suppose, on the contrary, that some connected subset H of the hereditarily locally connected continuum M is not locally connected at one of its points p. Then there exists a neighborhood E of p and an infinite sequence $p_1, p_2, p_3, \cdots$ of distinct points of $H \cdot E$ such that $\rho(p_i, p) < (1/2)\rho[p, F(E)]$ for every i, and no two points of this sequence lie in the same component of $K = H \cdot \bar{E}$. For each i let C_i denote the component of K containing p_i. Since, by (4), the components of K form a null family, it follows that for some i, $C_i \cdot F(E) = 0$. Then, applying (2), we obtain a neighborhood U of p_i such that $E \cdot C_i \subset U \subset E$ and $F(U) \cdot K \subset F(E) \cdot C_i = 0$. Since $F(U) \subset \bar{E}$, therefore we have $0 = F(U) \cdot K = F(U) \cdot H \cdot \bar{E} = F(U) \cdot H$. But then H is the sum of the two mutually separated sets $H \cdot U$ and $H - H \cdot U$, contrary to the fact that H is connected.

Corollaries. *Let M be any compact, metric, and hereditarily locally connected continuum and let K be any subset of M. Then:*

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* See R. L. Wilder, *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), p. 616. This theorem and its proof are included in the present paper for the sake of completeness of the treatment. The proof given is the author's own.

(a) *If C is any component of K and R is any open set, there exists an open set U such that $R \cdot C \subset U \subset R$ and $F(U) \cdot K \subset F(R) \cdot C$.* [Consequence of (2) and (3)].

(b) *If K is of dimension > 0 at any one of its points p, then p lies in a non-degenerate connected subset of K.* [Consequence of (2) and (3)].

(c) *Either K is of dimension zero, or it contains non-degenerate connected sets but no continua, or it contains continua.* [Consequence of (b)].

Remark. Corollary (c) states that the subsets of any hereditarily locally connected continuum fall into three mutually exclusive classes as follows: (α) sets containing continua, (β) punctiform sets which contain non-degenerate connected sets, and (γ) zero-dimensional sets. Sets of all three classes are known * to exist in hereditarily locally connected continua, and it is now definitely established that there are no others. This classification is actually a restrictive one, because it tells us, for example, that sets such as the totally disconnected one-dimensional sets are not to be found among the subsets of hereditarily locally connected continua. The classification is all the more restrictive in view of the theorem of Wilder [see (5) above] that every connected subset of such a continuum is locally connected.

3. LEMMA 1. *In any separable metric space R there exists a countable set of points D such that if p and q are any two points whatever which can be separated by some countable set, then p and q can be separated by some subset of D.*†

Proof. There exists a countable set of points $Q = x_1 + x_2 + x_3 + \cdots$ which is dense in R. Order all possible pairs of points x_i, x_j of Q such that x_i and x_j can be separated by some countable set into a sequence $P_1, P_2, P_3, \cdots$. For each n there exists a countable set of points E, which we may suppose closed,‡ which separates the two points x_i and x_j in R. There exists a positive real number a such that E also separates in R the point sets $V_a(x_i)$ and

* See similar remarks by the author concerning the subsets of regular curves in *Monatshefte für Mathematik und Physik*, Vol. 38 (1931), and note references therein to examples by Knaster, Kuratowski, Sierpinski, and Mazurkiewicz.

† It is evident from the proof of this lemma that the same argument suffices to establish the following general theorem: *If S is any class of closed subsets of a separable metric space R, there exists a countable sub-class $[S_i]$ of S such that each pair of points which may be separated by some set of the class S may also be separated by some set of the class $[S_i]$.* This general proposition together with some of its consequences will be considered by the author in a later paper.

‡ See Tietze, *Mathematische Annalen*, Vol. 88, p. 310.

$V_a(x_j)$, where $V_r(x)$ denotes in general the set of all points of the space whose distances from the point x are less than the positive real number r. With the aid of the Dedekind Cut-Postulate it is seen that there exists a number a_n, $0 < a_n \leq (1/2)\rho(x_i, x_j)$, such that for every positive number $a < a_n$, but for no number $> a_n$, there exists some countable set which separates the point sets $V_a(x_i)$ and $V_a(x_j)$. For each n let D_n denote some countable set which separates the point sets $V_{b_n}(x_i)$ and $V_{b_n}(x_j)$, where $b_n = a_n - 1/n$. Let $D = \sum_1^\infty D_n$. Then D has the required properties.

For let p and q be any two points which can be separated in R by some countable set E. We may suppose E closed, and hence there exists a positive number u such that E also separates the sets $V_u(p)$ and $V_u(q)$. Since Q is dense in R, it follows that there exists an integer $n > 8/u$ such that the points x_i and x_j of the pair P_n satisfy the relations $\rho(x_i, p) < u/8$ and $\rho(x_j, q) < u/8$. Hence $V_{u/2}(x_i) \subset V_u(p)$ and $V_{u/2}(x_j) \subset V_u(q)$, and therefore $a_n \geq u/2$. Thus $b_n = a_n - 1/n > a_n - u/8 > u/2 - u/8 > u/4$, because $1/n < u/8$. Hence $V_{b_n}(x_i) \supset p$ and $V_{b_n}(x_j) \supset q$, and therefore the set D_n, which is a subset of D, separates p and q in R.

Definitions. Any connected open subset of a locally connected space N will be called a *region* in that space. A region R is said to *join* two point sets A and B provided R contains at least one point of A and at least one point of B. Two regions R_1 and R_2 will be said to be *strongly separated* provided that they have no points and no boundary points in common, i. e., $\bar{R}_1 \cdot \bar{R}_2 = 0$.

LEMMA 2. *If N is any connected and locally connected metric space which has no cut point and A and B are any two mutually exclusive non-degenerate subsets of N, then there exist two strongly separated regions in N joining A and B.*

Proof. We may suppose that A and B are closed, for obviously they contain closed and non-degenerate subsets. Let p denote some point of A. Since $N - p$ is connected, there exists* a region R_{ab} joining A and B and such that $p \cdot \bar{R}_{ab} = 0$. There exists a region R_{ax} containing p and such that $\bar{R}_{ax} \cdot \bar{R}_{ab} = 0$. Thus there exist points x of N such that two strongly sepa-

* See R. L. Wilder, *Bulletin of the American Mathematical Society*, Vol. 34 (1928), pp. 649-655. It is only necessary to cover $N - p$ with a set of regions no one of which contains p or has p on its boundary and then take R_{ab} equal to the sum of the elements of a finite simple chain of these regions joining some point a of A and some point b of B.

rated regions R_{ab} and R_{ax} exist joining A and B and A and x respectively. Let S denote the set of all such points x of N. We shall show that $S = N$. Suppose this is not so. Then since obviously S is open in N and N is connected, it follows that some point y of $N - S$ is a limit point of S. Since $N - y$ is connected, it follows just as above in the case of p that there exist two strongly separated regions G_{ab} and G such that G_{ab} joins A and B and G contains y. The region G contains a point x of S; and hence there exist two strongly separated regions R_{ab} and R_{ax} joining A and B and A and x respectively.

Now let R_1 and R_2 respectively denote components of $R_{ab} - \bar{G} \cdot R_{ab}$ and $R_{ax} - \bar{G} \cdot R_{ax}$ each of which contains at least one point of A. Inasmuch as N is locally connected and y does not belong to S, it follows that $\bar{G} \cdot \bar{R}_1 \neq 0 \neq \bar{G} \cdot \bar{R}_2$. Let K denote the set $A + \bar{R}_1 + \bar{R}_2$, and let U be a component of $G_{ab} - K \cdot G_{ab}$ which contains at least one point of B. At least one point f of K is a limit point of U, because N is locally connected. Let Z denote one of the sets R_1 and R_2 such that $\bar{Z}$ does not contain f and let T denote the other one of these sets. Let g denote some point of $\bar{G} \cdot \bar{Z}$. Since f cannot belong to $\bar{G}$, there exist strongly separated regions U_f and U_g containing f and g respectively and such that $\bar{U}_f \cdot (\bar{G} + \bar{Z}) = 0$ and $\bar{U}_g \cdot (\bar{T} + \bar{U}) = 0$. Clearly $U + U_f + T$ contains a region V joining A and B and $Z + U_g + G$ is a region V_{ay} joining A and y, and the regions V and V_{ay} have no point in common. But V contains a region V_{ab} joining A and B and such that $\bar{V}_{ab} \subset V$; for it is only necessary to cover V with regions $[V_p]$ each lying together with its boundary wholly in V, and then take V_{ab} equal to the sum of the elements of a finite simple chain of these regions $[V_p]$ joining some point a of A and some point b of B. This is impossible, because the regions V_{ab} and V_{ay} are strongly separated and join A and B and A and y respectively, contrary to the fact that y does not belong to S. Therefore $S = M$. Accordingly S contains a point x of B, and thus there exist two strongly separated regions R_{ab} and R_{ax} joining A and B. Q. E. D.

4. THEOREM. *Every hereditarily locally connected, compact and metric continuum is a rational curve.*

Proof. Suppose, on the contrary, that some continuum M exists satisfying our hypothesis but which is not rational. By Lemma 1 there exists a countable subset D of M such that if any two points p and q can be separated in M by some countable set, then p and q can be separated by some subset of D. Now there exists at least one non-degenerate component C of $M - D$. For if every point of $M - D$ is a component of $M - D$, then by § 2, results

(2) and (3) or Corollary (b), it follows that $M - D$ is zero-dimensional at every point and hence that M is rational at every point of $M - D$, for D is countable. But D being countable, it follows that M is rational at all of its points, contrary to hypothesis. Thus there exists a non-degenerate component C of $M - D$.

Now no two points of $\bar{C}$ can be separated in $\bar{C}$ by any countable set of points. For if, on the contrary, some two points p' and q' of $\bar{C}$ can be separated in $\bar{C}$ by some countable set E, it follows that some two points p and q of C can be separated in $\bar{C}$ by E; and hence there exists a neighborhood R of p such that $\bar{R}$ does not contain q and $F(R) \cdot C$ is countable. By § 2, result (2), there exists a neighborhood U of p such that $U \subset R$ and $F(U) \cdot (M - D) \subset F(R) \cdot C$, and hence such that $F(U) \cdot (M - D)$ is countable. But $F(U) \cdot M = F(U) \cdot (M - D) + F(U) \cdot D$, and hence $F(U) \cdot M$ is countable. Thus p and q are separated in M by the countable set $F(U) \cdot M$, and therefore they can be separated in M by some subset of D. But this is impossible, because C is connected and contains both p and q but contains no point of D. Consequently no two points of $\bar{C}$ can be separated in $\bar{C}$ by any countable set.

For convenience of notation we set $\bar{C} = N$. Then N is a hereditarily locally connected continuum no two points of which can be separated in N by any countable set of points. Now the local separating points * of any connected subset H of M which are not rational points of H must be countable. For if G is any uncountable set of local separating points of H, then since, by (5) in § 2, H is locally connected, it follows that there exists a region R in H and an uncountable subset E of G every point of which is a cut point of R. Now it is a consequence of a theorem of the author's † that there exists a point p of E and a countable subset D of R such that p is a component of $R - D$. Then by § 2, Corollary (b), $R - D$ is zero-dimensional at p, and therefore both R and H are rational at p. Thus every such set G contains a point in which H is rational, and accordingly the local separating points of H which are not rational points of H are countable. Now since N is not rational in any of its points, it follows that the set D_0 of all local separating points of $N = N_0$ is countable. Hence $N_1 = N - D_0$ is connected, because D_0 cannot separate any two points of N. Now N_1 cannot be rational at any one of its points, because a point of rationality of N_1 would be also a point of rationality of N, since D_0 is countable. Thus it follows that the set

* The point p of a connected and locally connected set H is a local separating point of H provided that p is a cut point of some region in H.

† See my paper "Non-Separated Cuttings of Connected Sets," *Transactions of the American Mathematical Society*, Vol. 33 (1931).

D_1 of all local separating points of N_1 is countable. Hence $N_2 = N_1 - D_1$ is connected, and so on. For any ordinal number α of the first or second class, let us suppose we have defined the sets N_β and D_β for all ordinal numbers $\beta < \alpha$. Then $\sum_{\beta<\alpha} D_\beta$ is countable, and therefore $N - \sum_{\beta<\alpha} D_\beta$ is connected. Set $N - \sum_{\beta<\alpha} D_\beta = N_\alpha$ and let D_α denote the set of all local separating points of N_α. It follows just as above that D_α is countable. Thus we have defined the sets N_α and D_α for all ordinal numbers α of the first and second classes.

We shall now show that for some α, $D_\alpha = 0$. Suppose, on the contrary, that $D_\alpha \neq 0$ for every α of the first and second class. Then for each α there exists a point p_α of D_α, and p_α is a local separating point of N_α but not of N_β for any $\beta < \alpha$. Since the set of points $[p_\alpha]$ is uncountable and since for each α there exists an $\epsilon_\alpha > 0$ such that p_α is a cut point of the component of $N_\alpha \cdot V_{\epsilon_\alpha}(p)$ containing p_α, it follows that there exists some $\epsilon > 0$, a point p, and an uncountable subset $[p_{a_\alpha}]$ of $[p_\alpha]$, where $a_\beta < a_\alpha$ if $\beta < \alpha$, such that for each a_α, $\rho(p, p_{a_\alpha}) < \epsilon/4$ and p_{a_α} is a cut point of the component C_α of $N_{a_\alpha} \cdot V_\epsilon(p)$ which contains p_{a_α}. Now inasmuch as $C_1 - p_{a_1}$ has at least two distinct components, there exists at least one component F_1 of $C_1 - p_{a_1}$ and an infinite subset E_1 of $[p_{a_\alpha}]$ such that $E_1 \cdot F_1 = 0$. Let $p_{a_{n_2}}$ be the first point in the sequence $[p_{a_\alpha}]$ following p_{a_1} which belongs to E_1. Then, just as before, there exists at least one component F_2 of $C_{n_2} - p_{a_{n_2}}$ and an infinite subset E_2 of E_1 such that $E_2 \cdot F_2 = 0$. Let $p_{a_{n_3}}$ be the first point in the sequence $[p_{a_\alpha}]$ following $p_{a_{n_2}}$ which belongs to E_2. There exists a component F_3 of $C_{n_3} - p_{a_{n_3}}$ and an infinite subset E_3 of E_2 such that $E_3 \cdot F_3 = 0$, and so on. Continuing this process indefinitely we obtain an infinite sequence of sets $F_1, F_2, F_3, \cdots$. Now since for each i, F_i is a component of $C_{n_i} - p_{a_{n_i}}$, since $\rho(p, p_{a_{n_i}}) < \epsilon/4$, and since C_{n_i} is connected and locally connected and clearly $\bar{C}_{n_i} \cdot F[V_\epsilon(p)] \neq 0$, it follows at once that $\delta(F_i) > \epsilon/2$ for all i's. Now for each $j > i$, we have $N_{a_{n_j}} \subset N_{a_{n_i}} - p_{a_{n_i}}$; and therefore $C_{n_j} \subset C_{n_i} - p_{a_{n_i}}$. Thus since $p_{a_{n_j}} \subset E_j$ and $E_j \cdot F_i = 0$, it follows that $C_{n_j} \cdot F_i = 0$; and as $F_j \subset C_{n_j}$, therefore $F_i \cdot F_j = 0$ for every pair of integers i and j. Since for each i it is true that for every $j > i$, $C_{n_j} \subset C_{n_i} - p_{a_{n_i}}$, and hence that $C_{n_j} \cdot F_i = 0$ as above, and since F_i is a component of $C_{n_i} - p_{a_{n_i}}$ and C_{n_i} is locally connected, it follows that no point of F_i is a limit point of $\sum_{n=i+1}^{\infty} F_n$; and therefore no point of F_i is a limit point of $F - F_i$, where $F = \sum_1^\infty F_n$. But then for each i, F_i is a component of F, which is impossible by virtue of result (4) in § 2, because the diameter of every set F_i is $> \epsilon/2$. Thus the

supposition that $D \neq 0$ for every α leads to a contradiction. Accordingly there exists an α of the first or second class such that $D_\alpha = 0$ and hence such that the set N_α, which is $= N - \sum_{\beta<\alpha} D_\beta$, has no local separating point. Since $N - N_\alpha$ $(= \sum_{\beta<\alpha} D_\beta)$ is countable, therefore N_α is connected.

The set N_α, then, is connected and locally connected, [by (5) in § 2], and has no local separating point. Let A and B be two mutually exclusive open subsets of N_α such that $\bar{A} \cdot \bar{B} = 0$ and hence such that $\rho(A, B) > 0$. Since N_α has no cut point, it follows by Lemma 2 in § 3 that there exist two strongly separated regions R_1 and S_1 in the space N_α each of which joins A and B. Since N_α has no local separating point, S_1 can have no cut point. Thus by Lemma 2 there exist two strongly separated regions R_2 and S_2 in S_1, (S_1 considered as a space), each of which joins A and B, because $A \cdot S_1$ and $B \cdot S_1$ are non-degenerate sets. Likewise S_2 can have no cut point, and hence by the same reasoning it follows that there exist two strongly separated regions R_3 and S_3 in the space S_2 each of which joins A and B, and so on. Continuing this process indefinitely, we obtain an infinite sequence of sets $R_1, R_2, R_3, \cdots$ each of which is a region in the space N_α which joins A and B and such that each pair of these regions are strongly separated. But then for each n, R_n is a component of $\sum_1^\infty R_i$, which is impossible in view of result (4) in § 2, because for each n, R_n joins A and B and hence $\delta(R_n) \geq \rho(A, B)$.

Thus the supposition that our theorem is false leads to a contradiction, and accordingly the theorem is established.

In conclusion it will be noted that our theorem is equally valid for metric continua which are locally compact as for compact metric continua. This is evident at once, because any locally connected and locally compact metric continuum M which is not a rational curve clearly contains compact continuum which is not a rational curve, namely, a closed and compact region in M containing any point p of M in which M is not rational. And if M is hereditarily locally connected, this is impossible by our theorem.

THE JOHNS HOPKINS UNIVERSITY.

GENERALIZATIONS OF BICONNECTED SETS.

By P. M. Swingle.

Introduction.

The problem of the infinite divisibility of space is one which has interested philosophers ever since Zeno stated his paradoxes while geometry was having its origin in ancient Greece. It is a problem which the greatest philosophers of the past centuries gave thought to.

Hume insisted that the mind refused actually to subdivide further after a few divisions.* It seemed to him that the mind was unable to know parts of space, such as present day points, which were obtained by subdivision of a bounded space into more than a finite number of parts. For of such the mind could not have an impression. Kant came a little nearer to present day theory in that he apparently admitted that the mind could continue to subdivide. However in the world of phenomena he did not admit an infinite subdivision as a completed process. But he granted that there might be an unknowable world of noumena in which infinite divisibility existed.†

Thus, in the light of the historical development of mathematics, all kinds of sets, which do not admit of a type of infinite divisibility, are of interest, even though in their definition the type of divisibility objected to is used. Biconnected sets are such sets.‡ For they are connected sets which cannot be subdivided into two distinct § connected subsets. Here infinite divisibility into distinct subsets may exist according to present day mathematics. But infinite divisibility into distinct connected subsets does not exist.

In this paper the various definitions of biconnected sets will be generalized as well as various known theorems concerning them. While these

* David Hume, "A Treatise on Human Nature."

† Immanuel Kant, "Critique of Pure Reason."

‡ For definitions, theorems, and examples of biconnected sets see B. Knaster and C. Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2, pp. 206-253. See also J. R. Kline, "A Theorem Concerning Connected Sets," *Fundamenta Mathematicae*, Vol. 3, pp. 238-239. For an interesting example see R. L. Wilder, "A Point Set Which Has no True Quasi-Components and Which Becomes Connected upon the Addition of a Single Point," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 423-427. For further theorems see also R. L. Wilder, "On the Dispersion Sets of Connected Point Sets," *Fundamenta Mathematicae*, Vol. 4, pp. 214-228. For an unsolved problem see C. Kuratowski, *Fundamenta Mathematicae*, Vol. 3, p. 322 (19).

§ In this paper two sets A and B are said to be *distinct* if $A \times B = 0$.

generalizations will give sets which admit of an infinite divisibility, the main results obtained are for sets which admit only a finite subdivision into distinct connected subsets.

Due to the few known types of biconnected sets and the difficulty of developing the theory of such sets, a number of unsolved problems are stated in this paper in the hope that they will suggest further interesting sets or development of theory.

The results obtained will hold for any space in which the sets exist, unless otherwise stated.

Two Equivalent Generalizations.

Definition. An *n-divisible connected set,* where n is a given cardinal number, is a connected * set which is the sum of n but not of a greater number of distinct connected subsets. Such a set will be said to be *n-divisible.*

Definition. An *n-containing connected set,* where n is a given cardinal number, is a connected set which contains n but not a greater number of distinct connected subsets. Such a set will be said to be *n-containing.*

Examples of *biconnected sets, which are both one-divisible and one-containing,* have been given for the euclidean plane by B. Knaster and C. Kuratowski in their paper "*Sur les ensembles connexes.*" † In the example α there given let b be the point $(0,0)$, c the point $(1,0)$, and a the point $(1/2, 1/2)$, where a is the point which totally disconnects the biconnected set. The *notation* (bac), with necessary subscripts, will be used in this paper to denote such a set, wherever a, b, and c may be in the plane. The set (bac) will be understood to contain b, c, and whatever other possible points are desired below.

If $(b_i a_i c_i)$ $(i=1,2)$ are distinct except that $c_1 = c_2$, then $(b_1 a_1 c_1) + (b_2 a_2 c_2)$ is an example of both a two-divisible and a two-containing connected set. A simple continuous arc would be an example of an n-divisible and n-containing connected set, where n is the power of a countable infinity. Since in a euclidean space the greatest number of points therein contained is the power of the linear continuum, it is seen that such a space does not contain an n-divisible or n-containing set, where n is greater than the power of the linear continuum. But if n is the power of the linear continuum, a euclidean

* A set M will be said to be *connected* if it contains at least two points and for every two distinct non-vacuous subsets of M, whose sum is M, at least one of these contains a limit point of the other. By this definition a point will not be considered connected.

† *Loc. cit.*

space of dimensions greater than one is itself both an n-divisible and an n-containing connected set.*

LEMMA 1. *If the connected set M contains n distinct connected subsets, (N), where n is a given cardinal number, then, for any positive integer w not greater than n, M is the sum of w distinct connected subsets, (C), such that each set of (C) contains at least one set of (N).*

Let $N_1, N_2, \cdots, N_w$ be w of the sets of (N). Let C_1 be the maximal connected subset of $M-(N_2+N_3+\cdots+N_w)=Z$ which contains N_1. Let $M-C_1=M_1+M_2+\cdots+M_k$ separate.† It is necessary that k be less than w. For if not there exists an M_i, M_h say, which does not contain a point of $M-Z$, and so, as M_h+C_1 is connected, C_1 is not a maximal connected subset of Z. It is necessary then that each of the sets M_i $(i=1,2,\cdots,k)$, where k has its maximum value, be connected and contain a point of an N_g $(g=2,\cdots,w)$. Let C_1 and each M_i, which contains one and only one N_g, be each a set of (C). The remaining sets M_i can now be treated as M was above. Thus the sets of (C) are obtained.

The truth of the following corollary is now evident, giving a type of an "any to finite" property.

COROLLARY 1. *If the connected set M contains infinitely many distinct connected subsets, then M is the sum of w distinct connected subsets, where w is any positive integer.*

That lemma 1 does not hold if both n and w are the power of a countable infinity is seen from the following example. Let $(b_ia_ic_i)$ $(i=1,2,\cdots)$ be a countable infinity of biconnected sets, which are distinct except that $(b_ia_ic_i)\times(b_{i-1}a_{i-1}c_{i-1})=c_i=b_{i-1}$ for every i; let these biconnected sets have the further property that they have a simple continuous arc t as sequential limiting set, which has nothing common with any $(b_ia_ic_i)$. Let q be any point of t. The set $(b_1a_1c_1)+(b_2a_2c_2)+\cdots+q$ will be called *a set* (bq) in this paper. The set (bq) is an n-containing connected set, where n is a countable infinity, which contains the n distinct biconnected sets $(b_ia_ic_i)-b_i$. However (bq) is not n-divisible as there do not exist n distinct connected subsets of M, one of which contains q, of which M is the sum. This set is an example of a set defined as follows.

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* For a space containing more elements than the power of the linear continuum see F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914, p. 68. For other spaces see also pp. 284-290.

† By the notation $M_1+M_2+\cdots+M_k$ *separate* is meant that the sets M_i $(i=1,2,\cdots,k)$ are distinct, non-vacuous, sets, no one of which contains a limit point of the sum of the remaining ones.

Definition. A connected set M is said to be *finitely-divisible* if it is neither the sum of a maximum finite number nor of an infinite number of distinct connected subsets.

Problem 1. It is not determined in this paper if the first part of lemma 1 holds if n is the power of the linear continuum and if w is either the power of a countable infinity or $w = n$.

THEOREM 1. *In order that the connected set M be n-divisible, where n is a positive integer, it is necessary and sufficient that M be n-containing.**

The condition is necessary. For as M is n-divisible it contains n distinct connected subsets. Hence, if it is not n-containing, it must contain more than n, and so $n+1$, such subsets. Thus by lemma 1 it cannot be n-divisible.

The condition is sufficient. For, if M is n-containing, it is by lemma 1 the sum of n distinct connected subsets. And, if it is the sum of more than n such subsets, it is not n-containing.

COROLLARY 2. *If M is an n-divisible connected set lying in a locally compact metric space, where n is any positive integer, then M is punctiform.*†

For if M is not punctiform it contains a subcontinuum W. And in a locally compact metric space any vicinity which contains a point of W contains a subcontinuum of W. Therefore W must contain $n+1$ distinct connected subsets and so, by theorem 1, M is not n-divisible.

THEOREM 2. *If M is an n-divisible connected set, where n is any positive integer, then M is the sum of n distinct biconnected subsets.*

As M is n-divisible it is the sum of n distinct connected subsets, no one of which can be the sum of two distinct connected subsets, as M cannot be the sum of $n+1$ distinct connected subsets. Hence each of the distinct connected subsets, of which M is the sum, must be biconnected.

It is of interest to note that an n-divisible connected set may be the sum of less than n distinct biconnected subsets. For consider the three-divisible connected set $(b_1a_1c_1)+(b_2a_2c_2)+(b_3a_3c_3)$, where $b_1c_1=b_2c_2$, $(b_2a_2c_2)$ is obtained by rotating $(b_1a_1c_1)$ 180° about b_1c_1, $a_3c_3=b_2a_2$, and otherwise the three biconnected sets are distinct. Then this three-divisible connected set is the sum of the two distinct biconnected subsets $(b_3a_3c_3)$ and $[(b_1a_1c_1)+(b_2a_2c_2)]-b_2a_2$.

* For a proof of this theorem for $n=1$ see B. Knaster and C. Kuratowski, *loc. cit.*, p. 215, theorem 11.

† For a proof of this corollary for $n=1$ see B. Knaster and C. Kuratowski, *loc. cit.*, p. 216, theorem 14.

THEOREM 3. *If M is an n-divisible connected set, where n is any positive integer, then M contains at most $2n - 1$ points each of which disconnects M.**

As M is n-divisible it is the sum of n distinct biconnected subsets by Theorem 2. Let C_i $(i = 1, 2, \cdots, n)$ be these n sets. Hence each point q that disconnects M is contained in one and only one set C_i. And either q disconnects C_i or it does not. Of those points which disconnect both M and C_i there can be at most n, since there exists at most one point which disconnects a biconnected set.

Consider now the set K, each point of which disconnects M but does not disconnect the C_i which contains it. Consider all points of K that are limit points of C_1. They are of three classes, which we now proceed to consider.

Let p_1 be a point of K in C_1. Then $M - p_1 = M_1 + M_2$ separate, where M_1 contains the connected set $C_1 - p_1$ say. Now any set C_i which has a point in M_2 lies wholly in M_2. Let the class of those sets C_i that lie in M_2 and have p_1 as a limit point be denoted by G_1. No set C_i in G_1 has any other limit point then p_1 in C_1, nor does it contain a limit point of C_1, as M_1 and M_2 separate. Let p_2 be another point of K in C_1. Then $M - p_2 = H_1 + H_2$ separate, where $C_1 - p_2$ lies in H_1 say. As before denote by G_2 the class of those sets C_i in H_2 that have p_2 as a limit point. Since the observation made above concerning the sets of G_1 apply also to the sets of G_2, no set of G_1 lies also in G_2. As the number of sets C_i is finite, we proceed, in this manner, to a finite set of points $p_1, p_2, \cdots, p_{k_1}$ with which are associated, respectively, classes $G_1, G_2, \cdots G_{k_1}$ of sets C_i.

Let x_1 be a point of K in some C_h that is a limit point of C_1 and separates C_1 from C_h in M. Obviously C_h is not in any G_i, and C_h has no other limit point of C_1, nor does it have a limit point in C_1. Let F_1 be the set of all the sets C_i that have x_1 as a limit point, and are separated from C_1 by it. Proceed in this manner to get sets $F_2, F_3, \cdots, F_{k_2}$ corresponding to points $x_2, x_3, \cdots, x_{k_2}$, respectively. As before we see that the classes $F_1, F_2, \cdots, F_{k_2}$ are distinct, and are distinct from the G's. As every point of K that lies in C_1 is a point of the set $p_1, \cdots, p_{k_1}$, and every point that is a point of K and a limit point of C_1 and separates C_1 from the C_i containing it is a point of the set $x_1, \cdots, x_{k_2}$, any other point of K must be a point having neither of these properties. There may still be for instance a point that is a limit point of C_1 and in a C_j, but does not separate C_1 and C_j. Such points we consider next.

* For the case where $n = 1$ see J. R. Kline, *loc. cit.*; also see C. Kuratowski, *loc. cit.*

Let y_1 be such a point. It separates C_j from some C_k. Then with y_1 associate all sets C_i such as C_k; i. e., sets which have y_1 as a limit point and are separated from C_j by y_1; denote the class of such sets by T_1. No set of T_1 is in an F or a G. That it is not in a G is easy to see. If it were in an F, it would be separated from C_1 by a point x_m, which lies in a $C_n \neq C_j$ and hence is distinct from y_1. This is impossible.

Thus with every point of K that is a limit point of C_1 we associate at least one set C_i and in such a way that there is no overlapping between the sets associated with different points. Let Z_1 denote the point set consisting of all points in C_1, and all points in G's, F's, and T's, together with the C's which have limit points in these sets which are not contained in K.

To complete the proof we note that no point of $Z_1 + C_1$ that is in K separates $Z_1 + C_1$, and we can proceed from this set as we proceeded from C_1. In this manner we associate with each point of K at least one set $C_i \neq C_1$ and in such a way that overlapping is avoided. Consequently the number of points in K is less than or equal to $n - 1$.

Hence there exist at most $n + (n - 1) = 2n - 1$ points each of which disconnects M.

THEOREM 4. *If M is an n-divisible connected set, where n is a positive integer, then no connected subset C of M is irreducible connected about any subset N of C such that $C - N$ contains more than $2n - 1$ points.**

By theorem 1 there exists an integer q, not greater than n, such that C is q-containing. Let N be any set about which C is irreducibly connected. Then any point p of $C - N$ disconnects C otherwise $C - p$ is a proper connected subset of C containing N. Hence $C - N$ can contain at most $2n - 1$ points by theorem 3.

ANOTHER GENERALIZATION.

Definition. Let $1, 2, \cdots; \omega, \omega + 1, \omega + 2, \cdots; 2\omega, 2\omega + 1, \cdots; \cdots; \cdots, k$ be the set of the first k ordinal numbers and let $W_1, W_2 \cdots W_k$ be a set of k connected subsets of the connected set W. Then $W_1, W_2, \cdots, W_k$ is called a *k-convergable sequence of W* when $W_1 = W$, W_g contains W_i, where i runs over all ordinal numbers not greater than k and g over all less than i, and $W_g - W_{g+1}$ is a connected subset of W, but for every connected subset C of W_k, $W_k - C$ is either a point, a vacuous, or a totally disconnected point set.

* For the case where $n = 1$ see B. Knaster and C. Kuratowski, *loc. cit.*, p. 225. theorem 29.

Definition. A connected set W is an *n-convergable connected set,* where n is any cardinal number, if there exists an ordinal number k, the cardinal number of which is n, such that W contains a k-convergable sequence, but there does not exist an ordinal number q, the cardinal number of which is greater than n, such that W contains a q-convergable sequence. The set W will be called also *n-convergable.*

A biconnected set is an example of a one-convergable set.*

Definition. A connected set W is a *finitely-convergable connected set* or is *finitely-convergable* if it contains, for every positive integer N, an n-convergable sequence such that n is an integer $\geqq N$, but it does not contain a q-convergable sequence, where q is an infinite cardinal number.

The example (bq), given above, of a finitely-divisible connected set is also an example of a finitely-convergable set. However $(bq) - q$ is an example of a finitely-convergable but not of a finitely-divisible connected set.

The example $(bac) + bd$, where bd is an arc distinct from (bac) except for the point b, is an example of an n-convergable set, where n is the power of a countable infinity.

In the euclidean plane S let (t) be the set of line segments from $(i, 0)$ to $(i, 1)$, where i takes on all real values from zero to one. Let the sets of (t) be well ordered by Zermelo's postulate, obtaining the set (t_i). And let (bac) be a biconnected set which has no point common with any of the sets of (t). Let T be the set of points of S contained in neither (bac) nor in a set of (t) and let $(t)_j$ be the points of S contained in the first j sets of (t_i). Then there exists an ordinal number $k+1$, whose cardinal number n is the power of the linear continuum, such that $W_1 = S$, $W_2 = S - (t)_1$, $W_3 = S - (t)_2, \cdots$; $\cdots; \cdots, W_{j+1} = S - (t)_j \cdots; \cdots, W_k = T + (bac)$, and $W_{k+1} = (bac)$. As all the W_i's except W_{k+1} contain T it is seen that they are connected. And $W_i - W_{i+1} = t_i$ is connected as is also $W_k - W_{k+1} = T$. Hence $W_1, W_2, \cdots$, W_k, W_{k+1} is a $(k+1)$-convergable sequence of S. And as the power of the set of distinct points of S is the power c of the linear continuum, there does not exist an ordinal number q, whose power is greater than c, such that S contains a q-convergable sequence. Hence S, and similarly any euclidean space of dimension > 1, is a c-convergable connected set, where c is the power of the linear continum.

THEOREM 5. *If W is an n-convergable connected set, where n is a positive integer, then W is the sum of n distinct biconnected subsets.*

As W is n-convergable it contains an n-convergable sequence $W_1 = W$,

* B. Knaster and C. Kuratowski, *loc. cit.*, p. 215, theorem 11.

$W_2, \cdots, W_n$, where it is known that W_n is biconnected. Let $W_i - W_{i+1} = V_{n-i+1}$ $(i = 1, 2, \cdots, n-1)$ and let $W_n = V_1$. Let j be such that $W_{n-j+1} = V_1 + V_2 + \cdots + V_j$, where each of the sets $V_1, V_2, \cdots, V_j$ is a biconnected set. Assume that V_{j+1} is not biconnected and so is the sum of the two distinct connected subsets X and Y. Then as $W_{n-j} = W_{n-j+1} + V_{j+1} = W_{n-j+1} + (X + Y)$, either $X + W_{n-j+1}$ or $Y + W_{n-j+1}$ is connected. Take for example the case where $Z_2 = Y + W_{n-j+1}$ is connected. Let $Z_1 = W_{n-j}$ and $Z_g = W_{n-j+g-2}$ $(g = 3, 4, \cdots, j+2)$. Hence $Z_1, Z_2, \cdots, Z_{j+2}$ is a $(j+2)$-convergable sequence of the $(j+1)$-convergable connected set W_{n-j} which is a contradiction. It is then necessary that W be the sum of the n distinct biconnected sets $V_1, V_2, \cdots, V_n$.

The example given of an n-convergable set, where n is the power of a countable infinity shows that theorem 5 is untrue for such a set. And the following example shows that it is untrue for the case where n is the power of the linear continuum.

Example A. In the euclidean plane consider the straight line interval g from $(0,0)$ to $(1,0)$ and the set of line segments (t) from $(i,0)$ to $(i,1)$ where i takes on the irrational values from zero to one. Let $T = g + (t)$ and let (bac) have but the point b common with T where g contains b. It is seen that $T + (bac)$ is a c-convergable connected set, where c is the power of the linear continuum. It is seen further that T does not contain a biconnected set V, for V would have to contain more than one point of g and so there would exist more than one point which disconnects the biconnected set V, which is impossible. Hence it follows that $T + (bac)$ cannot be the sum of c distinct biconnected subsets.

Problem 2. Does there exist a cardinal number q such that a euclidean space is the sum of q distinct biconnected subsets? Such a space is the sum of c distinct indivisible subsets, that is points, where c is the power of the linear continuum.

LEMMA 2. *If the connected set W is the sum of the k, k a positive integer, distinct connected subsets $C_1, C_2, \cdots, C_k$, then there exist k connected subsets $W_1 = W, W_2, \cdots, W_k = C_k$, where W_i contains W_{i+1} $(i = 1, 2, \cdots, k-1)$ and $W_i - W_{i+1}$ is connected.*

As W is connected, either $W_k = C_k$ has a limit point in one of the sets $C_1, C_2, \cdots, C_{k-1}$ or one of these sets has a limit point in W_k. Hence let one of these sets, which has either of these properties, together with the set W_k form the set W_{k-1}. And now either W_{k-1} has a limit point in one of the

remaining sets of $C_1, C_2, \cdots, C_{k-1}$ or one of these sets has a limit point in W_{k-1}. Let one of these sets, having either property, together with W_{k-1} form the set W_{k-2}. Proceeding in this manner the truth of the lemma is seen.

THEOREM 6. *If W is an n-divisible connected set, where n is a positive integer, then W is an n-convergable connected set.*

As W is n-divisible, by theorem 2 it is the sum of n distinct biconnected subsets $C_1, C_2, \cdots, C_n$. Hence by lemma 2 it is seen that W contains an n-convergable sequence $W_1 = W, W_2, \cdots, W_n = C_n$, as C_n is biconnected. And as an n-divisible connected set contains at most n distinct connected sets, W does not contain a q-convergable sequence, where q is greater than n. Thus W is n-convergable.

As a simple continuous arc is n-divisible, where n is the power of a countable infinity, but is not n-convergable, it is seen that theorem 6 does not hold for such an n. And in Example A the set T is c-divisible but not c-convergable, where c is the power of the linear continuum.

Problem 3. Does there exist a finitely-divisible connected set which is not finitely-convergable? An example has been given above of a finitely-convergable set which is not finitely-divisible.

A theorem will be proven now for finitely-convergable connected sets which corresponds to theorem 5 for n-convergable sets, n finite.

THEOREM 7. *Let W be a finitely-convergable set. Then either W contains an infinite sequence of distinct biconnected sets $\cdots, M_3, M_2, M_1$ or for every integer k there exists an integer q greater than or equal to k such that W is the sum of the q distinct biconnected sets $M_q, M_{q-1}, \cdots, M_2, M_1$. The set $M_1 + M_2 + \cdots + M_t = H_t$ $(t = 1, 2, \cdots)$ may be taken connected and $W - H_t$ the sum of a finite number of maximal connected subsets such that, if C is one of these, then $H_t + C$ also is either the sum of a finite number or contains an infinite number of biconnected sets. And the set $W - H_\alpha$ does not contain a biconnected set which for every finite t, is contained in the maximal subset of $W - H_t$ which contains M_{t+1}.*

For any integer k there exists an integer n, greater than k, such that W contains an n-convergable sequence $W_1, W_2, \cdots, W_n$. Then, as in the proof of theorem 5, we obtain the sets $V_1, V_2, \cdots, V_n$, of which the first is known to be biconnected, and if it is assumed that the first j of these sets is biconnected but that V_{j+1} is not, one obtains the $(j+2)$-convergable sequence of W_{n-j} obtained in theorem 5. Let this sequence be $Z_1 = W_{n-j}$, $G_1 = W_{n-j+1} + Y$, $Z_2 = W_{n-j+1}$, $Z_3 = W_{n-j+2}, \cdots, Z_{j+1} = W_n$. If now G_1

$-Z_2$ is not a biconnected set, it can be treated as V_{j+1} was above, obtaining a $(j+3)$-convergable sequence $Z_1, G_1, G_2, Z_2, Z_3, \cdots, Z_{j+1}$. Proceeding in this manner it is seen that either there exists an integer h such that there exists a $(j+h+1)$-convergable sequence $Z_1, G_1, G_2, \cdots, G_h, Z_2, Z_3, \cdots, Z_{j+1}$, such that $G_h - Z_2$ is biconnected, where h may equal one, or one obtains the m-convergable sequence, where m is the power of a countable infinity, $W_1, W_2, \cdots, W_{n-j} = Z_1, G_1, G_2, \cdots; V_j$. As the latter is contrary to the fact that W is finitely-convergable, there exists the $(j+h+1)$-convergable sequence of which $G_h - Z_2 = M_{j+1}$ is biconnected. Hence by mathematical induction it follows that either W must be the sum of q distinct biconnected subsets, where q is greater than or equal to k, or W contains the infinite sequence of distinct biconnected subsets $\cdots, M_3 = V_3, M_2 = V_2, M_1 = V_1$. From the derivation it is seen that H_t, t finite, is connected and $W - H_t$ is always the sum of a finite number of maximal connected subsets, of which, if C is one, $H_t + C$ is either the sum of a finite number or contains an infinite number of distinct biconnected subsets. And $W - H_\infty$ does not contain a biconnected subset B which for every finite t, is contained in the maximal connected subset of $W - H_t$ which contains M_{t+1}, for $W_1, W_2 = W - H_1, \cdots; B$ or a similar sequence, is an m-convergable sequence, where m is the power of a countable infinity, which is impossible.

This theorem suggests the existence of the following interesting and more complicated example of a finitely-convergable set. Let (bq) be a set similar to the example of a finitely-convergable set given above. Let (b_iq_i) $(i = 1, 2, \cdots, 6)$ be six such sets distinct except that $(b_1q_1) \times (b_2q_2)$, $(b_2q_2) \times (b_3q_3)$, $(b_2q_2) \times (b_4q_4)$, $(b_4q_4) \times (b_5q_5)$, and $(b_1q_1) \times (b_6q_6)$ say each contain one and only one point, which is neither a q_i nor is it a point which totally disconnects one of the biconnected subsets. Then $(b_1q_1) + (b_2q_2) + \cdots + (b_6q_6)$ is a finitely-convergable set. The theorem further suggests the following problem.

Problem 4. Is a finitely-convergable connected set ever the sum of a finite number of distinct biconnected subsets?

The following example of an n-convergable set, where n is the power of a countable infinity, is of interest in this connection.

Example B. Let $(baca')$ be the biconnected set (bac) together with the biconnected set $(ba'c)$ obtained by rotating (bac) 180° about bc. Let $(b_ia_ic_ia_i')$ $(i;\ 1, 2, \cdots)$ be an infinite number of such sets, whose sum is bounded, which are distinct except that $a_1 = a_2 = a_3 = \cdots$, and let (bac) be a biconnected set such that bc of (bac) contains $a_1' + a_2' + \cdots$, but (bac) contains nothing else common with the sets $(b_ia_ic_ia_i')$. Then $W = (b_1a_1c_1a_1') + (b_2a_2c_2a_2') + \cdots + (bac)$ is both an n-convergable and an

n-divisible connected set, where n is the power of a countable infinity. It has the interesting property that it is the sum of q distinct biconnected subsets, where q is either any integer $\geqq 2$ or is a countable infinity.

Example C. In Example B let (bac) have the further property that $a = a_1$. The resulting set W is still both n-divisible and n-convergable and is the sum of a countable infinity of distinct biconnected subsets. However it is no longer the sum of a finite number of distinct biconnected subsets.

These examples suggest the following problems.

Problem 5. If for every integer n, greater than one, the connected set M is the sum of n distinct biconnected subsets, is M then the sum of infinitely many such distinct subsets? In previous theorems conditions have been given which cause a connected set to be the sum of a finite number of distinct biconnected subsets. Here it is asked what conditions cause a connected set to be the sum of infinitely many such subsets.

Problem 6. If the connected set M is the sum of a finite number of distinct biconnected subsets but is not the sum of an infinite number of distinct connected subsets, does there exist a finite n such that M is n-divisible? This problem is of interest in connection with theorem 2.

Problem 7. If the connected set M does not contain a maximum finite number of distinct connected subsets, must it contain an infinite number of such subsets, i. e., does there exist a *finitely-containing connected set?*

n-Convergable Sets.

A number of other theorems will now be proven concerning n-convergable sets, where n is a positive integer. In theorem 6 it was shown that if a connected set is n-divisible it is n-convergable. Thus we have the following problem.

Problem 8. Does there exist an n-convergable connected set, where n is a positive integer, which is not n-divisible?

The following theorems will be of interest in connection with this and other problems.

Theorem A. *Let W be an n-convergable connected set, where n is a positive integer, which is not also an n-divisible set, if such a set exists. Let C_i $(i = 1, 2, \cdots, n)$ be the n distinct biconnected subsets of which W is the sum and let N_j $(j = 1, 2, \cdots, n + k)$ be a finite number, greater than n, of distinct connected subsets of which W is the sum. Let C be any C_i and N be any N_j. Then* (1) *N is not biconnected;* (2) *if M is any connected*

subset of W which is the sum of more than n distinct connected subsets, as N is, then $C \times M \neq 0$; (3) $C - N \times C$ is totally disconnected; and (4) $N - C \times N$ is totally disconnected if it contains more than one point.

It follows at once that (1) is true, since by lemma 2 W would contain an $(n+k)$-convergable sequence otherwise.

Assume that $C \times M$, in (2), is vacuous. As $W - C$ is the sum of a finite number of maximal connected subsets, one of these, Z say, contains M. Hence by lemma 1 Z is the sum of more than n distinct connected subsets. Thus it follows that W is also, and one of these distinct connected subsets is the biconnected set C, which is a contradiction according to (1). It also follows from (1) that N is a set such as M.

Assume that $C - N \times C = (N + C) - N$ contains the maximal connected subset K, which must be biconnected as C contains it. Then $(N + C) - K$ is connected and contains N, and, as by (2) N contains a point of every C_i, $W - K$ must be connected. Since $W - K$ contains N, by lemma 1 it is the sum of more than n distinct connected subsets, and so a contradiction with (1) is obtained as K is biconnected.

Assume that $N - C \times N = (N + C) - C$ contains the maximal connected subset X. Then by (1) X cannot be the sum of a finite number of distinct connected subsets, one of which is biconnected; for $W - X$, which contains C and the connected subset $(N + C) - X$, is connected, since by (2) C contains a point of every N_i; and $W - X$ is the sum of more than n distinct connected subsets by lemma 1, since it contains an N_i. Hence X must be the sum of more than n distinct connected subsets. But this is impossible by (2) as $X \times C = 0$.

LEMMA A. *If W is an n-convergable connected set, where n is a positive integer, but W is not n-divisible, then there does not exist a finite subset which disconnects W.*

Assume that the set Q, which contains q points, disconnects W. Let g be an integer greater than both n and q. Hence as W is not n-divisible it is necessary by theorem A (1) that W be the sum of g distinct connected subsets N_i $(i = 1, 2, \cdots, g)$. Thus one of the N_i's, N say, does not contain a point of Q. Let $W - Q = M_1 + M_2$ separate, where M_1 contains N. As by theorem A, N contains a point of each biconnected set C_j $(j = 1, 2, \cdots, n)$, of which W is the sum, M_1 does also. Let C be a C_j which contains also points of M_2. Hence $C - C \times Q = K_1 + K_2$ separate, where M_1 contains K_1 and M_2 contains K_2. As $C - N \times C$, which is totally disconnected by theorem A, contains $K_2 + Q \times C$, $K_2 + Q \times C = Z_1 + Z_2 + \cdots + Z_g$ sepa-

rate, where $Z_g \times Q = 0$. Hence $C = (K_1 + Z_1 + Z_2 + \cdots + Z_{g-1}) + Z_g$ separate, which is a contradiction. Thus no finite subset disconnects W.

THEOREM 8. *If W is an n-convergable connected set, where n is a positive integer, then W contains at most $2n - 1$ points each of which disconnect W.*

If W is not n-divisible the truth of the theorem follows from lemma A. And if it is n-divisible it follows from theorem 3.

The truth of the following corollary is now evident.

COROLLARY 3. *If W is an n-convergable connected set, where n is a positive integer, then W is not irreducibly connected about any set N such that $W - N$ contains more than $2n - 1$ points.*

In the previous theorems on n-convergable connected sets, where n is a positive integer, the full power of the definition of such sets has not been used. Only properties have been made use of which are given by the following definition.

Definition. A connected set W will be said to be *n-convergable on a finite range,* where n is a positive integer, if W contains an n-convergable sequence but does not contain an $(n+1)$-convergable sequence.

The theorems proved this far for n-convergable connected sets hold for all sets n-convergable on a finite range. It is evident that an n-convergable set is n-convergable on a finite range. But we have the following problem, since a set n-convergable on a finite range might contain a q-convergable sequence, where the power of q is a transfinite cardinal number.

Problem 9. Does there exist a set n-convergable on a finite range which is not n-convergable?

Two theorems will now be proven which use in their proof more than is given apparently in the definition of sets n-convergable on a finite range.

THEOREM 9. *If B is a biconnected subset of an n-convergable connected set W, where n is a positive integer, then W is the sum of a finite number, less than or equal to n, of distinct biconnected subsets, a number of which form a connected set Z containing B, such that $Z - B$ is either vacuous, a point, or a totally disconnected set.*

By theorem 5 it is seen that the theorem is true unless $W - B$ contains a maximal connected subset.

Assume then that $W - B$ contains the maximal connected subset T. Then W is the sum of the two distinct connected subsets T and $W - T$, the latter of which contains B. Consider for example the case where T is not the sum of a finite number of distinct connected subsets one of which is bi-

connected. Then T is the sum of two distinct connected subsets U_1 and V_1. Consider for example the case where $U_1 + (W - T)$ is connected, since if it is not, $V_1 + (W - T)$ is. Also U_1 is the sum of two distinct connected subsets U_2 and V_2, where U_2 is such say that $U_2 + (W - T)$ is connected. Proceeding in this manner one obtains the k-convergable sequence, where k is a countable infinity, $W_1 = W$, $W_2 = U_1 + (W - T)$, $W_3 = U_2 + (W - T)$, $\cdots$; B. As this is a contradiction it follows that T, and likewise the set composed of all maximal connected subsets of $W - B$, must be the sum of a finite number, less than n, of distinct biconnected subsets. Let (B) represent the sum of these distinct biconnected subsets of $W - B$. Hence $W - (B) = Z$ is connected and so is composed of a finite number of distinct biconnected subsets. And it is evident now that $Z - B$ is either vacuous, a point, or a totally disconnected point set.

THEOREM 10. *Let W be an n-convergable connected set, where n is a positive integer. Let B be a biconnected and C a connected subset of W. Then* (1) *no connected subset of W contains more than n distinct connected subsets where one of them contains a biconnected subset;* (2) *either C is the sum of not more than $n - 1$ distinct connected subsets or $C \times B \neq 0$;* (3) *either C and $C + B$ are the sum of not more than n distinct biconnected subsets or $B - B \times C$ is vacuous or totally disconnected;* (4) *either $(C + B)$ is the sum of not more than n distinct connected subsets or $C - B \times C$ is totally disconnected; and* (5) *if C is not the sum of distinct biconnected subsets then $C - C \times B$ contains at most one biconnected subset F and $C - F$ is totally disconnected.*

Assume that W contains the connected subset K which is the sum of more than n distinct connected subsets one of which contains the biconnected subset E. Then by lemma 1, W is the sum of more than n distinct connected subsets, $U_1, U_2, \cdots, U_g$, where U_g contains E. Hence there exists, by lemma 2, the sequence of connected sets $W_1 = W$, W_2, $\cdots$, $W_g = U_g$, where $W_i - W_i + 1$ $(i = 1, 2, \cdots, g - 1)$ is connected, but by (1) of theorem A it is not biconnected. Thus the k-convergable sequence $W_1, W_2, \cdots, W_{g-1}$, $Z_1, Z_2, \cdots$; E is obtained, where k is a countable infinity, $W_{g-1} - W_g$ contains $Z_j - Z_{j+1}$ $(j = 1, 2, \cdots)$ and Z_j contains W_g. As this is a contradiction, (1) is true.

Assume that C is the sum of at least n distinct connected subsets and that $C \times B = 0$. Then by lemma 1, W is itself the sum of more than n distinct connected subsets, one of which contains the biconnected set B. As this is contrary to (1) it is seen that (2) is true. Also it is seen that if $C \times B = 0$, $C + B$ is the sum of not more than n distinct connected subsets.

Assume that $B - B \times C = (B + C) - C$ contains the maximal connected, and so biconnected, subset E. Then, as $E \times C = 0$, by (2) C is the sum of not more than $n - 1$ distinct connected subsets. And as $(B + C) - E$ is connected, and has nothing common with the biconnected set E, by (2) it must be the sum of not more than $n - 1$ distinct connected subsets. Thus by (1) $B + C$ is the sum of not more than n distinct biconnected subsets. Hence (3) is true.

Assume that $C - B \times C = (B + C) - B$ contains the maximal connected subset F. Then, as $F \times B = 0$, by (2) F is the sum of not more than $n - 1$ distinct connected subsets and so of less than n distinct biconnected subsets. Similarly $(B + C) - F$ is the sum of distinct biconnected subsets and so $B + C$ is the sum of not more than n distinct biconnected subsets by (1). Hence (4) is true.

Assume that C is not the sum of distinct biconnected subsets and that $C - B \times C$ contains the biconnected subset F. Then by (4), as $C + F = C$, $C - F$ is totally disconnected. Hence (5) is true.

Problem 10. Does an n-convergable connected set, where n is a positive integer, contain a connected subset which contains no biconnected subset?

Other Theorems.

Theorem 11. *Any connected set M, in a locally compact metric space, which contains both a subcontinuum C and a biconnected subset B which is locally connected at a point q, contains also a k-convergable sequence, where k is a countable infinity.*

There exists a region R_1 containing q, and so containing a biconnected subset E, and a region R_2 containing a point of C such that $R_1' \times R_2' = 0$ and $R_2' \times C$ contains a subcontinuum K. There exists a proper subcontinuum K_1 of K such that $K - K_1$ contains a subcontinuum. Let T_1 be a maximal connected subset of $M - K_1$ which contains a subcontinuum of $K - K_1$. Then $M - T_1$ is connected and contains K_1. Hence M is the sum of the two distinct connected subsets T_1 and $M - T_1$ one of which, T_1 say, contains E and a subcontinuum C_1. Proceeding as above it can be shown that T_1 is the sum of two distinct connected subsets T_2 and $T_1 - T_2$, one of which, T_2 say, contains E and a subcontinuum C_2. Thus it can be shown that M contains the k-convergable sequence $W_1 = M$, $W_2 = T_1$, $W_3 = T_2, \cdots ; E$.

Corollary 4. *If a finitely-convergable set M, in a locally compact metric space, contains a biconnected subset which is locally connected at a point, then M is punctiform.*

Problem 11. Is a finitely-convergable or finitely-divisible set, in a locally compact metric space, always punctiform?

LEMMA 3. *Any connected set which is the sum of a finite number of distinct connected subsets, one of which is biconnected, is not irreducibly connected about a finite point set.*

Assume that the connected set M is irreducibly connected about the finite subset H. But as a biconnected set is disconnected by at most one point there exists in the biconnected subset of M a point q, not contained in H, such that $M - q$ is connected. Hence the lemma is true.

COROLLARY 5. *A finitely-convergable set M is not irreducibly connected about a finite subset.*

This follows at once from lemma 3.

LEMMA 4. *Any connected set M is either the sum of a finite number of distinct connected subsets, one of which is biconnected, or it contains a connected subset which is the sum of a countable infinity of distinct connected subsets.*

If M is not the sum of a finite number of distinct connected subsets, one of which is biconnected, it is the sum of two distinct connected subsets U_1 and V_1. And U_1 is the sum of two distinct connected subsets U_2 and V_2, where V_2 is such say that $V_1 + V_2$ is connected since if it is not $V_1 + U_2$ must be. Proceeding in this manner it is seen that the theorem is true.

THEOREM 12. *A finitely-divisible connected set M is not irreducibly connected about a finite subset.*

Assume that M is irreducibly connected about the finite subset Q, which contains q points. Then by lemmas 3 and 4 it is seen that M contains a connected subset H which is the sum of a countable infinity of distinct connected subsets. Then $M - H = M_1 + \cdots + M_{q+1}$ separate, as M is not the sum of a countable infinity of distinct connected subsets. Say for example that $M_1 + M_2 + \cdots + M_q$ contains $Q - H \times Q$. Then $H + M_1 + M_2 + \cdots + M_q$ is a proper connected subset of M containing Q. As this is impossible under our assumption, it is seen that the theorem must be true.

OHIO STATE UNIVERSITY.

SOME METRIC PROPERTIES OF DESCRIPTIVE PLANES.

By J. L. Dorroh.

Introduction.

This paper is concerned with the study of certain properties of planes in which Axioms I-VIII of Veblen's *System of Axioms for Geometry* * are satisfied. Such a plane will be called a ***descriptive plane.*** A point P of a descriptive plane S will be said to be a limit point of a subset L of S if and only if every triangle of S which incloses † P also incloses a point of L distinct from P.

From the results obtained in Part I, it may be concluded that *a necessary and sufficient condition that a descriptive plane S be metric is that S contain a countable set of distinct points which has a limit point.* In Part IV, it is shown that *not every descriptive plane is metric.*

A conclusion which may be drawn from Part II is that *a necessary and sufficient condition that a descriptive plane S be in one-to-one continuous correspondence with an everywhere dense subset of the number-plane is that S contain a separable segment.* In Part III, it is shown that *not every metric descriptive plane is separable.*

Part I.

Let S denote a descriptive plane which contains a countable set of distinct points which has a limit point. The first of the following theorems can be established by means of projections.

* O. Veblen, "A System of Axioms for Geometry," *Transactions of the American Mathematical Society*, Vol. 5 (1904), pp. 343-384.

† A polygon will be said to inclose a point P if its interior contains P. The term "polygon" is used, in this paper, in the sense of "simple polygon" as defined by Veblen, *loc. cit.*, p. 363, Definition 9. For a definition of the interior of a triangle, see Veblen's Definition 5, *ibid.*, p. 345. A simple polygon separates the plane into just two domains (Veblen's Theorem 28, *loc. cit.*); if K is a polygon which separates the plane into the domains D_1 and D_2, just one of these domains—say D_1—is a subset of the sum of the interiors of a finite number of triangles; D_1 will be called the interior of K, and D_2 will be called the exterior of K.

THEOREM 1. *If A and B are two distinct points of S, there exists a sequence of points $P_1, P_2, P_3, \cdots$, such that:*

(1) AP_iB* $(i = 1, 2, 3, \cdots)$.
(2) $AP_{i+1}P_i$.
(3) A is the sequential limit point of the P_i.

THEOREM 2. *S is metric.*

In order to establish Theorem 2 a method for defining distance in S will be indicated.

Let A, B, C, D, Q denote distinct points of S such that A, C, D are non-collinear, ABC and DCQ. Let $c_1, c_2, c_3, \cdots$ denote the positive rational fractions less than or equal to $\frac{1}{2}$ ($c_1 = \frac{1}{2}$, and $c_i \neq c_j$ if $i \neq j$) which in their lowest terms have for their denominators integral powers of 2. For each positive integer i a point p_i will be selected to correspond to c_i, the p_i being selected so that:

(1) Ap_iQ.
(2) *If* $c_i < c_j$, then Ap_ip_j.
(3) If c_k is the lower limit of a subset $[c]_k$ of the c_i, then p_k is a limit point of the subset of the p_i which correspond to the fractions in $[c]_k$.
(4) A is a limit point of the p_i.

A method for so selecting the p_i will now be described. Let p_1 denote a point in the order Ap_1Q. Let $g_1, g_2, g_3, \cdots$ denote a set of segments of the line AQ such that g_i contains A, g_1 does not contain p_1, g_i contains the end-points of g_{i+1}, and A is the only point common to the g_i.† Let P_{21} denote a point in g_2 such that $AP_{21}p_1$. Let P_{21} correspond to 1/4. For each positive integer $n > 1$ let $c_{n1}, c_{n2}, \cdots, c_{n2^{n-2}}$ denote the positive rational fractions less than $\frac{1}{2}$ which in their lowest terms have the denominator 2^n, and let the second subscripts be chosen so that $c_{ni} < c_{ni+1}$ $(0 < i < 2^{n-2})$. If points $P_{n1}, P_{n2}, \cdots, P_{n2^{n-2}}$ have been put into correspondence with these fractions (P_{ni} corresponding to c_{ni}) by a process previously described, let H_{ni} denote a set of segments $h_1, h_2, h_3, \cdots$ of the line AQ such that h_i contains P_{ni}, h_1 contains no point selected to correspond to a positive fraction less than or equal to $\frac{1}{2}$ with the denominator 2^r $(0 < r < n + 1)$ except P_{ni}, h_i contains the end-points of h_{i+1}, and P_{ni} is the only point common to the h_i. Let the points which have been put into correspondence with fractions which

* If A, B, C are points, "ABC" used as a statement means A, B, C are in the order $\hat{A}BC$.

† That such a set of segments exists is a consequence of Theorem 1.

have for their denominators powers of 2 not greater than the n-th be denoted by $Q_{n1}, Q_{n2}, \cdots, Q_{n2^{n-1}}$ in such a way that $AQ_{ni}Q_{ni+1}$ $(0 < i < 2^{n-1})$. For each i $(0 < i < 2^{n-1}+1)$, let $P_{(n+1)i}$ denote a point such that $Q_{ni-1}P_{(n+1)i}Q_{ni}$ $(Q_{n0}=A)$ and such that $P_{(n+1)i}$ is in h'_{n+1} where h'_{n+1} denotes g_{n+1} if $i=1$ and h'_{n+1} denotes the $n+1$-st segment H_{jk} if $Q_{ni-1}=P_{jk}$ for some j and some k $(1<j<n+1; 0<k<2^{j-2}+1)$. Let $P_{(n+1)i}$ correspond to $c_{(n+1)i}$. Let $[P]$ denote the set of all the points obtained by continuing this process for $n=3, 4, 5, \cdots$, together with p_1 and P_{21}. Let p_i denote the point of $[P]$ which corresponds to c_i $(i=1, 2, 3, \cdots)$.*

For each point x of AQ † let $f(A,x)=f(x,A)$ denote a number determined as follows:

(1) $f(A,x)=0$ if $x=A$.

(2) If $x\neq A$, let $[p]_x$ denote the set of all points p of the p_i such that Apx, and let $[c]_x$ denote the set of all fractions c such that a point of $[p]_x$ corresponds to c; let $f(A,x)$ denote the upper limit of $[c]_x$.

Then if AxQ, $f(A,x)>0$, and if $Axx'Q$, $f(A,x')-f(A,x)\geqq 0$; indeed, if both x and x' belong to $[P]$, this difference is positive. Also, for each positive integer k there exist points $P_1, P_2, \cdots, P_{2^{k-1}}$, all of which belong to $[P]$, such that $f(A,P_i)=i(\frac{1}{2})^k$.

If V and W are points such that AVC and $W=A$, $W=V$, or AWV, let $f(V,W)=f(W,V)$ be defined as follows: let $T_{DW}(V)$ denote the projection through D of V on QW, and let $T_{CW}(V)$ denote the projection through C of $T_{DW}(V)$ on AQ. Let $f(V,W)=f[A,T_{CW}(V)]$. Then $f(V,W)\geqq 0$, and is zero if and only if $W=V$. If Z is also a point of the segment AC, and if WVZ, then $f(V,W)\leqq f(W,Z)\leqq \frac{1}{2}$. For each positive number e there exists a subinterval s of AC containing W such that W is not a limit point of $AC-s$ and such that if z is a point of s then $f(W,z)<e$. Also, if s denotes a subinterval of AC which contains W and such that W is not a limit point of $AC-s$, then there exists a positive number d such that if y is any point of $AC-C-s$, then $f(W,y)>d$.

If V and W are points of AB, let $\phi(V,W)=\phi(W,V)$ denote the upper

* That A is a limit point of the p_i is a consequence of the fact that it follows from the definition of limit point that *a point A of a line l is a limit point of a subset L of l if and only if every segment of l which contains A contains a point of L distinct from A*, and the p_i have been so chosen that every segment of the line AQ which contains A contains one of the p_i.

† If A and Q are distinct points, the Symbol "AQ" unmodified by a word such as "line," "segment," "ray," etc. means the interval AQ; that is, A and Q together with all points X such that AXQ.

limit of $|f(Z,W)-f(Z,V)|$ for all points Z of $AC-C$.* The function ϕ is a distance for points of AB; that is it satisfies the following conditions:

(1) If V and W are points of AB $\phi(V,W)>0$ if $V\neq W$, and $\phi(V,W)=0$ if $V=W$. For if $V\neq W$ $|f(Z,W)-f(Z,V)|>0$ for $Z=V$, and if $V=W$, $f(Z,W)=f(Z,V)$ for each Z.

(2) If V, Y, W are points of AB, then $\phi(V,W)+\phi(W,Y)\geqq\phi(V,Y)$, for $f(Z,V)-f(Z,Y)=f(Z,V)-f(Z,W)+f(Z,W)-f(Z,Y)$, and hence $|f(Z,V)-f(Z,Y)|\leqq|f(Z,V)-f(Z,W)|+|f(Z,W)-f(Z,Y)|$.

(3) If V is a point of AB and is not a limit point of a subset M of AB, then there exists a positive number e such that if z is any point in M, then $\phi(V,z)>e$. For there exists a subinterval s of AB which contains V and which contains no point of M distinct from V and such that V is not a limit point of $AB-s$; hence there exists a positive number e such that if z is in M, then $f(V,z)>e$. By letting $Z=V$, it is seen that $\phi(V,z)$ is at least as great as $f(V,z)$.

(4) If V is a point of AB and is a limit point of a subset M of AB, then for each positive number e there exists a point x of M such that x is distinct from V and $\phi(V,x)<e$.

To establish (4) it will be assumed that AVB. The modifications of the argument necessary to establish the property for $V=A$ and $V=B$ will be pointed out at its conclusion. Let k denote a positive integer such that $(\frac{1}{2})^k<e/4$. Let $P_1, P_2, \cdots, P_{2^{k-1}}$ denote points of $[P]$ such that $f(A,P_i)=i(\frac{1}{2})^k$, then AP_iP_{i+1} $(1\leqq i<2^{k-1})$. Let $F_{CV}(P_i)$ denote the projection through C of P_i on QV, and let x_i denote the projection through D of $F_{CV}(P_i)$ on VC. Then $Vx_1x_2\cdots x_{2^{k-1}}C$. If x is a point of Vx_1 and Z is also in Vx_1, then $f(Z,V)\leqq(\frac{1}{2})^k$ and $f(Z,x)\leqq(\frac{1}{2})^k$, and hence $|f(Z,V)-f(Z,x)|\leqq(\frac{1}{2})^k$. Let b_i denote the intersection of CP_{i-1} $(i=1, 2, \cdots, 2^{k-1})$ with $DF_{CV}(P_i)$, where $P_0=A$. On account of the order of the P_i and the order DCQ, and since the P_i are all on the A-side of the line DQ, the common part of the interiors of angles x_iDx_{i+1} and b_iQV is interior to angle $P_{i-1}CP_{i+1}$ $(1\leqq i\leqq 2^{k-1}-1)$. Let b_i' denote the projection through Q of b_i on VC. Then for $1\leqq i\leqq 2^{k-1}-1$ if x is a point of Vb_i' and Z is in x_ix_{i+1}, VxZ (or $Z=x$), $T_{CV}(Z)$ is in P_iP_{i+1}, and $T_{Cx}(Z)$ is in $P_{i-1}P_{i+1}$. Hence $(i-1)(\frac{1}{2})^k\leqq f(Z,x)\leqq(i+1)(\frac{1}{2})^k$ and $i(\frac{1}{2})^k$

* Cf. E. W. Chittenden's definition of écart in terms of a *Hahn function* which he defines in terms of voisinage in his paper "On the Equivalence of Écart and Voisinage," *Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 161-166.

$\leqq f(Z,V) \leqq (i+1)(\frac{1}{2})^k$, and hence $|f(Z,V)-f(Z,x)| \leqq 2(\frac{1}{2})^k$. If Z is in $x_{2^{k-1}}C - C$ and x is in $Vb'_{2^{k-1}}$, then $T_{CV}(Z)$ is in $P_{2^{k-1}}Q$ and $T_{Cx}(Z)$ is in $P_{2^{k-1}-1}Q$, and hence $(2^{k-1}-1)(\frac{1}{2})^k \leqq f(Z,x) \leqq \frac{1}{2}$ and $f(Z,V) = \frac{1}{2}$. Then if X_c denotes a point such that VX_cb_i' $(i = 1, 2, \cdots, 2^{k-1})$, $|f(Z,V) - f(Z,x)| \leqq 2(\frac{1}{2})^k$ if Z is in $VC - C$ and x is in VX_c.

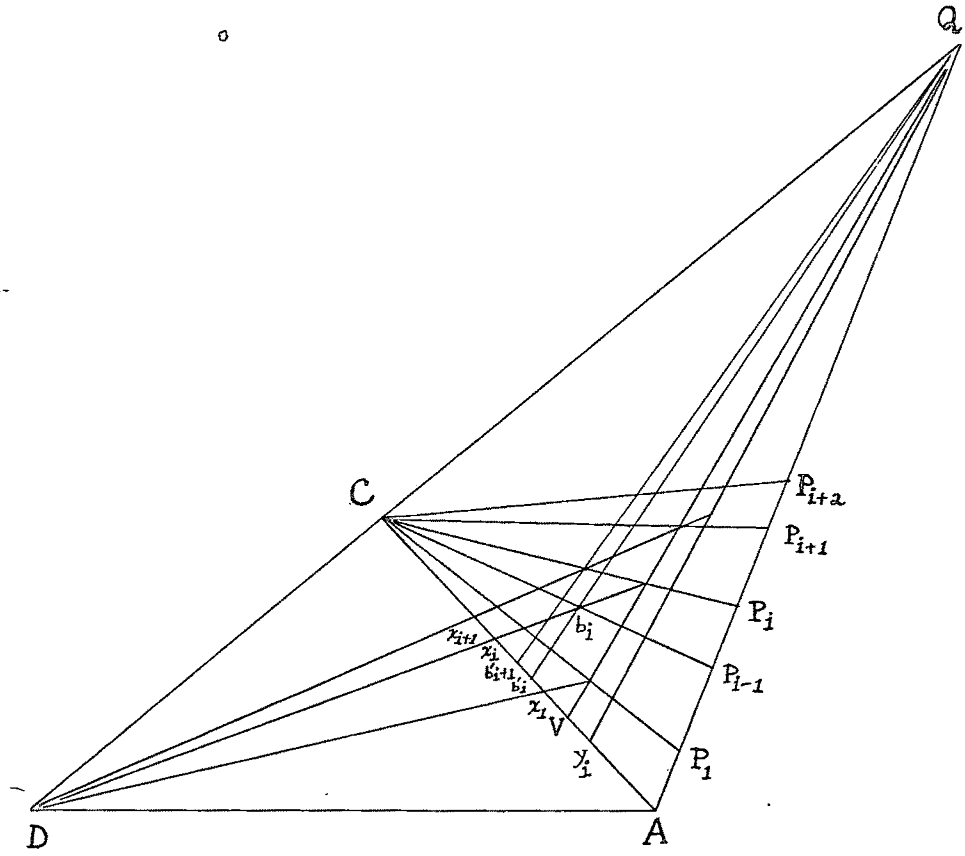

FIG. 1
(for the determination of X_c and Y_c).

Let V_0 denote the projection through D of V on AQ. If AV_0P_1 or if $V_0 = P_1$, then if Z is in VA, $f(Z,V) \leqq (\frac{1}{2})^k$, and if x is a point of VX_c' (where X_c' denotes a point in the orders $VX_c'C$ and $DX_c'P_2$), $f(Z,x) \leqq 2(\frac{1}{2})^k$.

If AP_1V_0, X_c' will be selected differently. Let j denote the greatest integer for which AP_jV_0, and let $a_1, a_2, \cdots, a_j$ denote the points on VA

such that QQ_ia_i, where Q_i is the intersection of VV_0 with CP_i. Let Q'_{i+1} denote the intersection of Qa_i with CP_{i+1} ($P_{j+1} = Q$ if $j = 2^{k-1}$). Let s_i denote the interval $a_{i-1}a_i$ ($i = 1, 2, \cdots, j+1$), where $a_0 = V$ and $a_{j+1} = A$. Let x'_{i+1} denote a point of VC such that $Dx'_{i+1}Q'_{i+1}$. Then if Z is in s_i and Vxx'_{i+1} ($x'_{j+1+1} = C$ if $j \geqq 2^{k-1} - 1$), $(i-1)(\frac{1}{2})^k \leqq f(Z, V) \leqq i(\frac{1}{2})^k$ and

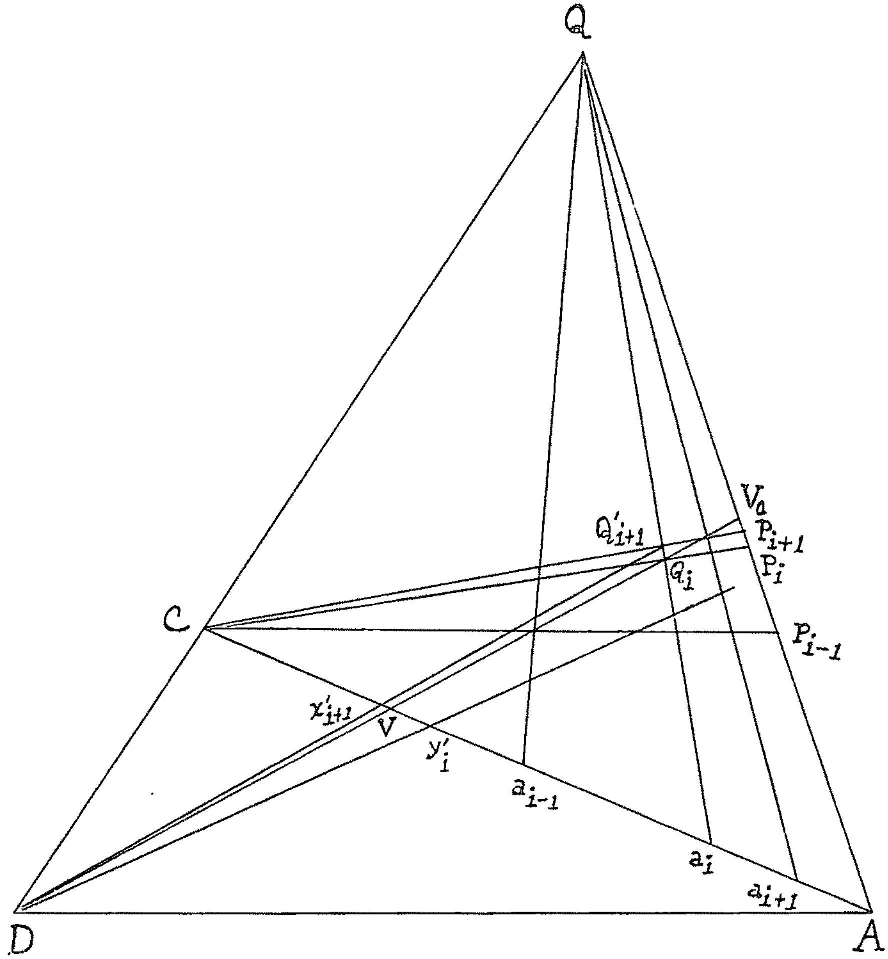

FIG. 2
(for the determination of X_e' and Y_e').

$(i-1)(\frac{1}{2})^k \leqq f(Z, x) \leqq (i+1)(\frac{1}{2})^k$. Let X_e' denote a point such that $VX_e'x'_{i+1}$ for $i = 1, 2, \cdots, j+1$. Then $| f(Z, V) - f(Z, x) | \leqq 2(\frac{1}{2})^k$ if VxX_e' and Z is in AV.

Let X_e^* denote a point such that $VX_e^*X_e'$ and $VX_e^*X_e$. Then if x is in VX_e^*, $\phi(V,x) \leqq 2(\frac{1}{2})^k < e$.

In the determination of X_e a set of points $x_1, x_2, \cdots, x_{2^{k-1}}$ was defined. Let y_0 denote a point such that Vy_0A and the segment Qy_0 intersects the ray Dx_1 within the angle P_1CP_2. For $i = 1, 2, \cdots, 2^{k-1} - 1$ let y_i denote a point in the order Vy_iA and such that the segment Qy_i intersects the ray Dx_{i+1} within the angle $P_{i+1}CP_{i+2}$ $(P_{2^{k-1}+1} = Q)$. Let $y_{2^{k-1}} = A$. Let Y_e denote a point such that VY_ey_i $(i = 0, 1, 2, \cdots, 2^{k-1})$. Then if Z is $VC - C$ and x is in VY_e, $|f(Z,V) - f(Z,x)| \leqq 2(\frac{1}{2})^k$, for if Z is in x_ix_{i+1} $(x_0 = V$ and $x_{2^{k-1}+1} = C)$, and $Z \neq C$, and x is in Vy_i, then $i(\frac{1}{2})^k \leqq f(Z,V) \leqq (i+1)(\frac{1}{2})^k$ and $i(\frac{1}{2})^k \leqq f(Z,x) \leqq (i+2)(\frac{1}{2})^k$.

Employing the notation used in the determination of X_e', suppose that V_0 is in AP_3; then if both Z and x are in AV, $|f(Z,V) - f(Z,x| \leqq 3(\frac{1}{2})^k$, for neither $f(Z,V)$ nor $f(Z,x)$ is greater than $3(\frac{1}{2})^k$. Suppose, secondly, that V_0 is not in AP_3, then $j > 2$. For $2 \leqq i \leqq j$, let y_i' denote a point of the segment AV such that the common part of the interiors of angles VDy_i' and a_iQa_{i+1} is within the angle $P_{i-1}CP_{i+1}$. If both x and Z are in Va_2, then neither $f(Z,V)$ nor $f(Z,x)$ is greater than $2(\frac{1}{2})^k$. If x is in Vy_i' and Z is in a_ia_{i+1} $(2 \leqq i \leqq j)$, then neither $f(Z,V)$ nor $f(Z,x)$ is outside the limits $(i-1)(\frac{1}{2})^k$ and $(i+1)(\frac{1}{2})^k$. If V_0 is in AP_3, let $Y_e' = A$. If V_0 is not in AP_3, let Y_e' denote a point such that $VY_e'a_2$ and $VY_e'y_i'$ for each i $(2 \leqq i \leqq j)$. Then if Z is in AV and x is in VY_e', $|f(Z,V) - f(Z,x)| \leqq 3(\frac{1}{2})^k$.

Let Y_e^* denote a point such that $VY_e^*Y_e$ and $VY_e^*Y_e'$, then if x is in VY_e^*, $\phi(V,x) \leqq 3(\frac{1}{2})^k < e$. Let s denote the segment $X_e^*Y_e^*$, then s contains V and a point of M distinct from V, and if x is a point of s, $\phi(V,x) < e$.

If $V = A$, only the point X_e need be determined to establish the property (4) of ϕ; if $V = B$, only the point Y_e' is required.

The distance ϕ for AB will now be used to define a distance for all of S. Let B_0 denote a point such that DBB_0 and AB_0Q, and let R_t and R_v denote points such that $AR_tR_vB_0$. Let L_1 denote the point set consisting of the triangle ABD and its interior. If a is a point of L_1, let t_a denote the projection through R_t of a on AB, and let v_a denote the projection through R_v of a on AB. If b is also a point of L_1, let $d_1(a,b) = d_1(b,a) = \phi(t_a, t_b) + \phi(v_a, v_b)$. Then d_1 is a distance for L_1. Let K_x, K_y, K_z denote three non-collinear points within the triangle ABD, and let L_2 denote the point set consisting of the triangle ABD and its exterior. If p is a point of L_2, let w_p denote the intersection of the ray K_wp $(w = x, y, z)$ with the triangle ABD. If q is also a point of L_2, let

$$d_2(p,q) = d_2(q,p) = (1/3)[d_1(x_p,x_q) + d_1(y_p,y_q) + d_1(z_p,z_q)].$$

Then d_2 is a distance for L_2, and, for points of the triangle ABD, d_2 is identical with d_1. If, now, p and q denote any two points of S, let $d(p,q) = d(q,p)$ be defined as follows:

(1) If both points belong to L_1, $d(p,q) = d_1(p,q)$.
(2) If both points belong to L_2, $d(p,q) = d_2(p,q)$.
(3) If p is in the interior of the triangle ABD and q is in its exterior, let $d(p,q)$ denote the lower limit of $[d_1(p,x) + d_2(x,q)]$ for all points x of the triangle ABD. Then d is a distance for S.

Part II.

Let S^* denote a descriptive plane which contains two points A_0 and B_0 such that the segment A_0B_0 is separable. The assumption that a segment of S^* is separable is not stronger than the assumption that the interior of some triangle of S^* is separable, for it is easily seen, by means of projections, that if the interior of a triangle of a descriptive plane is separable, then each of its sides is separable.

Theorem 3. *Every line † of S^* is separable.*

Proof. Let C denote a point such that A_0, B_0, C are non-collinear. Let D and E denote points in the orders CDA_0 and CB_0E, respectively. Let H denote a countable point set everywhere dense on A_0B_0. Let T denote the set of all points X of the ray B_0E such that X is collinear with D and a point of H. The ray B_0E is a subset of $\bar{T}$. Hence it is separable. Similarly the ray B_0C is separable, and therefore so is every segment of it. In view of the argument just given it follows that every line intersecting the line B_0C is separable. If l is a line not intersecting the line B_0C, there exists a line intersecting both l and B_0C and since this line is separable so is l.

Corollary. *If A and B are two distinct points, there exists, on the ray AB, a countable set of points $P_1, P_2, P_3, \cdots$ such that AP_nP_{n+1} and such that if x is any point of the ray AB then either $x = A$ or AxP_n for some n.*

Theorem 4. *If A and B are two distinct points and a and b are two distinct real numbers such that $a < b$, there exists between the points of AB and a subset of the numbers of the interval (a,b) a one-to-one reciprocal*

† In this paper, the word "line," unmodified, is used in the sense of "straight line."

correspondence in which every rational number of (a, b) corresponds to some point of AB, A corresponds to a, B corresponds to b, and if c and f are numbers of (a, b) corresponding to points p and q, respectively, of AB and $a < c < f < b$, then $ApqB$.

Proof. Let $p_1, p_2, p_3, \cdots$ denote a countable subset of the segment AB everywhere dense on AB and such that $p_i \neq p_j$ if $i \neq j$. Let $c_1, c_2, c_3, \cdots$ denote the set of all the rational numbers of (a, b) distinct from a and from b, $c_i \neq c_j$ if $i \neq j$. Let p_1 correspond to c_1. Let p_{21} and p_{22} denote the p_i with smallest subscripts such that $Ap_{21}p_1p_{22}B$, and let c_{21} and c_{22} denote the c_i with smallest subscripts such that $a < c_{21} < c_1 < c_{22} < b$. Let p_{21} correspond to c_{21} and p_{22} to c_{22}. If j points have been put into correspondence with j numbers in this way, let the points be denoted by $p_{j1}, p_{j2}, \cdots, p_{jj}$ so that $Ap_{ji}p_{ji+1}p_{ji+2}B$ $(1 \leqq i \leqq j-2)$, and let the corresponding c_i be denoted by $c_{j1}, c_{j2}, \cdots, c_{jj}$, using the same subscripts for corresponding points and numbers. Let $k = j + 1$ additional points $p_{k1}, p_{k2}, \cdots, p_{kk}$ and k additional numbers $c_{k1}, c_{k2}, \cdots, c_{kk}$ be selected as follows: p_{k1} is the p_i with smallest subscript such that $Ap_{k1}p_{j1}$; p_{kk} is the p_i with smallest subscript such that $p_{jj}p_{kk}B$, and, for $1 < n < k$, p_{kn} is the p_i with smallest subscript such that $p_{jn-1}p_{kn}p_{jn}$; c_{k1} is the c_i with smallest subscript such that $a < c_{k1} < c_{j1}$; c_{kk} is the c_i with smallest subscript such that $c_{jj} < c_{kk} < b$, and, for $1 < n < k$, c_{kn} is the c_i with smallest subscript such that $c_{jn-1} < c_{kn} < c_{jn}$. Let p_{kn} correspond to c_{kn} $(1 \leqq n \leqq k)$. Let A correspond to a and B to b. If q is a point in the order AqB and q is not one of the p_i, let q correspond to the number which is the lower limit of the rational numbers corresponding to the p_i which are in qB.

Definition. If A, B are distinct points and C, D are distinct points, the term "a segment from AB to CD" means a segment which has one of its end-points on the segment AB and the other on the segment CD and which has no other point in common either with AB or with CD.

Definition. If A, B, C are three non-collinear points, a set G of segments from AB to BC is said to fill the interior of the triangle ABC if no two segments of G have a point (or an end-point) in common and every point of the interior of the triangle ABC belongs to some segment of G.

Definition. If A, B, C are three non-collinear points and D and E are points in the orders ADB and BEC, respectively, a set H of segments from AD to CE is said to fill the interior of the quadrilateral $ADEC$ if no two

segments of H have a point (or an end-point) in common and every point of the interior of the quadrilateral belongs to some segment of H.

In order to show the existence of a set G of segments from AB to BC filling the interior of the triangle ABC, let F denote a point such that ACF, and let G consist of all the segments from AB to BC which are subsets of lines through F. In order to show the existence of a set H of segments from AD to CE filling the interior of the quadrilateral $ADEC$, let K denote a point in the order EDK † and let H denote the set of all segments h from AD to CE such that h is a subset of a ray which either starts at F and is interior to the angle KFA or starts at K and is interior to the angle FKE or coincides with the ray KF.

The definitions just given will be of use in proving the following theorem:

THEOREM 5. *If O is a point of S^*, there exists between the points of S^* and an everywhere dense subset of the number-plane a one-to-one continuous correspondence in which the image of each line of S^* which contains O is an everywhere dense subset of a line of the number plane through the origin, and the image of each line of S^* is an everywhere dense subset of an open curve of the number-plane.*‡

Proof. Let A_1 and B_1 denote two distinct points each distinct from O and such that O, A_1, B_1 are non-collinear. Let C_1 and D_1 denote points in the orders A_1OC_1 and B_1OD_1, respectively. Let $A_2, A_3, A_4, \cdots; B_2, B_3, B_4, \cdots; C_2, C_3, C_4, \cdots; D_2, D_3, D_4, \cdots$ denote sequences of points on the rays OA_1, OB_1, OC_1, OD_1, respectively, and such that each of the sequences $A_1, A_2, A_3, A_4, \cdots; B_1, B_2, B_3, B_4, \cdots; C_1, C_2, C_3, C_4, \cdots; D_1, D_2, D_3, D_4, \cdots$ has, with respect to the corresponding ray, the properties stated in the Corollary to Theorem 3. Let the points of OA_1 correspond to numbers of the interval $(0, 1)$ in the manner indicated in Theorem 4 and so that A_1 corresponds to 1. For each positive integer i let the points of A_iA_{i+1} correspond to numbers of the interval $(i, i+1)$ in the same manner and so that A_i corresponds to i. Let the points of A_1B_1 correspond in this way to numbers of the interval $(0, 1)$ so that A_1 corresponds to 0, and let the points of B_1C_1 correspond in the same manner to numbers of the interval $(1, 2)$, B_1 corresponding to 1. If p is a point of C_1D_1, and q is a point of A_1B_1 collinear with O and p, let x

† In addition, let K be on the D-side of the line AC, and let F be in the order ACF and within the angle ADE.

‡ Cf. R. L. Moore, "On a Set of Postulates Which Suffice to Define a Number-Plane," *Transactions of the American Mathematical Society*, Vol. 16 (1915), pp. 27-32.

[illegible] number to which q corresponds ($x=0$ if $q=A_1$), and [illegible] [illegible] 1 to $2+x$. If p is a point of $D_1A_1-A_1$, and q is a point [illegible] B_1C_1 [illegible] with O and p, let x denote the number to which q corresponds, and [illegible] correspond to $2+x$. In this way, the points of C_1D_1 correspond [illegible] [illegible] of the interval $(2, 3)$, and the points of $D_1A_1-A_1$ to numbers of [illegible] D_1 corresponding to 3.

[illegible] each rational number u $(0 \leq u < 4)$ let r_u denote the ray [illegible] is the point on the quadrilateral $A_1B_1C_1D_1$ which corresponds [illegible] [illegible] u let [illegible] $p_{u2}, p_{u3}, \cdots$ denote a sequence of points on [illegible] that $O_{\overline{1}}$ [illegible] and such that any point of the ray r_u except O [illegible] [illegible] some [illegible]. Let $p_{0i}=A_i$, $p_{1i}=B_i$, $p_{2i}=C_i$, $p_{3i}=D_i$.

[illegible] the interiors of the triangles A_1OB_1, B_1OC_1, C_1OD_1 be [illegible] [illegible] G_1, G_2, G_3 of segments from OA_1 to OB_1, from OB_1 to OC_1, [illegible] OC_1 to OD_1, respectively. For each positive number x to which there [illegible] [illegible] a point of the ray OA_1 let q_x denote that point. For $x < 1$, let [illegible] [illegible] the segment of G_1 which has q_x for one end-point. Let g_2 [illegible] [illegible] of G_2 which has an end-point in common with g_1, and let [illegible] [illegible] of G_3 which has an end-point in common with g_2. [illegible] [illegible] the segment from the end-point of $g_{3,1/2}$ on OD_1 to $q_{1/2}$ (this [illegible] is [illegible] to the triangle D_1OA_1). Let q_{j1} $(1 \leq j < 4)$ denote the end-point [illegible] Op ($p_{1}=B_1$, $p_2=C_1$, $p_3=D_1$).

For each positive integer n, by definition, $p_{(4-1/n)}$ denotes the [illegible] [illegible] which corresponds to $4-1/n$. Let $q_{(4-1/2),1/2}$ denote the intersection [illegible] with [illegible]. Let G_4 denote a set of segments from OD_1 to [illegible] which includes the segment $q_{3,1/2}q_{(4-1/2),1/2}$ and which fills the interior [illegible] triangle D_1O_1 [illegible]. That G_4 may contain the segment specified and [illegible] [illegible] of the triangle follows from the fact that it may consist [illegible] [illegible] filling the interior of the triangle $q_{3,1/2}Oq_{(4-1/2),1/2}$ plus [illegible] [illegible] plus a set of segments filling the interior of the quadrilateral $D_1q_{3,1/2}q_{(4-1/2),1/2}p_{(4-1/2)}$.

Let g_4 denote the segment of G_4 which has an end-point in common [illegible]. Let [illegible] denote the end-point of g_4 on $Op_{(4-1/2)}$. Let g'_{5} [illegible] g' [illegible] denote the segments $q_{(4-1/2),1/3}q_{1/3}$ and [illegible] respectively, and let $g'_{5,1/2}$ denote the segment $q_{(4-1/2),1/2}q_{1/2}$. Let [illegible] [illegible] $q_{(4-1/3),2/3}$ denote the intersections of $Op_{(4-1/3)}$ with g'_5, [illegible] g' [illegible] [illegible] respectively. Let G_5 denote a set of segments from $Op_{(4-1/2)}$ to [illegible] which fills the interior of the triangle $p_{(4-1/2)}Op_{(4-1/3)}$ and which [illegible] the subsegments of $g'_{5,1/3}$, $g'_{5,1/2}$, $g'_{5,2/3}$ which are within this triangle.

If for an integer n and for each integer m $(1 < m < n)$ a set G_{2+m} of segments from $Op_{[4-1/(m-1)]}$ to $Op_{(4-1/m)}$ filling the interior of the triangle whose vertices are the end-points of the segments just named has been selected, in the manner described for $m = 2$, 3, let $c_{n1}, c_{n2}, \cdots, c_{nN}$ denote the proper rational fractions which in their lowest terms have a denominator $\leqq n$ (N denotes the number of these fractions), and let $g'_{2+n,c_{nj}}$ denote the

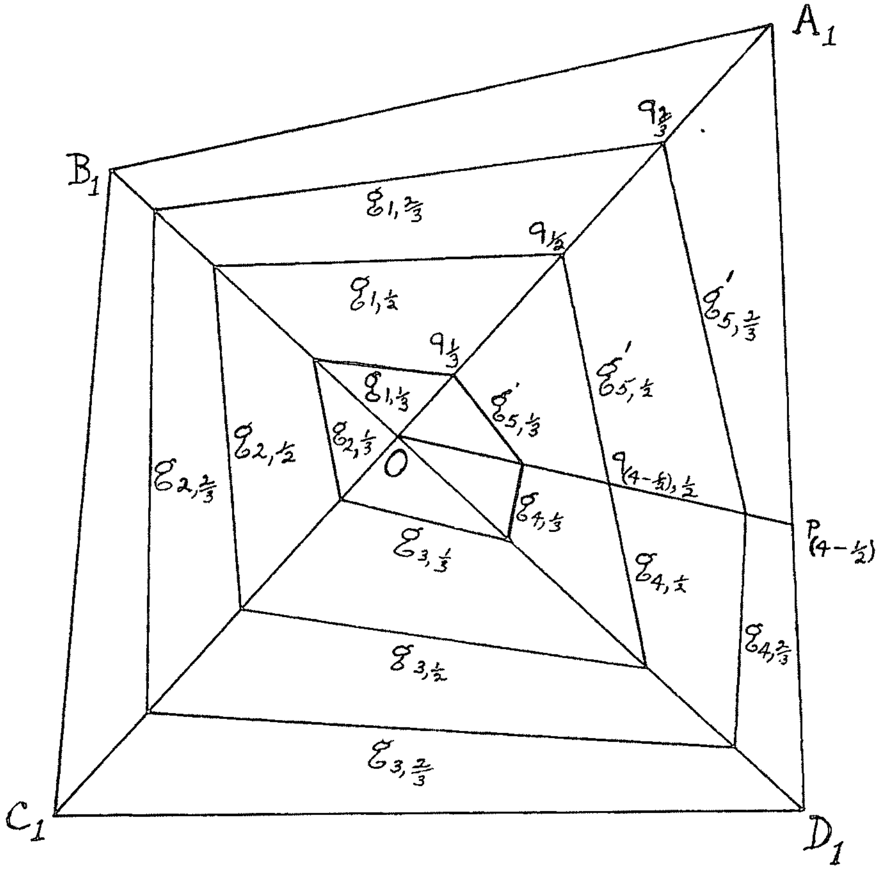

FIG. 3

segment $q_{[4-1/(n-1)],c_{nj}}q_{c_{nj}}$ $(1 \leqq j \leqq N)$, where $q_{(4-1/m),x}$ denotes the end-point of $g_{2+m,x}$ on $Op_{(4-1/m)}$ ($g_{2+m,x}$ is the segment of G_{2+m} which has an end-point in common with $g_{2+m-1,x}$). Let G_{2+n} denote a set of segments from $Op_{[4-1/(n-1)]}$ to $Op_{(4-1/n)}$ which fills the interior of the triangle $p_{[4-1/(n-1)]}Op_{(4-1/n)}$ and which contains the subsegments of the $g'_{2+n,c_{nj}}$ within this triangle. Let $g_{2+n,x}$ denote the segment of G_{2+n} which has an end-point in common with $g_{2+n-1,x}$.

If c is a proper rational fraction which in its lowest terms has the denominator m, the point set $g_{1c} + g_{2c} + g_{3c} + \cdots + g'_{2+m,c}$ plus the end-points of these segments is a polygon which has q_c as a vertex or as an inner point of one of its sides, for q_c is a common end-point of g_{1c} and $g'_{2+m,c}$, and in the sets of segments described in the preceding paragraph g_{kc} for $k \geqq 2 + m$ is a subsegment of $g'_{2+m,c}$. For each real number x to which there corresponds a point of the segment OA_1, let I_{ix} denote the interval obtained from g_{ix} by adding to the latter its end-points. For each such number x, let $K_x = \sum_{i=1}^{i=\infty} I_{ix}$. Then q_x is a limit point of the common part of K_x and the interior of the triangle D_1OA_1, and no other point of the ray OA_1 is a limit point of K_x. The subset of K_x within the triangle D_1OA_1 will be called a broken line from OD_1 to OA_1 and q_x and q_{3x} will be called its end-points; the collection of all such broken lines—that is, for all values of x corresponding to points of OA_1—will be said to fill the interior of the triangle D_1OA_1.

For each integer t $(t > 1)$ let T denote the number of rational numbers greater than or equal to 0 and less than 4 which are integers or which in their lowest terms have a denominator less than or equal to t. Let these numbers be denoted by $b_{t1}, b_{t2}, \cdots, b_{tT}$ in such a way that $b_{ti} < b_{ti+1}$ $(0 < i < T)$. Then $b_{t1} = 0$. Let M_1 denote the quadrilateral $A_1B_1C_1D_1$ and let M_t denote the polygon with the vertices $P_{t1}, P_{t2}, \cdots, P_{tT}$ which are points determined as follows: on the ray $r_{b_{ti}}$ there has been selected a sequence of points $p_{b_{ti}j}$ $(j = 1, 2, 3, \cdots)$ such that $Op_{b_{ti}j}p_{b_{ti}j+1}$ and such that any point of the ray except O is between O and one of these p's. Let P_{ti} denote the first of the $p_{b_{ti}j}$ outside of M_{t-1}. Then $P_{t1} = A_t$.

It is to be noted that M_t incloses M_{t-1}. It may be seen that the rays from O through the vertices of M_t intersect M_{t-1} in such a way that if Q_{ti} denote the point on M_{t-1} such that $OQ_{ti}P_{ti}$, then the interval $Q_{ti}Q_{ti+1}$ $(Q_{tT+1} = Q_{t1})$ is a subinterval of a side of M_{t-1}. Let V_{ti} denote the quadrilateral with vertices $P_{ti}, P_{ti+1}, Q_{ti+1}, Q_{ti}$ $(P_{tT+1} = P_{t1})$. The interiors of the V_{ti} are mutually exclusive. Let the interior of V_{ti} $(0 < i < T)$ be filled with a set of segments from $Q_{ti}P_{ti}$ to $Q_{ti+1}P_{ti+1}$. Let the interior of the quadrilateral V_{tT} be filled with a set of broken lines from $Q_{tT}P_{tT}$ to $Q_{t1}P_{t1}$, in a manner analogous to that in which the interior of the triangle D_1OA_1 was filled with broken lines from OD_1 to OA_1; that is, with a set of broken lines each consisting of a finite or infinite number of segments such that each point of the segment $Q_{tT}P_{tT}$ and each point of the segment $Q_{t1}P_{t1}$ is an end-point of one and only one broken line of the set, each point of the in-

terior of V_{tT} belongs to one and only one element of the set, and such that the set has the property now to be defined: for each real number x $(t-1<x<t)$ to which there corresponds a point of the segment $A_{t-1}A_t$, let I_{t1x} denote the interval obtained by adding its end-points to the segment of the set filling the interior of V_{t1} which has q_x for one end-point, for $1<i<T$ let I_{tix} denote the interval obtained by adding its end-points to the segment of the set filling the interior of V_{ti} which has an end-point in common with $I_{t,i-1,x}$; let I_{tTx} denote the point set obtained by adding its end-points to the broken line of the set filling the interior of V_{tT} which has an end-point in common with $I_{t,T-1,x}$, then q_x belongs to I_{tTx}. If $0<t-1<x<t$, let K_x denote the point set defined as follows: $K_x=\sum_{i=1}^{i=T} I_{tix}$. For each positive integer x, let K_x denote the polygon M_x.

Then if p is any point of S^* except O, there is one and only one positive number x such that p belongs to K_x; for each p let x_p denote this number. The point in which the ray Op intersects the quadrilateral $A_1B_1C_1D_1$ corresponds to a number, positive or zero and less than 4, by virtue of the correspondence between the points of this quadrilateral and a subset of the numbers of the interval $(0,4)$; let a_p denote this number. Let p correspond to that point of the number-plane which has the polar coördinates (r_p,ϕ_p) where $r_p=x_p$ and $\phi_p=\frac{1}{2}\pi a_p$, and let O correspond to the pole.

That the subset of the number-plane which corresponds to the points of S^* is everywhere dense in the number-plane is evident from the fact that it contains all the points whose polar coördinates are such that r is rational and ϕ is a rational multiple of $\pi/2$. It can be seen from the manner in which the numbers x_p and a_p are determined by a point p of S^* that p is a sequential limit point of a sequence of distinct points $p_1, p_2, p_3, \cdots$ if and only if $\lim\limits_{i=\infty} x_{p_i}=x_p$ and $\lim\limits_{i=\infty} a_{p_i}=a_p$, unless $p=O$ in which case it is only necessary that $\lim\limits_{i=\infty} x_{p_i}=0$. Hence the correspondence is continuous. If p is a point of S^* distinct from O, the points of the line Op correspond to an everywhere dense subset of the line in the number-plane whose equation is $\phi=\frac{1}{2}\pi a_p$. Let l denote a line of S^* which does not contain O, and let L denote the image of l in the number plane. It may be noted that if x' is any positive real number, then there exists a number $x>x'$ such that x corresponds to a point of the ray OA_1 and l intersects K_x; hence L is not bounded. Let A and B denote two distinct points of l such that AB contains no point of the ray OA_1. The image of AB in the number-plane can be covered with a simple chain of regions each region of which is the interior of a simple closed curve of diameter less than any previously assigned positive

[illegible], for AB can be covered with a simple chain of quadrilaterals [illegible] [illegible] are subintervals of the K_r and of lines through O. Th[illegible] [illegible] may be chosen so that their images are of diameter less than [illegible] [illegible] each incloses at least one point of AB, and since the image of each K [illegible] subset of the circle $r = x$, the images of these quadrilaterals are subsets [illegible] closed curves composed of segments of lines through the pole a[illegible] [illegible] with center at the pole. If e is a positive number and [illegible] [illegible] the type described covering the image of AB, then there exists [illegible] such chain covering the image of AB and such that the boundary [illegible] each region of the second chain is a subset of some region of C' and [illegible] [illegible] less than e. It follows, since $\overline{L}$ contains at most one point [illegible] [illegible] that any two points of $\overline{L}$ belong to a continuum lying [illegible] [illegible] is a continuum. That the omission of any point of $\overline{L}$ divides [illegible] mutually separated connected point sets may be seen by observing that [illegible] through the pole and a point of L divides the number-plane [illegible] mutually separated domains each of which contains a connected [illegible] [illegible] connectedness of the indicated subsets of $\overline{L}$ can be established [illegible] way in which the connectedness of L was shown). Hence [illegible] [illegible]. Furthermore, $\overline{L}$ has the property that no line through [illegible] [illegible] more than one point of it, and if a line whose equation is [illegible] ([illegible] and n integers) intersects $\overline{L}$ the point of intersection corresponds to [illegible] [illegible].

Part III.

Definition. If the points of a line l of a descriptive plane [illegible] two sets S_1 and S_2 such that no point of one set is between two points [illegible] other and no point P is between every point of S_1 distinct from P [illegible] [illegible] of S_2 distinct from P, the pair of point sets S_1 and S_2 [illegible] gap [illegible] and will be said to divide l into S_1 and S_2.

Definition. If g is a gap in a line l of a descriptive plane which [illegible] the point sets S_1 and S_2 and p and q are points of l, pgq [illegible] [illegible] p and q are in S_1 (S_2) and there exists a point q' in S_2 (S_1) [illegible] [illegible] pqg [illegible] mean that p is in one of the sets S_1, S_2 and q [illegible]. [illegible] triangle will be said to inclose g if it incloses a point of S_1 [illegible] [illegible] of S_2.

A finite set of polygons $Z_1, Z_2, \ldots, Z_n$ is said to be a simple chain [illegible] interior of Z_i has a point in common with the interior of Z_j [illegible] [illegible] if and only if $j = 1$. A set Z of polygons is said to cover a [illegible] [illegible] and only if every point of M is inclosed by some polygon of Z.

In his paper "Concerning a Non-Metrical Pseudo-Archimedean Axiom,"* R. L. Moore employed a pseudo-archimedean axiom which may be briefly stated: *if g is a gap in a line l and A and B are two distinct points on the same side of the line l, then if a triangle incloses g it incloses a point C on the opposite side of l from A and B such that the triangle ABC incloses g.*

Definition. If it is true of a descriptive plane that this pseudo-archimedean axiom is satisfied at no gap in that plane, then the plane is said to be "quasi-complete."

There will now be exhibited an example of a quasi-complete, metric, descriptive plane which is complete and in which it is possible to define congruence of segments in such a way that Moore's set C of congruence axioms † is satisfied, but which is not separable nor locally compact.

The descriptive plane is one described by Hilbert in "Ueber den Satz von der Gleichheit der Basiswinkel im Gleichschenkligen Dreieck" ‡; the definition of congruence, however, is different from the one used by Hilbert. The points of the plane are all the pairs (x, y) of elements of the set T of all the ideal numbers of the form $\alpha = a_0 t^n + a_1 t^{n+1} + \cdots$, where t is a parameter, the number of terms in α is finite or infinite, a_0 $(a_0 \neq 0)$, $a_1, \cdots$ are any real numbers, and n is any integer, positive, negative or zero. An element of T is positive if the corresponding a_0 is positive; $x > y$ if $x - y$ is positive; $y < x$ means $x > y$. To obtain a congruence satisfying Moore's axioms C, let AB be congruent to DE if and only if

$$[(x_A - x_B)^2 + (y_A - y_B)^2]^{1/2} = [(x_D - x_E)^2 + (y_D - y_E)^2]^{1/2}. \text{ §}$$

Collineation and order are defined in terms of the elements of T just as they are defined in the number-plane in terms of real numbers. Let H denote this plane.

In order to show that H *is not separable,* it suffices to establish the existence of a non-separable segment in H. For each real number b $(0 < b < 1)$ let s_b denote the segment whose end-points are $(b, 0)$ and $(b + t, 0)$.

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* *Bulletin of the American Mathematical Society*, Vol. 22 (1916), pp. 225-236.

† See R. L. Moore, "Sets of Metrical Hypotheses for Geometry," *Transactions of the American Mathematical Society*, Vol. 9 (1908), pp. 487-512. Concerning the equivalence, in a descriptive plane, of C and Hilbert's congruence axioms, see Moore's Theorem 1, *ibid.*, § 6, p. 504. That the proposition that every segment has a middle point is a consequence of Moore's order and congruence axioms has been shown by the present author in his paper "Concerning a Set of Metrical Hypotheses for Geometry," *Annals of Mathematics* (2), Vol. 29 (1928), pp. 229-231.

‡ *Grundlagen der Geometrie*, Anhang II, Teubner, 5. Auflage, 1922.

§ T contains $(\alpha)^{1/2}$ if $a_0 > 0$ and n is even.

No two of these segments have a point in common, for if $0 < b < b' < 1$, then $b' > b + t$. Each of these segments is a subset of the segment whose end-points are $(0, 0)$ and $(1, 0)$, and hence the latter segment, since it contains an uncountable set of mutually exclusive segments, is not separable.

If H satisfied the pseudo-archimedean axiom quoted, it would follow from the results of Moore's paper on this axiom that H is separable; hence H does not satisfy this axiom. If it can be shown that all the gaps in H must satisfy this axiom if one does, then it follows that H *is quasi-complete*. A translation of axes disturbs neither collineation nor order in H, and Hilbert has shown the existence of transformations of H into itself which disturb neither collineation nor order and which transform any given line through the origin into any other given line through the origin. Hence, in order to show that H is quasi-complete, it suffices to show that if g is any gap in the x-axis and g_1 is a particular gap in the x-axis, then there exists a transformation of H into itself which disturbs neither order nor collineation and which transforms g into g_1.

Let g denote a gap in the x-axis.* It may be supposed that one of the sets into which g divides the axis consists entirely of points whose abscissas are positive, for, if this is not the case, consider the gap h which divides the axis into the two sets each of which consists of the points whose abscissas are the negatives of the abscissas of the points in one of the two sets into which g divides the axis. Let S_1 and S_2 denote the sets into which g divides the axis, and let S_2 denote that one of these sets which consists entirely of points whose abscissas are positive. Let V denote the subset of T which consists of all the elements v of T such that $(v, 0)$ is in S_1, and let W denote the subset of T consisting of all the elements w of T such that $(w, 0)$ is in S_2. Then every element of T belongs either to V or to W, every element of W is greater than any element of V, and there is neither a greatest element of V nor a least element of W. Hence V contains some positive elements. Let V^* denote the subset of V which consists of all the positive elements of V.

Let N denote the greatest integer—if one exists—such that if $v = a_0t^n + a_1t^{n+1} + \cdots$ is in V^* then $n \geqq N$. Such an integer N exists, for suppose, on the contrary, that if m is any integer there exists an element $v = a_0t^n + a_1t^{n+1} + \cdots$ of V^* such that $n < m$; since v is in V^* $a_0 > 0$, and it follows that V contains an element greater than any preassigned element of T, contrary to hypothesis. Then V^* contains elements of the form $v = a_0t^N + a_1t^{N+1} + \cdots$. Let V_N denote the set of all elements of V^* of this form, then if v is an element of V there exists an element v' of V_N such that

* Every line of H contains a gap, for H is non-archimedean.

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$v' > v$. Let A_i $(i = 0, 1, 2, 3, \cdots)$ denote the set of all real numbers a such that a is the coefficient of the $N + i$-th power of t in some element of V_N. For each positive or zero integer i for which A_i has a finite upper limit, let $\bar{A}_i$ denote that upper limit. There is at least one value of i for which A_i fails to have a finite upper limit, for if all these upper limits are finite, the element $\bar{A}_0 t^N + \bar{A}_1 t^{N+1} + \bar{A}_2 t^{N+2} + \cdots$ of T is either the greatest element of V or the least element of W, contrary to hypothesis. Let k denote the least integer, positive or zero, for which A_k has no finite upper limit. Let $b = 0$ if $k = 0$, and if $k > 0$ let $b = \bar{A}_0 t^N + \bar{A}_1 t^{N+1} + \cdots + \bar{A}_{k-1} t^{N+k-1}$. Suppose that b is in V. Let $v_j = b + j t^{N+k}$ for each positive integer j, and let $z_j = b + (1/j) t^{N+k-1}$; then if v is in V and w is in W there exists an integer j such that $w > z_j > v_j > v$ (v_j is in V and z_j is in W for all j). Suppose, secondly, that b is not in V, then $k > 0$. Let r denote the least positive or zero integer less than k for which $c = \bar{A}_0 t^N + \bar{A}_1 t^{N+1} + \cdots + \bar{A}_r t^{N+r}$ does not belong to V. Let $w_j = c - j t^{N+r+1}$, and let $v_j' = c - (1/j) t^{N+r}$. Then w_j is in W and v_j' is in V, and if v is in V and w is in W there exists a positive integer j such that $w > w_j > v_j' > v$.

Let V_1 denote the subset of T consisting of all elements v of T such that $v < j t^2$ for some positive integer j, and let W_1 denote the subset of T consisting of all elements w of T such that $w > (1/j) t$ for some positive integer j. Then every element of T belongs either to W_1 or to V_1, neither of these two sets contains a greatest element of itself nor a least element of itself, and each element of W_1 is greater than any element of V_1. Let g_1 denote the gap in the x-axis determined by the point sets S^*_1 and S^*_2 consisting of all the points of the x-axis whose abscissas are in V_1 and all the points of the x-axis whose abscissas are in W_1, respectively. A transformation of H into itself which disturbs neither collineation nor order and under which g_1 is the image of g is the transformation under which the image of a point (x, y) is the point (x', y'), where x' and y' are determined as follows:

(1) If b is in V, $x' = (x - b)/t^{N+k-2}$, and $y' = y/t^{N+k-2}$.

(2) If b is in W, $x' = (c - x)/t^{N+r-1}$, and $y' = -y/t^{N+r-1}$.

In case (1), S_1 is transformed into S^*_1, and, in case (2), S_1 is transformed into S^*_2.

A complete, compact space is separable.† Since no segment of H is

† F. Hausdorff, *Grundzüge der Mengenlehre* (1914).

separable, H is not locally separable. It follows that if H is complete, then H is not locally compact.

The plane H is complete. If $z = a_0t^n + a_1t^{n+1} + \cdots$, let the function $f(z)$ be defined as follows:

$$f(z) = 1, \quad \text{if} \quad n \leqq 1;$$
$$f(z) = 1/n, \quad \text{if} \quad n > 1,$$

and let $f(0) = 0$. If A is a point of H and B is a point of H, let $d(A,B) = d(B,A) = f(x_A - x_B) + f(y_A - y_B)$. Then d is a distance for points of H; that is, it satisfies the three conditions:

(1) $d(A,B) = 0$ if $A = B$, and $d(A,B) > 0$ if $A \neq B$.

(2) $d(A, C) = f(x_A - x_B + x_B - x_C) + f(y_A - y_B + y_B - y_C) \leqq f(x_A - x_B) + f(x_B - x_C) + f(y_A - y_B) + f(y_B - y_C) = d(A, B) + d(B, C)$, for it follows from the definition of $f(z)$ that $f(z + z') \leqq f(z) + f(z')$.

(3) A point A is a sequential limit point of a sequence of distinct points A_i $(i = 1, 2, 3, \cdots)$ if and only if $\lim\limits_{i \to \infty} d(A, A_i) = 0$. Let $A = (x_A, y_A)$ and let $A_i = (x_i, y_i)$. That d satisfies the second part of condition (3) is verified by observing that if A is a sequential limit point of the A_i, then either all but a finite number of the points $(0, y_i)$ are identical with $(0, y_A)$ or $(0, y_A)$ is the sequential limit point of the points $(0, y_i)$; similarly for the point $(x_A, 0)$ and the points $(x_i, 0)$. Hence if e is any positive element of T, there exists a positive integer m such that if $i > m$ then $-e < x_A - x_i \pm (y_A - y_i) < e$, and therefore the least exponent of t in $x_A - x_i$ or in $y_A - y_i$ is not less than the least exponent of t in e. To verify that d satisfies the first part of condition (3), note that if $\lim d(A, A_i) = 0$, then for each positive integer n the rectangle with vertices $(x_A + t^n,\ y_A + t^n)$, $(x_A + t^n,\ y_A - t^n)$, $(x_A - t^n,\ y_A - t^n)$, $(x_A - t^n,\ y_A + t^n)$ incloses all but a finite number of the A_i, and note that if a triangle incloses A it also incloses at least one of these rectangles. To show that H is complete, let $p_1, p_2, p_3, \cdots$ $[p_i = (x_i, y_i)]$ denote a sequence of distinct points such that if h is any positive real number there exists a positive integer m such that if k is any positive integer then $d(p_m, p_{m+k}) < h$, then there exists a point $p_0 = (x_0, y_0)$ which is a sequential limit point of the p_i. For each integer $r > 1$, let m_r denote a positive integer such that $d(p_{m_r}, p_{m_r+k}) < 1/r$ for every positive integer k. Then if $x_{m_r} - x_{m_r+k} = a_0t^n + a_1t^{n+1} + \cdots$ and $y_{m_r} - y_{m_r+k} = b_0t^{n'} + b_1t^{n'+1} + \cdots$, it follows that $n > r$ and $n' > r$. If $x_{m_2} = 0$ or if the lowest exponent of t in x_{m_2} is greater than 2, let $x_0 = 0 \cdot t$

$+ a_3't^3 + a_4't^4 + \cdots$ (where a_r' is the coefficient of t^r in x_{m_r}). If $x_{m_2} = a_0 t^N + a_1 t^{N+1} + \cdots$ where $N \leqq 2$, let $x_0 = a_0 t^N + a_1 t^{N+1} + \cdots + a_2't^2 + a_3't^3 + a_4't^4 + \cdots$ where a_r' is defined as before. Let y_0 denote the element of T similarly determined by the y_{m_r}.

Part IV.

Let T denote an uncountable, well-ordered sequence of parameters t such that no countable subsequence of T runs through T. If t is an element of T, let $t^0 = 1$. A polynomial with real coefficients in a finite number of the elements of T will be said to be equal to 0 if and only if the expression obtained by replacing the different elements of T occurring in it by independent real variables is identically zero. An expression $at_1^{n_1}t_2^{n_2}\cdots t_s^{n_s}$ (the exponents are positive or zero integers) is said to be *lower* than the expression $bt_1^{m_1}t_2^{m_2}\cdots t_s^{m_s}$ $(a \neq 0 \neq b)$ if the first of the differences $n_s - m_s, n_{s-1} - m_{s-1}, \cdots, n_2 - m_2, n_1 - m_2$ which is not zero is negative. Then a non-zero polynomial with real coefficients in a finite number of elements of T may be represented by $a_1Y_1 + a_2Y_2 + \cdots + a_kY_k$ where the Y_i $(i = 1, 2, \cdots, k)$ represent power products of a finite number of elements of T, Y_i is lower than Y_{i+1} $(0 < i < k)$, and the a_i $(a_1 \neq 0)$ are real numbers. Such a polynomial will be said to be less than zero if $a_1 < 0$ and greater than zero if $a_1 > 0$. If P and Q are such polynomials and $P > 0$ and $Q > 0$, then $PQ > 0$ and $P + Q > 0$, for the lowest term in PQ is the product of the lowest term in P by the lowest term in Q, and the lowest term in $P + Q$ is the lowest term in P, the lowest term in Q, or it is $(a_1 + b_1)Y_1$ where a_1Y_1 is the lowest term in P and b_1Y_1 is the lowest term in Q. The statement $P > Q$ means $P - Q > 0$. Then if F is also a polynomial of the same sort as P and Q, it follows from $P > Q$ and $Q > F$ that $P > F$, for if $P - Q > 0$ and $Q - F > 0$, then $P - F = P - Q + Q - F > 0$.

Let R denote the field composed of all rational expressions r such that r contains only a finite number of elements of T, has real coefficients, and does not involve division by zero. Then an element r of R is either a polynomial or it can be expressed as the quotient of two polynomials of R. The relation "greater than" has already been defined for the polynomials of R. If r is an element of R which is not a polynomial, let $r = P/Q$ where P and Q are polynomials of R, then $r > 0$ will mean either $P > 0$ and $Q > 0$ or $P < 0$ and $Q < 0$; $r = 0$ means $P = 0$ and $Q \neq 0$. If r and r' are elements of R, $r > r'$ means $r - r' > 0$. If r, r', r'' are elements of R, there exists

in R a polynomial $Q > 0$ and polynomials P, P', P'' such that $r = P/Q$, $r' = P'/Q$, and $r'' = P''/Q$. Hence if $r > r'$ and $r' > r''$, then $r > r''$. Let $r' < r$ mean $r > r'$. If r is an element of R and r' is an element of R, $r' \neq r$ means $r - r' \neq 0$. In respect of rational operations and relations *greater than* and *less than* the elements of R satisfy all of the rules 1-16 given by Hilbert on pages 35 and 36 of *Grundlagen der Geometrie*. It is this fact which justifies the statement made below that the plane G, there defined, is a descriptive plane.

Let G denote the plane whose points are all the pairs (x, y) of elements of R, in which a line consists of all the points whose coördinates satisfy an equation of the form $Ax + By + C = 0$ (A, B, C are elements of R and not both A and B are zero), and in which three collinear points N, N', N'' are said to be in the order $NN'N''$ when:

(1) Their abscissas x, x', x'', respectively, are such that $x > x' > x''$ or $x < x' < x''$ if $B \neq 0$ in the equation of the line containing the three points.

(2) Their ordinates y, y', y'', respectively, are such that $y > y' > y''$ or $y < y' < y''$ if $B = 0$. Then G is a descriptive plane, but *G is not metric,* for the segment between the points $(0, 0)$ and $(1, 0)$ contains no countable set of points of which $(0, 0)$ is a limit point. For let $[N]$ denote a countable set of points of this segment and let $[t]_N$ denote the set of all elements t of T such that t occurs in the abscissa of some point of $[N]$, then $[t]_N$ is countable, and, by hypothesis, there exists an element t' of T such that t' follows t in T if t is in $[t]_N$. It follows that t', as an element of R, is less than the abscissa of any point of $[N]$, and hence the point $(t', 0)$ is between $(0, 0)$ and every point of $[N]$.

THE UNIVERSITY OF TEXAS.

CONCERNING METRIC COLLECTIONS OF CONTINUA.†

By J. H. Roberts.‡

The notion *upper semi-continuous collection of bounded continua,* introduced by R. L. Moore,§ has been very fruitful in the realm of analysis situs. In particular, Moore has shown that if G is an upper semi-continuous collection of bounded continua ¶ filling a plane S, then if no element of G separates S the collection G is itself homeomorphic with S.‖ If the restriction that no element of G separates S is removed then G is †† homeomorphic with some cactoid.††

I have attempted to extend the idea *upper semi-continuous collection* to apply to collections whose elements were not all bounded. In my paper ‡‡ "Concerning Collections of Continua not all Bounded," I suggest a definition. I there show that if G is an upper semi-continuous *and metric* collection of continua filling a plane S and no element of G separates S, then G is homeomorphic with a *subset* of some cactoid.§§ In the present paper it is shown that without the restriction that no element of G separates S, G is homeomorphic with a subset of some cactoid.

Definition. If $h_1, h_2, h_3, \cdots$ denotes any sequence of elements of a collection G of continua, and for each i the element h_i contains points P_i and Q_i then, granting that the sequence $P_1, P_2, P_3, \cdots$ has a sequential limit point P in an element g_P of G, the collection G is said to be *upper semi-continuous* provided every limit point of $\sum_{i=1}^{\infty} Q_i$ lies in g_P.

† Presented to the Society, under a different title, June 21, 1929.

‡ National Research Fellow.

§ "Concerning Upper Semi-Continuous Collections of Continua," *Transactions of the American Mathematical Society,* Vol. 27 (1925), pp. 416-428.

¶ Throughout this paper a *point* is considered as a special type of *continuum.*

‖ With respect to a suitable definition of limit element. See Moore, *loc. cit.*

†† Moore, "Concerning Upper Semi-continuous Collections," *Monatshefte für Mathematik und Physik,* Vol. 36 (1929), pp. 81-88. A *cactoid* is a bounded continuous curve M lying in space of three dimensions and such that (a) every non-degenerate maximal cyclic subset of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M.

‡‡ *American Journal of Mathematics,* Vol. 52 (July, 1930). Hereafter this paper will be referred to as C. C.

§§ In fact, except for the case where exactly one element of G is unbounded, G is homeomorphic with a subset of the plane.

Definition. If G is an upper semi-continuous collection of continua, then an element g of G is said to be a *limit element* of a set K of elements of G if g contains a point P which is a limit point of the point set obtained by adding together all elements of K except g.

The preceeding definitions apply to collections all of whose elements lie in a space S for which the notion *limit point* has been defined. Note that the elements of an upper semi-continuous collection of continua are mutually exclusive.

THEOREM 1. *If G is an upper semi-continuous and metric collection of continua filling a continuous curve* † *M, then G is itself a continuous curve.*

Proof. It is easily seen that G is separable, connected, and locally connected. I shall show that it is also locally compact. Let g be any element of G. There exists a sequence of points $P_1, P_2, P_3, \cdots$ lying in g such that every point of g is a limit point of $\sum_{i=1}^{\infty} P_i$. Let d and δ denote distance functions for the spaces M and G, respectively. I shall define, for each i, a number d_i. If the M-domain $S(P_i, 1/2)$ ‡ is compact, then let d_i be $1/2$. Otherwise, let d_i be $1/2$ the upper limit of d where $S(P_i, d)$ is compact. Now let K_n be the point set $\sum_{i=1}^{n} S(P_i, d_i)$, and let R_n be the set of elements of G which contain at least one point of K_n. For every n the set R_n is compact and contains g. If for every n the element g is a limit element of $G - R_n$, then there is a sequence of elements $g_1, g_2, g_3, \cdots$ such that $\delta(g, g_n) < 1/n$, and g_n does not belong to R_n. Then as limit of $\delta(g, g_n) = 0$ there is, for each n, a point Q_n in g_n such that $\sum_{i=1}^{\infty} Q_i$ has a limit point Q in g. Let D be a compact M-domain containing the point Q. Obviously as M is metric there is an integer n such that $S(P_n, d_n)$, and therefore K_n, contains Q. But as K_n is a domain containing a limit point Q it must contain, for some m greater than n, the point Q_m. Then g_m belongs to K_m, and we have reached a contradiction, which shows that for some n the element g is not a limit element of the set $G - R_n$. Thus the compact set R_n contains a domain containing g.

THEOREM 2. *If G is an upper semi-continuous and metric collection of*

† In the present paper a space M is said to be a continuous curve if it is metric, connected, locally connected, separable, and locally compact.

‡ That is, the set of all points of M whose distance from P_i is less than $1/2$.

continua filling a plane S, then each cyclic element † *of the continuous curve G is homeomorphic with a subset of a sphere.*

Proof. Let M be any non-degenerate cyclic element of G. If g is an element of M it follows, since $M - g$ is connected, that just one complementary domain E_g of g in the plane S contains points belonging to elements of $M - g$. Let g^* be $S - E_g$. If g_1 and g_2 are distinct elements of M then $g_1{}^*$ and $g_2{}^*$ are distinct. Let M^* be the collection of continua containing g^* for every element g of M. Now M^* fills the plane S, for if P is a point of S and g_P—the element of G containing P—does not belong to M, then since M is a cyclic element of G, some element g of M separates g_P and $M - g$ in S. Thus g_P is a subset of g^*. It follows that M^* is an upper semi-continuous and metric collection of continua, homeomorphic with M, and such that no element of M^* separates S, and every point of S belongs to some element of M^*. Then by theorems 3 and 4 of C. C. it follows that M^*, and therefore M, is homeomorphic with a subset of a sphere.

THEOREM 3. *If every cyclic element of a continuous curve G is homeomorphic with some subset of a sphere, then G itself is homeomorphic with a subset of some cactoid.*

Proof. Let E_1 and E_2 be distinct cyclic elements of G, and let X be a simple cyclic chain ‡ of G between E_1 and E_2. Let A_1A_2 denote an arc such that A_i lies in E_i, but does not lie in any other cyclic element of G except when E_i is degenerate $(i = 1, 2)$. Let K be the set of points of G which separate A_1 and A_2. Then by a theorem of Whyburn's,§ for every maximal segment s of $A_1A_2 - (K + A_1 + A_2)$ there is a unique cyclic element of G containing s. Let $s_1, s_2, s_3, \cdots$ denote the maximal segments of $A_1A_2 - (K + A_1 + A_2)$. Let π be a continuous one to one correspondence throwing A_1A_2 onto the interval $0 \leqq x \leqq 1$, $y = 0$, $z = 0$. For each i let F_i be a sphere with $\pi(\bar{s}_i)$ as diameter. Then the set M $[M = \sum_{i=1}^{\infty} F_i + (K + A_1 + A_2)]$

† By a *cyclic element* of a continuous curve M is meant (a) a maximal cyclic curve of M, (b) a cut point of M, or (c) an end point of M. See G. T. Whyburn, "Concerning the Structure of a Continuous Curve," *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194.

‡ See Whyburn, *loc. cit.* A point set X is said to be a *simple cyclic chain* of G between E_1 and E_2 provided that X is connected and contains E_1 and E_2 and is the sum of the elements of some collection of the cyclic elements of G, and furthermore no proper connected subset of X containing E_1 and E_2 is the sum of the elements of such a collection.

§ "Some Properties of Continuous Curves," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 305-308, theorem II.

is a cactoid. Moreover, as any two distinct points of a sphere can be thrown into any two distinct points of the sphere by a continuous transformation of the sphere into itself, it follows that the cyclic chain X is homeomorphic with a subset of the cactoid M.

Now it is easily seen that there exists a sequence $Q_1, Q_2, Q_3, \cdots$ of distinct cyclic elements of G such that every point of G is a limit point of $\sum_{i=1}^{\infty} Q_i$. Now for every m and n $(m \neq n)$ G contains a unique simple cyclic chain between Q_m and Q_n. If i, n_i, and m_i are positive integers, let C_i denote the simple cyclic chain of G between Q_{n_i} and Q_{m_i}. Let $n_1 = 1$ and let $m_1 = 2$. For i greater than 1 the integers n_i and m_i will be defined by induction as follows: Having defined integers $n_1, n_2, \cdots, n_i$ and $m_1, m_2, \cdots, m_i$ $(i \geqq 1)$ let n_{i+1} be the smallest integer k such that Q_k does not belong to $\sum_{j=1}^{i} C_j$. Let m_{i+1} be the integer k such that Q_k is the first element of the chain from $Q_{n_{i+1}}$ to Q_{n_1} which belongs to the set $\sum_{j=1}^{i} C_j$. Clearly every point of G not an end point belongs, for some i, to the chain C_i. It has been shown that every simple cyclic chain of G is homeomorphic with a subset of some cactoid.

It can be shown that there exist cactoids $D_1, D_2, D_3, \cdots$, and correspondences $\pi_1, \pi_2, \pi_3, \cdots$ such that (1) π_i is a continuous one to one correspondence between $\sum_{j=1}^{i} C_j$ and a subset of D_i, (2) if P is a point of C_i then $\pi_i(P) = \pi_{i+1}(P) = \pi_{i+2}(P) = \cdots$, (3) the cactoid D_i is a subset of D_{i+1}, (4) for every i and k $(i < k)$ the components of $D_k - D_i$ can be ordered $H_1, H_2, \cdots, H_n$ so that if P_j $(j \leqq n)$ denotes the limit point of H_j which belongs to D_i, and K_j denotes the sum of the sets $\bar{H}_1, \bar{H}_2, \cdots, \bar{H}_{j-1}$ which do not contain P_j, then the diameter of H_j is less than the smaller of the numbers $1/4^i$ and $d(P_j, K_j)/4^i$, and (5) if $i < j < k$, H is a component of $D_k - D_j$, and P is the limit point of H which belongs to D_j, then if P is not in D_i the diameter of H is less than $d(P, D_i)/4^i$.

Let D denote $\sum_{i=1}^{\infty} D_i$. From property 4 above it follows that the continuum D is a continuous curve. If T is a point which belongs to D_i and is a limit point of $D_{i+1} - D_i$, then for every k $(k > i)$ let g_{Tk} denote any component of $D_k - D_i$ with T as limit point. Then (see properties 4 and 5) the set $g_{Tk} + g_{T(k+1)} + g_{T(k+2)} + \cdots$ has no limit point in D_i except T. Moreover, if T_1 and T_2 are distinct points which belong to D_i and are limit points of $D_k - D_i$ $(k > i)$ then no point is a limit point of both the sets $g_{T_1k} + g_{T_1(k+1)} + g_{T_1(k+2)} + \cdots$ and $g_{T_2k} + g_{T_2(k+1)} + g_{T_2(k+2)} + \cdots$. It fol-

lows that every simple closed curve belonging to $\bar{D}$ belongs, for some i, to some topological sphere of the cactoid D_i. Obviously no point lies within any sphere of $\bar{D}$. Thus $\bar{D}$ is a cactoid.

Let P be a point of G not belonging, for any i, to C_i, and let P_1, P_2, $P_3, \cdots$ denote a sequence of points converging to P such that, for each i, P_i belongs to $\sum_{j=1}^{i} C_j$. Obviously for every i the points of a subsequence V of $P_1, P_2, P_3, \cdots$, where V contains all but a finite number of points of the original sequence, belong to a single component of $G - \sum_{j=1}^{i} C_j$. Then the sequence $\pi_1(P_1)$, $\pi_2(P_2)$, $\pi_3(P_3), \cdots$ converges to a point P'. Moreover P' is uniquely determined by the point P. If P is a point of C_i then let P' be $\pi_i(P)$. Thus for every point P of G there has been defined a point P' belonging to $\bar{D}$. The correspondence π which is such that $\pi(P) = P'$ is a continuous one to one correspondence between G and a subset of the cactoid $\bar{D}$. Thus theorem 3 is established.

COROLLARY. *If G is any upper semi-continuous and metric collection of continua filling a plane, then G is homeomorphic with a subset of some cactoid.*

THEOREM 4. *If G is an upper semi-continuous and metric collection of continua filling a plane S, and is such that* (1) *no element of G which is a bounded continuum in S separates S, and* (2) *every maximal domain D of bounded elements of G has at least two elements of G on its boundary, then G is homeomorphic with a subset of the plane.*

Proof. From theorem 1 it follows that the hyperspace G is a continuous curve each of whose cyclic elements is homeomorphic with a subset of a sphere. But in view of the hypotheses of the present theorem, and theorem 3 of C. C., it is clear that no cyclic element of the hyperspace G is a sphere. Thus each cyclic element of G is homeomorphic with a subset of a plane. Now no element g of G which is a bounded continuum in S separates S. Then g does not separate G. Hence if M is a cyclic element of G, those elements of M which are limit elements of $G - M$ must be unbounded continua in S. But a correspondence π throwing M into a set as described in theorem 4 of C. C. throws every unbounded element of G into a point which is arcwise accessible from the complement of $\pi(M)$. Thus theorem 4 can be proved by a modification of the proof of theorem 3.

THE UNIVERSITY OF PENNSYLVANIA.

THE CYCLIC AND HIGHER CONNECTIVITY OF LOCALLY CONNECTED SPACES.

BY G. T. WHYBURN.

1. *Introduction.* In this paper we shall consider connected and locally connected, separable metric spaces which we shall denote by the letter M. By a *region* in such a space is meant any connected open set of points. The point set $M - \bar{R}$ will be called the exterior of the region R. A region will be said to *join* two point sets A and B provided that $A \cdot R \neq 0 \neq B \cdot R$. Two regions R and S are said to be *strongly separated* provided that $\bar{R} \cdot \bar{S} = 0$. The point p is a cut point of a space M provided $M - p$ is not connected. A point x is called an end point of M provided there exists a monotonic decreasing sequence $[U_i]$ of neighborhoods of x, such that the boundary of U_i is a single point and such that $x = \prod_1^\infty \bar{U}_i$. A subset N of M is called a *nodule* of M if N is a connected subset of M which contains more than one point, has no cut point and is saturated in M with respect to these properties. The cut points, end-points, and nodules of a space M are called the *nodular elements* of M.*

If R is a region, $F(R)$ will denote the boundary of R, i. e., the set of points $\bar{R} - R$. For each point p and each positive number r, $S_r(p)$ and $V_r(p)$ will denote respectively the set of all points whose distances from p are equal to and less than r; $U_r(p)$ will denote the component of $V_r(p)$ which contains p.

This paper is devoted to the study of what might be termed, broadly speaking, the connectivity properties of spaces M and their relation to similar known properties of continuous curves, i. e., spaces M which are locally compact. The results found effect, in many cases, real generalizations of the known connectivity properties of continuous curves to the more general spaces M. Before proceeding with the theorems, however, we give in the next section an example which illustrates the complexity which may arise even in a greatly restricted space M.

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* [illegible] M [illegible] their nodular elements, see the author's paper in the *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 926–943.

2. *Example.* There exists a space M which has no cut point, is both an absolute G_δ* and an absolute F_σ*, and yet is not cyclicly connected.*

Let a and b respectively denote the points $(1, 0)$ and $(-1, 0)$, and let ab denote the interval $(-1, 1)$. For each positive integer n let C_n denote the semi-circle $y = [(1 - 1/n)^2 - x^2]^{1/2}$. Let $\mathfrak{M} = ab + \sum_1^\infty C_n$. Then clearly $\mathfrak{M}$ has all the desired properties and yet a and b lie together on no simple closed curve in $\mathfrak{M}$.

Thus it is seen that although the property of arcwise connectivity of continuous curves extends † also to spaces M which are absolute G_δ's, the cyclic connectivity property of continuous curves without cut points does not extend to absolute G_δ spaces M without cut points, even though M is at the same time an absolute F_σ.

It is interesting to note that even in this example $\mathfrak{M}$ there does exist two mutually exclusive regions R and S such that $\bar{R} \cdot \bar{S} = a + b$.

3. *The quasi-divisibility of spaces M.*

(3.1). Theorem. *If p is any non-cut point of M, there exists a monotone decreasing sequence $R_1, R_2, R_3, \cdots$ of regions such that $p = \prod_1^\infty \bar{R}_i$, $\lim_{i\to\infty} \delta[p + F(R_i)] = 0$ and for each i the exterior G_i of R_i is connected and $F(G_i) = F(R_i)$.*

Proof. Let q be any point of $M - p$. For each $r < \rho(p, q)$, let $G_r(q)$ be the component of $M - \overline{U(p)}$ containing q, let $X_r = F[G_r(q)]$, and let $R_r(p)$ be the component of $M - X_r$ containing p. Then clearly we have

$$\text{(i)} \qquad X_r = F[G_r(q)] = F[R_r(p)] \subset F[U_r(p)] \subset S_r(p).$$

Now if $r_1 < r_2 < \rho(p, q)$, we have

$$\text{(ii)} \qquad G_{r_2}(q) + X_{r_2} = \overline{G_{r_2}(q)} \subset G_{r_1}(q),$$

and therefore

$$\text{(iii)} \qquad R_{r_1}(p) \subset R_{r_2}(p),$$

for $R_{r_1}(p) \cdot X_{r_2} = 0$ by (ii).

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* A G_δ set is one which is the product of some family of open sets. An F_σ set is one which is the sum of a countable number of closed sets. A G_δ or F_σ set is called *absolute* provided it is a G_δ or F_σ in every metric space in which it is topologically contained. A space M is said to be *cyclicly connected* if every two points of M lie together on a simple closed curve of M.

† See R. L. Moore, *Bulletin of the American Mathematical Society*, abstract, Vol. 33 (1927), p. 141; also K. Menger, *Monatshefte für Mathematik und Physik*, Vol. 36 (1930), p. 210.

Now for all save a countable number of r's, we have $R_r(p) + X_r + G_r(q) = M$. For let E be the set of all numbers $e < \rho(p, q)$ such that a component D_e of $M - X_e$ other than $R_e(p)$ and $G_e(q)$ exists. Then for $e_1 < e_2$ it follows by (ii) and (iii) since $F[D_{e_2}] \subset X_{e_2}$ that $D_{e_2} \subset G_{e_1}(q)$ and therefore that $D_{e_1} \cdot D_{e_2} = 0$. Hence E must be countable, for the sets D_e are open. Thus there exists a sequence of numbers $r_1, r_2, r_3, \cdots$, converging monotonically to zero such that for each i, $M = R_{r_i}(p) + X_{r_i} + G_{r_i}(q)$. Now if x is any point of $M - p$, there exists a connected set I containing $q + x$ such that $\bar{I} \cdot p = 0$, for p is not a cut point of M. Hence there exists an integer i such that $q + x \subset I \subset G_{r_i}$. Therefore $p = \prod_1^\infty \bar{R}_{r_i}(p)$. Thus if for each i, we set $R_i = R_{r_i}(p)$, $G_i = G_{r_i}(q)$, the sets $[R_i]$ and $[G_i]$ have all the necessary properties for the theorem. $\lim_{i \to \infty} \delta[p + F(R_i)] = 0$, because

$$p + F(R_i) = p + X_{r_i} \subset \overline{V_{r_i}(p)} \text{ and } \delta[V_{r_i}(p)] \leq 2r_i.$$

COROLLARY (3.1a). *If H is the sum of a finite number of connected sets and $\bar{H} \cdot p = 0$, then there exists an integer i such that $\bar{H} \subset G_i$ and hence $\bar{H} \cdot \bar{R}_i = 0$.*

COROLLARY (3.1b). *The regions $R_1 = R_{r_1}(p)$, $R_2 = R_{r_2}(p), \cdots$, may be chosen so that for every n, $| r_n - 1/n | < 1/n^2$; or more generally, if $a_1, a_2, \cdots$ is any sequence of numbers $< \rho(p, q)$ which converges monotonically to zero, then, so that for each n, $| r_n - a_n | < 1/n^2$.*

COROLLARY (3.1c). *If M is locally compact, then each non-cut point of M is contained in arbitrarily small regions whose exteriors (also whose complements) are connected.**

Note. Corollary (3.1a) yields a very simple proof for the theorem † that the set G of all non-cut points of a space M is a G_δ set (or that the set F of all cut points is an F_σ set). For let $\sum_1^\infty p_i$ be a countable set of points dense in M. For each $n > 0$, let G_n be the set of all points x of M such that x lies in some region R whose exterior is connected and contains every point

* For theorems closely related to this corollary, see H. M. Gehman, *Proceedings of the National Academy of Sciences*, Vol. 14 (1928), pp. 431-433, and W. L. Ayres, *Monatshefte für Mathematik und Physik*, Vol. 36 (1929), pp. 139-140. Indeed, the two theorems in this section of the present paper may be regarded as generalizations [illegible]

† See G. T. Whyburn, *loc. cit.*, result (1.3). For the case of compact spaces M, see Zarankiewicz, *Fundamenta Mathematicae*, Vol. 9, p. 163. For a simple proof of Zarankiewicz's theorem, see W. L. Ayres, *Fundamenta Mathematicae*, Vol. 16.

except possibly one of $\sum_1^n p_i$. Then clearly G_n is open, and it follows at once from Corollary (3.1a) that $G = \prod_1^\infty G_n$.

(3.2) THEOREM. *If p is any point of a space M, there exists a monotone decreasing sequence $R_1, R_2, \cdots$ of regions such that $p = \prod_1^\infty \bar{R}_i$, $\lim_{i\to\infty} \delta[p + F(R_i)] = 0$ and for each i the exterior G_i of R_i is the sum of a finite number, n_i, of connected sets and $F(G_i) = F(R_i)$. Furthermore n_i does not exceed either i or the number of the components of $M - p$, and no two of these n_i sets lie together in the same component of $M - p$.*

Proof. Let $M_1, M_2, M_3, \cdots$ be the components of $M - p$. For each n the set $M_n + p$ is connected and locally connected and has p for a non-cut point. By virtue of (3.1) and its proof it follows that there exists a sequence of number $d_1, d_2, \cdots$ converging monotonically to zero and for each n there exists a sequence of regions (relative to $M_n + p$) $R_1^n, R_2^n, \cdots$ such that $p = \prod_{i=1}^\infty \bar{R}_i^n$ and for each i the exterior, G_i^n, in $M_n + p$ of R_i^n is connected and the boundary $X^n_{d_i}$ of R_i^n and G_i^n is a subset of $S_{d_i}(p)$. Now set $R_1 = M - \bar{G}_1^1$, $R_2 = M - (\bar{G}_2^1 + \bar{G}_2^2), \cdots$, $R_n = M - \sum_{i=1}^n \bar{G}_n^i, \cdots$, it being understood that if there are only a finite number, k, of components of $M - p$, then for all n's $> k$, and all i's, $M_n = R_i^n = G_i^n = X_{d_i}^n = 0$. Then for each n, $F(R_n) = \sum_{i=1}^n X_{d_n}^i \subset S_{d_n}(p)$ and $M - \bar{R}_n = G_n = \sum_{i=1}^n G_n^i$; and it follows at once that the sets $R_1, R_2, \cdots$ satisfy all the required conditions in the theorem.

COROLLARY (3.2). *If p is any point of a space M and H is a closed subset of $M - p$ which is the sum of a finite number of connected sets, then there exists a region R containing p whose exterior contains H and is the sum of a finite number of connected sets.*

4. *Locally divisible spaces M.*

Definition. A space M will be said to be *locally divisible in the point p* provided that p is contained in arbitrarily small regions whose exteriors are connected or are the sum of a finite number of connected sets according as p is a non-cut point or a cut point of M; a space which is locally divisible in each of its points will be said to be *locally divisible.*†

† It should be noted that this notion is a localization of a stronger property than that of *divisibility* as introduced by W. A. Wilson. See Wilson, *Bulletin of the American Mathematical Society*, Vol. 36 (1930), p. 85.

Examples are easily constructed of spaces M which are not locally divisible—indeed which are not locally divisible in any one of their points.

(4. 1) THEOREM. *In order that M be locally divisible in p it is necessary and sufficient that p be contained in arbitrarily small regions whose complements are connected or are the sum of a finite number of connected sets.*

(4. 2) THEOREM. *Any space M may be transformed by a biunivalued and continuous transformation into a connected, locally connected, separable metric space M^* which is locally divisible.*

Proof. Let G denote the collection of all regions in M whose exteriors and whose complements are the sum of a finite number of connected sets. Let M^* denote the space whose points are exactly the points of M but in which limit point is defined by means of the system of neighborhoods G, i. e., the point p^* of M^* is a limit point of the point set N^* in M^* if and only if every region of the collection G which contains p^* contains at least one point of N^* distinct from p^*. Now since every set E^* in M^* which is open in M^* is also an open set E in M, it follows that M^* may be regarded as the image of M under a biunivalued and continuous transformation T. It remains to show that M^* is connected, locally connected, separable, metric, and locally divisible. Now M^* is connected, locally connected and separable because it is the image under T of M, and M has these properties. To show that M^* is metric we need only prove that it is regular and perfectly separable. It is regular, because if p^* is any point of M^* and U^* is any neighborhood of p^*, there exists a region E^* of G containing p^* and lying in U^*. Since $M - E$ is the sum of a finite number of connected sets, there exists, by corollary (3. 2), a region V of G containing p whose exterior contains $M - E$. Therefore, $\bar{V} \subset E$; whence $\bar{V}^* \subset E^*$, which proves M^* regular. To show that M^* is perfectly separable, let $p_1, p_2, p_3, \cdots p_n \cdots$ be a countable sequence of points dense in M. Arrange the positive rational number into a sequence $r_1, r_2, r_3, \cdots r_i \cdots$. For each n and each i, arrange all possible point sets Q such that Q is the sum of a finite number of the components of $M - \overline{V_{r_i}(p_n)}$ into a sequence $Q_1(i,n), Q_2(i,n), \cdots, Q_j(i,n), \cdots$. For each j, i, and n and for each point x of $U_{r_i}(p_n)$, there exists, by corollary (3. 2), a region G_x of the collection G containing x but containing no point of $Q_j(i,n)$. By the Lindelöf theorem there exists a countable collection $[G_k(j,i,n)]_{k=1}^{\infty}$ of the [illegible] whose sum covers $U_{r_i}(p_n)$. Let E [illegible] the collection [illegible] $[G_k(j,i,n)]$, for all n's, i's, j's and k's. Then E is countable and [illegible] equivalent to the system G. For let p be any point and let R_0 be any region of G

containing p. There exists an i such that $V_{4r_i}(p) \subset R_g$. There exists an n such that $p_n \subset U_{r_i}(p)$ and hence such that

$$U_{r_i}(p) \subset U_{2r_i}(p_n) \subset V_{2r_i}(p_n) \subset V_{4r_i}(p) \subset R_g.$$

Then since $M - R_g$ is the sum of a finite number of connected sets, there exists a j such that $Q_j(i', n) \supset M - R_g$, where i' is an integer such that $r_{i'} = 2r_i$. There exists a k so that $G_k(j, i', n)$ contains p but contains no point of $Q_j(i', n)$ and hence no point of $M - R_g$. Thus we have

$$p \subset G_k(j, i', n) \subset R_g,$$

which proves E equivalent to G, inasmuch as E is a subcollection of G. Therefore M^* is a perfectly separable, regular, Hausdorff space and is then metrisable by the theorem of Alexandroff–Urysohn–Tychonoff.† We suppose a distance function defined in M^*. Clearly M is locally divisible, because for each ϵ the set $V^*_\epsilon(p^*) = V^*$ is open in M^* for each point p^* in M^*; and thus there exists a region R_g of the collection G which contains p and lies in V. Hence $p^* \subset R^*_g \subset V^*_\epsilon(p^*)$. Whence $\delta(R^*_g) < 2\epsilon$, and $M^* - \bar{R}^*_g$ is the sum of a finite number of connected sets.

5. *Local end points.*

Definition. A point p which is an end point of at least one region in M is called a *local end point* of M. It is obvious that if M is compact, its local end points are identical with its end points. For general M-spaces, however, this is not true. For example, the points a and b of the space $\mathfrak{M}$ described in § 3 are local end points but not end points of $\mathfrak{M}$.

(5.1) LEMMA. *If in a space M having no cut point, A, B_1 and B_2 are mutually exclusive closed sets such that A is non-degenerate, and there exist regions G_1 and G_2 joining A and B_1, and A and B_2 respectively such that $G_1 \cdot B_2 = G_2 \cdot B_1 = 0$, then there exist two strongly separated regions R_α and R_β joining A and B_1, and A and B_2 respectively, such that $R_\alpha \cdot B_2 = R_\beta \cdot B_1 = 0$.*

Proof. Let S_1 denote the set of all points x of M such that strongly separated regions R_a and R_x exist joining A and B_1, and x and B_2 respectively, such that $R_a \cdot B_2 = R_x \cdot B_1 = 0$; similarly let S_2 be the set of all points x of M such that strongly separated regions R_a and R_x exist joining A and B_2, and x and B_1 respectively, and such that $R_a \cdot B_1 = R_x \cdot B_2 = 0$. It follows at once from our hypothesis that the sets S_1 and S_2 are non-vacuous and contain B_1 and B_2 respectively, and that their sum S is an open set. We

† See articles by these authors in *Mathematische Annalen*, Vols. 92-95.

shall show that $S = M$. If this is not so, then there exists at least one point y of $M - S$ which is a limit point of S. Since y is not a cut point of M, since A contains more than one point, and y does not belong to $B_1 + B_2$, it follows that there exist strongly separated regions R^* and G such that R^* joins A and $B_1 + B_2$ and G contains y but $\bar{G} \cdot (B_1 + B_2) = 0$. The region G contains at least one point x of S, and thus there exist strongly separated regions R_a and R_x such that either R_a joins A and B_1 but contains no point of B_2 and R_x joins x and B_2 and contains no point of B_1, or R_a joins A and B_2, R_x joins x and B_1, and $R_a \cdot B_1 = R_x \cdot B_2 = 0$. The two cases are entirely alike, so we shall suppose the former. Let R_1 and R_2, respectively, denote components of $R_a - R_a \cdot \bar{G}$ and $R_x - R_x \cdot \bar{G}$ containing points of B_1 and B_2 respectively. Then since y does not belong to S, it follows at once that $\bar{R}_1 \cdot \bar{G} \neq 0 \neq \bar{R}_2 \cdot \bar{G}$. Let $K = \bar{R}_1 + \bar{R}_2 + B_1 + B_2$, and let R denote a component of $R^* - R^* \cdot K$ which contains at least one point of A. Then K contains at least one limit point p of R. Either $p \subset \bar{R}_1 + B_1$ or $p \subset \bar{R}_2 + B_2$. Here again the two cases are alike, so we shall treat only the former. Let q denote a point of $\bar{R}_2 \cdot \bar{G}$, and let U_p and U_q be strongly separated regions, containing p and q respectively, such that $\bar{U}_p \cdot (\bar{R}_2 + \bar{B}_2 + \bar{G}) = 0$ and $\bar{U}_q \cdot (\bar{R}_1 + B_1 + \bar{R}) = 0$. Clearly $R + U_p + R_1$ contains a region W joining A and B_1, and $R_2 + U_q + G$ is a region, say Q, joining y and B_2 and we have $W \cdot Q = 0$, $W \cdot B_2 = Q \cdot B_1 = 0$. Now obviously W contains a region V joining A and B_1 and such that $\bar{V} \subset W$. But then Q and V are strongly separated and join A and B_1, and y and B_2, respectively, and $V \cdot B_2 = Q \cdot B_1 = 0$, contrary to the fact that y does not belong to S. Thus the supposition that $S \neq M$ leads to a contradiction. Accordingly, S contains a point s of A, and thus there exist two strongly separated regions $R_a = R_\alpha$ and $R_x = R_\beta$ joining A and B_1, and A and B_2, respectively, such that $R_\alpha \cdot B_2 = R_\beta \cdot B_1 = 0$.

Corollary (5.1a). *If the space M as in the Lemma is imbedded in a space M_0, also an M-space, and if $B_1{}^0$ and $B_2{}^0$ are subsets of M_0 such that $\bar{B}_1{}^0 \cdot M = B_1$ and $\bar{B}_2{}^0 \cdot M = B_2$ and ϵ is any positive number, then there exist strongly separated regions R_1 and R_2 in M_0 joining A and B_1, and A and B_2, respectively, such that $R_1 \cdot B_2{}^0 = R_2 \cdot B_1{}^0 = 0$ and every point of $R_1 + R_2$ is at a distance $< \epsilon$ from some point of M.*

Corollary (5.1b). *If H and K are non-degenerate mutually exclusive subsets of a space M which has no cut point, then there exist two strongly* [illegible]

(5.2). Theorem. *If the point p is not a local cut point of a space M,*

then there exist two regions R_1 *and* R_2 *such that* $R_1 \cdot R_2 = p$ *and the sets* $R_1 + p$ *and* $R_2 + p$ *are connected and locally connected.*

Proof. We may suppose p is neither a cut point nor a local separating point of M, for in these cases the theorem is obvious. Hence there exists a nodule N of M containing p. Let a and b be any two points of $N - p$ and let U_1 denote the component containing p of $N \cdot V_{\epsilon_1}(p)$, where $\epsilon_1 < 1$ and also $\epsilon_1 < \rho(p, a + b)$. Likewise, we may suppose p is neither a cut point nor an end point of U_1, for it is seen at once that if a point p is a local end point of some nodule of a given space M, then p is a local end point of the whole space. Hence there exists a nodule N_1 of U_1 which contains p. Since N has no cut point, then by virtue of the preceding lemma and its corollaries there exist two strongly separated regions S^*_a and S^*_b in M joining a and N_1, and b and N_1, respectively. Let S_a and S_b, respectively, denote the components of $S^*_a - \bar{N}_1 \cdot S^*_a$ and $S^*_b - \bar{N}_1 \cdot S^*_b$ which contain a and b respectively. Now in case $\bar{S}_a \cdot N_1$ and $\bar{S}_b \cdot N_1$ or either is non-vacuous, then let A_1 and B_1 respectively denote single points of these sets. All cases which may arise here, however, are only simpler than that in which these two sets are vacuous; hence we shall suppose that $\bar{S}_a \cdot N_1 = \bar{S}_b \cdot N_1 = 0$. Then let a_1 and b_1 respectively denote points of $\bar{S}_a \cdot \bar{N}_1$ and $\bar{S}_b \cdot \bar{N}_1$; and let G_{a_1} and G_{b_1} denote strongly separated regions in M of diameter $< 1/2$ containing a_1 and b_1 respectively and such that $\bar{G}_{a_1} \cdot (\bar{S}_b + p) = \bar{G}_{b_1} \cdot (\bar{S}_a + p) = 0$. Let U_2 be a region containing p and of diameter $< 1/2$ such that $\bar{U}_2 \cdot \overline{(S_a + S_b + G_{a_1} + G_{b_1})} = 0$, and let N_2 be the (necessarily non-degenerate) nodule of U_2 which contains p. Now let N^*_1 be the nodule of the set $N_1 + G_{a_1} + G_{b_1}$ which contains N_1 and let $A_1 = \bar{S}_a \cdot N^*_1$ and $B_1 = \bar{S}_b \cdot N^*_1$. Let $A_1^0 = S_a$, $B_1^0 = S_b$. Then A_1 and B_1 are mutually exclusive closed subsets of N^*_1 and they contain a_1 and b_1 respectively but contain no points of $\bar{N}_2$. Also A_1 does not separate N_2 and B_1 in N^*_1 nor does B_1 separate N_2 and A_1 in N^*_1; for there exists a point x of N_1 in G_{b_1} which can be joined to N_2 by a region R in N_1 having no limit points in A_1, and hence $R + G_{b_1} \cdot N^*_1$ is a connected set in N^*_1 joining N_2 and B_1 and having no limit points in A_1, and similarly for the sets N_2 and A_1. Hence by the lemma and corollary (5.2b), there exist in M two strongly separated regions $S^*_{a_1}$ and $S^*_{b_1}$ joining N_2 and A_1, and N_2 and B_1, respectively, and such that $B_1^0 \cdot \bar{S}^*_{a_1} = A_1^0 \cdot S^*_{b_1} = 0$, and such that every point of these two regions is at a distance $< 1/2$ from some point of N^*_1, and hence so that $\bar{S}^*_{a_1} \cdot \bar{S}_b = \bar{S}^*_{b_1} \cdot \bar{S}_a = 0$. Let S_{a_1} and S_{b_1}, respectively, denote components of $S^*_{a_1} - \bar{N}_2 \cdot S_{a_1}$ and $S^*_{b_1} - \bar{N}_2 \cdot S^*_{b_1}$ containing points of A_1 and B_1 respectively. Again we shall suppose, without real loss of generality, that $\bar{S}_{a_1} \cdot N_2 = \bar{S}_{b_1} \cdot N_2 = 0$. Let a_2 and b_2 be points of $\bar{S}_{a_1} \cdot \bar{N}_2$ and $\bar{S}_{b_1} \cdot \bar{N}_2$ respec-

and b respectively, each of diameter $< 1/3$, and such that

$$\bar{G}_{a_2} \cdot (\bar{S}_a + \bar{S}_{a_1} ...) = \bar{G}_{b_2} \cdot (\bar{S}_b + \bar{S}_{b_1} ...) = 0.$$

Let U_3 be a region of diameter $< 1/3$ containing p and such that

$$\bar{U}_3 \cdot (\bar{S}_a + \bar{S}_{a_1} + \bar{S}_b + \bar{S}_{b_1} + \bar{G}_{a_2} + \bar{G}_{b_2}) = 0.$$

and let N_3 be the nodule of U_3 which contains p. Then just as above it follows that there exist in M two strongly separated regions $S^*_{a_2}$ and $S^*_{b_2}$ joining S_{a_1} and N_3, and S_{b_1} and N_3, respectively, and such that

$$\bar{S}^*_{a_2} \cdot (\overline{S_b + S_{b_1}}) = \bar{S}^*_{b_2} \cdot (\overline{S_a + S_{a_1}}) = 0,$$

and every point of $S^*_{a_2} + S^*_{b_2}$ is at a distance $< 1/3$ from some point of N^*_2, where N^*_2 is the nodule of the set $N_1 + G_{a_2} + G_{b_2}$ containing N_2. Let S_{a_2} and S_{b_2} be components, $\cdots$, and so on. Continue this process indefinitely. Let $R_1 = S_a + \sum\limits_1^\infty S_{a_n}$ and $R_2 = S_b + \sum\limits_1^\infty S_{b_n}$. Then R_1 and R_2 have the desired properties for the theorem. For, every point of $\sum\limits_n^\infty (S_{a_i} + S_{b_i})$ is at a distance $\leq 3/n$ from p and hence it follows that $\bar{R}_1 \cdot \bar{R}_2 = p$ and that $R_1 + p$ and $R_2 + p$ are locally connected.

COROLLARY (5.2). *In order that the point p of a space M should be a local end point of M, any one of the following conditions is necessary and sufficient:*

1) *that there should not exist two regions R_1 and R_2 such that $R_1 \cdot \bar{R}_2 = p$ and $R_1 + p$ and $R_2 + p$ are locally connected.*

2) *that there should not exist two mutually exclusive simply infinite chains† of regions each converging to p.*

3) *that p should not be a cut point of any connected and locally connected subset of M.*

6. *A Characterization of end points.*

Definition. If a and p are points of a space M, the subset C of M will

(6.1). Lemma. *If p is any point of a space M and a is any point of $M - p$, there exists a simply infinite chain of regions from a converging to p.*

Proof. Let N_1 denote the component of $M - p$ containing a. Let $d_1 = 1/2\,\rho(p, a)$, and let x_1 be a point of the set $U_{d_1}(p) \cdot N_1$. Since p is a non-cut point of $N_1 + p$, there exists a region R_1 such that $a + x_1 \subset R_1 \subset N_1$ and, $p \cdot \bar{R}_1 = 0$. Let $d_2 = 1/2\,\rho(p, \bar{R}_1)$, and let x_2 be a point of $U_{d_2}(p) \cdot N_2$, where N_2 is the component of $U_{d_1}(p) \cdot N_1$ containing x_1. Likewise there exists a region R_2 such that $x_1 + x_2 \subset R_2 \subset N_2$ and $p \cdot \bar{R}_2 = 0$. Let $d_3 = 1/2\,\rho(p, \bar{R}_2)$ and let x_3 be a point of $U_{d_3}(p) \cdot N_3$, where N_3 is the component of $U_{d_2}(p)$ containing x_2. There exists a region R_3 such that $x_2 + x_3 \subset R_3 \subset N_3$ and $p \cdot \bar{R}_3 = 0$, and so on. Continuing this process indefinitely, it is clear that the set $C = R_1 + R_2 + \cdots$ is the desired chain of regions from a converging to p. For, $\delta(p + \sum_n^\infty R_i) \leq \delta(p + N_n) \leq \delta[U_{d_n}(p)] = 2d_n \leq d_1/2^{n-1}$, and if $j - i > 1$, $N_j \supset R_j$ but $N_j \subset U_{d_{j-1}}(p)$, therefore $N_j \cdot R_i = 0$.

(6.2) Theorem. *If the point p of the space M is not an end point of M, then there exist two regions R_1 and R_2 such that $\bar{R}_1 \cdot \bar{R}_2 = p$.*

Proof. If p is not a local end point, the theorem is obvious in view of proposition (5.2). Thus we may suppose p is a local end point but is neither an end point nor a cut point of M. Hence there exists a nodule N of M containing p, and p is a local end point of N. We now consider N as the space.

First we treat the case where N is locally divisible in p. There exists an $\epsilon > 0$ such that p is an end point of the component U_0 of $V_\epsilon(p)$ containing p. There exists a region U such that p is an end point of U and the exterior and complement of U are connected and $\bar{U} \subset U_0$. There exists a region R_1 which is a subset of U and is the sum of a simply infinite chain of regions $C_1, C_2, \cdots$ converging to p, so that $R_1 + p$ is locally connected at p, and also so that $\bar{R}_1 \subset U$. For each i, let x_i be a point of $C_i \cdot C_{i+1}$ and let $B_i = \sum_i^\infty C_j$. There exists a monotone decreasing sequence $G_1, G_2, \cdots$ of regions, subsets of U_0, such that $p = \prod_1^\infty \bar{G}_i \cdot U_0$, and for each i, $F(G_i) \cdot U_0 = p_i$, a single point. There exists an n_1 such that $G_{n_1} + p_{n_1}$ does not contain x_1. Since $F(G_{n_1}) \cdot F(U_0)$ is necessarily $\neq 0$, for p_{n_1} cannot cut N, it follows that $G_{n_1} \cdot F(U) \neq 0$. Let y_1 be a point of $G_{n_1} \cdot F(U)$. There exists an $n_2 > n_1$ such that $G_{n_2} + p_{n_2}$ does not contain y_1 or x_2. Let $E_0 = N - \bar{U}$, let K_1 be the component of $G_{n_1} - G_{n_1} \cdot S_1(\bar{B}_1)$ containing y_1, where we suppose our unit 1 so chosen that $\rho(y_1, \bar{B}_1) > 1$; and let $E_1 = E_0 + K_1$. Just as above

it follows that $G_{n_2} \cdot F(U) \supset$ some point y_2. There exists an $n_3 > n_2$ such that $G_{n_3} + p_{n_3}$ does not contain y_2 or x_3. Let K_2 be the component of $G_{n_2} - G_{n_2} \cdot S_{1/2}(\bar{B}_2)$ containing y_2 and let $E_2 = E_1 + K_2$. Let y_3 be a point of $G_{n_3} \cdot F(U)$. There exists an $n_4 > n_3$ such that G_{n_4} does not contain y_3 or x_4. Let K_3 be the component of $G_{n_3} - G_{n_3} \cdot S_{1/3}(\bar{B}_3)$ containing y_3 and let $E_3 = E_2 + K_3$, and so on. Continue this process indefinitely. Then since for each i, $p_{n_i} \subset B_{i+1}$ (for p_{n_i} separates x_i and p in U_0), it follows that for each i, $K_i \cdot G_{n_{i+1}} = 0$; and hence $K_i \cdot K_j = 0$ for $i \neq j$, and also $\bar{K}_i \cdot S_{1/i}(\bar{B}_i) \neq 0$. Hence if $R_2 = \sum_1^\infty E_n = E_0 + \sum_1^\infty K_i$, it is seen at once that R_2 is a region (since E_0 is connected) and that $\bar{R}_2 \supset p$ but $R_2 \cdot R_1 = 0$. It remains to show that $\bar{R}_1 \cdot \bar{R}_2 = p$. Now for each i, $F(E_i) \cdot \bar{R}_1 = 0$ and hence $\bar{R}_1 \cdot \sum_1^\infty F(E_i) = 0$. Now

$$F(R_2) \subset \sum_1^\infty F(E_i) + \prod_1^\infty \bar{G}_{n_i}, \quad \text{for} \quad E_i - E_{i-1} \subset K_i \subset G_{n_i}.$$

Hence

$$\bar{R}_1 \cdot F(R_2) \subset \bar{R}_1 \cdot \sum_1^\infty F(E_i) + \bar{R}_1 \cdot \prod_1^\infty \bar{G}_{n_i} \subset 0 + U \cdot \prod_1^\infty \bar{G}_{n_i} = p,$$

and therefore $\bar{R}_1 \cdot \bar{R}_2 = p$, for $R_1 \cdot R_2 = 0$.

Now in case N is not locally divisible in p, then by Theorem (4.2), there exists a biunivalued and continuous transformation T of N into a set N^* which is locally divisible in $p^* = T(p)$, and furthermore such that every region in N^* whose complement and exterior are connected is the image of a region in N. Now by virtue of the case treated above, there exist two regions R^*_1 and R^*_2 in N^* such that $\bar{R}^*_1 \cdot \bar{R}^*_2 = p^*$, for clearly p^* is not an end point of N^*. These regions may be chosen so that their complements and exteriors are connected. Set $R_1 = T^{-1}(R^*_1)$, $R_2 = T^{-1}(R^*_2)$. Then R_1 and R_2 are regions in N and indeed $\bar{R}_1 \cdot \bar{R}_2 = p$. For obviously $\bar{R}_1 \cdot \bar{R}_2 \subset p$, since T is biunivalued and continuous; and then if $\bar{R}_1 \cdot \bar{R}_2 \neq p$, it must be true that for either R_1 or R_2, say R_1, $\bar{R}_1 \cdot p = 0$; and hence by corollary (3.1a) there exists a region U in N containing p but such that $U \cdot \bar{R}_1 = 0$ and such that $U^* = T(U)$ is a region in N^*. Clearly this is impossible, for then $U^* \cdot R^*_1 = 0$, contrary to the fact that p^* is a limit point of R^*_1. Therefore $\bar{R}_1 \cdot \bar{R}_2 = p$, and our theorem is proved.

Corollary (6.2a). *In case M is locally divisible in the non-end point* [illegible] *connected.*

COROLLARY (6.2b). *In order that the point p of a space M be an end point of M, either of the following conditions is necesary and sufficient:*

1). *that there should not exist two regions R_1 and R_2 such that $\bar{R}_1 \cdot \bar{R}_2 = p$.*

2). *that p should not be a cut point of any connected subset of M.*

7. *Generalized three-point theorem.* The theorem of Ayres † that any three points $p_1, p_2, p_3, \cdots$ of a cyclicly connected continuous curve C taken in any order p_i, p_j, p_k lie on an arc $p_i p_j p_k$ in C will be called *the three-point theorem.* We now prove an analogus theorem for spaces M.

(7.1). THEOREM. *If p is not a local end point of a set M having no cut point, then if a and b are any two distinct points of $M - p$, there exist regions R_a and R_b containing a and b respectively, and such that $\bar{R}_a \cdot \bar{R}_b = p$ and $R_a + p$ and $R_b + p$ are locally connected.*

Proof. In case p is not a local separating point of M, this theorem has been proved in the proof of Theorem (5.2). Hence we may suppose p is a local separating point of M. Thus there exists a region K containing p but neither a nor b and two points c and d which belong to different components K_c and K_d, respectively, of $K - p$. There exists a region L such that $\bar{L} \subset K$ and such that the points a, b, c, and d all lie together in a single component C^* of $M - \bar{L}$. Now there exist components L_c and L_d of $L - p$ which are subsets respectively of K_c and K_d. By virtue of (5.1) there exist two strongly separated regions Q_a and Q_b joining a and $L_c + L_d$, and b and $L_c + L_d$ respectively. Let $W = \overline{L_c + L_d + p}$ and let N_a, N_b and C' denote the components of $Q_a - Q_a \cdot W$, $Q_b - Q_b \cdot W$ and $M - W$ respectively which contain a, b, and C^*, respectively. Now W contains at least one limit point of each of the sets N_a and N_b. And if for one of these sets, say N_a, p is the only limit point of N_a in W, then set $R_a = N_a$ and define R_b as follows: either $\bar{L}_c$ or $\bar{L}_d$, say $\bar{L}_c$, contains a point $q \neq p$ which is a limit point of N_b; let U_q be a region containing q and such that $\bar{U}_q \cdot \bar{N}_a = 0$; then set $R_b = N_b + U_q + L_c$. In this case clearly the sets R_a and R_b thus defined have the desired properties for the theorem. If this is not the case, then W contains points x and y, distinct from p, which are limit points of N_a and N_b respectively. Now if there is any choice of the points x and y such that x belongs to one of the sets $\bar{L}_c$ and $\bar{L}_d$ and y to the other one, say $x \subset \bar{L}_c$ and $y \subset \bar{L}_d$, then we define the sets R_a and R_b as follows: let U_x and U_y be strongly separated regions containing x and y respectively and such that

$$\bar{U}_x \cdot (\bar{N}_b + \bar{L}_d) = \bar{U}_y \cdot (\bar{N}_a + \bar{L}_c) = 0.$$

† See *Bulletin de l'Academie Polonaise des Sciences et des Lettres*, 1928, pp. 127-142.

Then $N_c + U_x$ and $N_b + U_y$ are mutually exclusive regions joining a and x, and b and y, respectively; clearly these regions contain strongly separated regions E_a and E_b joining a and x, and b and y, respectively. Then if $R_a = E_a + L_c$ and $R_b = E_b + L_d$, the sets R_a and R_b have the desired properties.

Now if no such choice of x and y is possible, then both x and y belong to one of the sets $\bar{L}_c$ and $\bar{L}_d$, say $\bar{L}_c$, and no point of $\bar{L}_d - p$ is a limit point of $N_a + N_b$. Now since $d \subset C^*$, it follows that at least one point z of $\bar{L}_d - p$ is a limit point of C. Let U_z be a region containing z and such that $U_z \cdot (\bar{N}_a + \bar{N}_b + \bar{L}_c) = 0$. Let f be a point of $U_z \cdot C$ and let C_f be the component of $C - C \cdot (\bar{N}_a + \bar{N}_b)$ containing f. Either $\bar{N}_a \cdot C$ or $\bar{N}_b \cdot C$, say $\bar{N}_a \cdot C$, (since the two cases obviously are alike) contains a point $g \neq p$ which is a limit point of C_f. Let U_g be a region containing g and such that $\bar{U}_g \cdot \bar{N}_b + \bar{L}_c) = 0$. Then U_g contains a point h of C_f, and C_f contains a region V_{fh} joining f and h and such that $\bar{V}_{fh} \subset C_f$ and hence so that $\bar{V}_{fh} \cdot (\bar{N}_b + \bar{L}_c) = 0$. Now let U_y be a region containing y and such that

$$U_y \cdot (\bar{N}_a + \bar{U}_g + \bar{V}_{fh} + \bar{U}_z + \bar{L}_g) = 0.$$

Then $\quad G_a = N_a + U_g + V_{fh} + U_z + L_d \quad$ and $\quad G_b = N_b + U_y + L_c$

are mutually exclusive regions joining a and L_d, and b and L_c, respectively, such that $G_a \cdot L_c = G_b \cdot L_d = 0$. The regions G_a and G_b contain regions V_a and V_b, respectively, joining a and L_d, and b and L_c, respectively, and such that $\bar{V}_a \subset G_a$ and $\bar{V}_b \subset G_b$. Hence if $R_a = V_a + L_d$ and $R_b = V_b + L_c$, then $\bar{R}_a \cdot \bar{R}_b = p$ and $R_a + p$ and $R_b + p$ are locally connected sets. Thus our theorem is established.

It is clear that essentially the same argument may be applied to prove the following slightly more general theorem.

(7.2). THEOREM. *If p is a non-local end point of a space M having no cut point and which is imbedded in a space M_0, if A and B are mutually exclusive closed subsets of M such that A does not separate p and B and B does not separate p and A in M, and if A_0 and B_0 are subsets of M_0 such that $A_0 \cdot M = A$ and $B_0 \cdot M = B$, then there exist regions R_1 and R_2 in M_0 such that $R_1 \cdot A \neq 0$, $R_2 \cdot B \neq 0$, $\bar{R}_1 \cdot B_0 = \bar{R}_2 \cdot A_0 = 0$, $\bar{R}_1 \cdot \bar{R}_2 = p$, and $R_1 + p$ and $R_2 + p$ are locally connected.*

8. GENERALIZED CYCLIC CONNECTIVITY THEOREM. *If a and b are any [illegible] points of a space M which has no cut point, then there exist two regions R_1 and R_2 such that $\bar{R}_1 \cdot \bar{R}_2 = a + b$ and $R_1 + a + b$ and $R_2 + a + b$ are locally connected sets.*

Proof. Now if $M-(a+b)$ is not connected, the theorem is obvious, for in this case we may take R_1 and R_2 as any two distinct components of $M-(a+b)$. Hence we may suppose that $M-(a+b)=(M-a)-b$ is connected. Now there exist two mutually exclusive regions G_1 and G_2 such that $\bar{G}_1\cdot\bar{G}_2=a$ and $\bar{G}_1\cdot b=\bar{G}_2\cdot b=0$ and G_1+a and G_2+a are locally connected. Since $M-(a+b)$ is connected, it follows that there exists a region G_3 joining G_1 and G_2 and such that $\bar{G}_3\cdot(a+b)=0$. Since clearly a is neither a cut point nor an end point of the set $G_1+G_2+G_3+a$, it therefore lies in a nodule N of $G_1+G_2+G_3+a$, and $N\cdot b=0$. Let x and y be any two points of N. Then by (7.1) there exist two regions R_x and R_y containing x and y respectively and such that $\bar{R}_x\cdot\bar{R}_y=b$ and R_x+b and R_y+b are locally connected sets. Let S_x and S_y respectively denote components of $R_x-R_x\cdot\bar{N}$ and $R_y-R_y\cdot\bar{N}$, respectively, such that $\bar{S}_x\cdot\bar{S}_y=b$ and $\bar{S}_x\cdot\bar{N}\neq 0\neq\bar{S}_y\cdot\bar{N}$. Now it will be seen at once that all cases in which $\bar{S}_x\cdot N$ or $\bar{S}_y\cdot N$ are non-vacuous are only simpler than that in which both of these sets are vacuous. Hence we may assume that $\bar{S}_x\cdot N=\bar{S}_y\cdot N=0$. Let x' and y' be points of $\bar{S}_x\cdot\bar{N}$ and $\bar{S}_y\cdot\bar{N}$ respectively, and let G_x and G_y be strongly separated regions containing x' and y' respectively and such that $\bar{G}_x\cdot(\bar{S}_y+a)=\bar{G}_y\cdot(\bar{S}_x+a)=0$. Let N^* be the nodule of the set $N+G_x+G_y$ which contains N, let $A_0=S_x$, $B_0=S_y$, $A=N^*\cdot A_0$ and $B=N^*\cdot B_0$. Now A does not separate the sets B and a in N^*, for $G_y\cdot N^*+N$ is a connected subset of N^* which contains both a and the point y' of B but contains no point of A. Similarly, B does not separate the sets A and a in N^*. Thus since $A=A_0\cdot N^*$ and $B=B_0\cdot N^*$, it follows by Theorem (7.2) that there exist two regions T_x and T_y containing points of A and B respectively and such that $\bar{T}_x\cdot\bar{T}_y=a$, $\bar{T}_x\cdot B_0=\bar{T}_y\cdot A_0=0$, and T_x+a and T_y+a are locally connected sets. Then if $R_1=R_x+S_x$ and $R_2=T_y+S_y$ it is seen at once that $\bar{R}_1\cdot\bar{R}_2=a+b$ and R_1+a+b and R_2+a+b are locally connected sets. This completes the proof of the theorem.

The theorem just proved is a generalization of the theorem,† which we call the *cyclic connectivity theorem*, that any locally compact space M which has no cut point is cyclicly connected. For clearly such a space M can have no local end point. Hence for any two points a and b, the regions R_1 and R_2 exist as in our theorem. Then R_1+a+b and R_2+a+b contain arcs

† See G. T. Whyburn, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38; and W. L. Ayres, *American Journal of Mathematics*, Vol. 51 (1929), p. 590. For a simple proof of this theorem, see the author's paper, "On the Cyclic Connectivity Theorem", appearing in the *Bulletin of the American Mathematical Society*.

$(ab)_1$ and $(ab)_2$, respectively, from a to b; and obviously $(ab)_1 + (ab)_2$ is a simple closed curve in M.

COROLLARY (8a). *If the space M has no local separating point, then for each pair of points a and b of M, there exists an infinite sequence $R_1, R_2, R_3, \cdots$ of mutually exclusive regions such that for each i, $R_i + a + b$ is connected and locally connected and for each i and j, $i \neq j$, $\bar{R}_i \cdot \bar{R}_j = a + b$.*

Proof. Clearly M can have neither cut points nor local end points. Hence by the theorem in this section there exist two mutually exclusive regions R_1 and S_1 such that $\bar{R}_1 \cdot \bar{S}_1 = a + b$ and $R_1 + a + b$ and $S_1 + a + b$ are locally connected. Now S_1 can have no local separating points, and therefore it follows that a and b are not local end points of $S_1 + a + b$. Hence, by the theorem just established, there exist in S_1 two regions R_2 and S_2 such that $\bar{R}_2 \cdot \bar{S}_2 = a + b$ and $R_2 + a + b$ and $S_2 + a + b$ are locally connected. Similarly, S_2 can have no local separating point and there exist in S_2 two regions R_3 and S_3 such that $\bar{R}_3 \cdot \bar{S}_3 = a + b$ and $R_3 + a + b$ and $S_3 + a + b$ are locally connected, and so on. Continuing this process indefinitely, obviously the sets $R_1, R_2, R_3, \cdots$ so obtained have all the desired properties.

9. *Higher connectivity of absolute G_δ M-spaces.*

(9.1) THEOREM. *If the space M has no cut point and is an absolute G_δ set, then every two non-local end points a and b of M lie on a simple closed curve in M. Thus every nodular G_δ-space M which has no local end points is cyclicly connected.*

For by the theorem in § 8, there exist regions R_1 and R_2 in M such that $\bar{R}_1 \cdot \bar{R}_2 = a + b$ and $R_1 + a + b$ and $R_2 + a + b$ are locally connected; and since each of the sets $R_1 + a + b$ and $R_2 + a + b$ is also a G_δ space M, these sets contain † arcs $(ab)_1$ and $(ab)_2$ respectively from a to b. Clearly $(ab)_1 + (ab)_2$ is a simple closed curve in M containing $a + b$.

It should be noted that this theorem is also a generalization of the cyclic connectivity theorem.

(9.2). THEOREM. *If the G_δ-space M has no local separating points, then each pair of points a and b of M can be joined in M by a continuum T which is the sum of a set of **c** independent arcs from a to b, i. e., $T = \sum_{0<x<1} axb$, where axb is an arc in M from a to b and if $x \neq y$, $axb \cdot ayb = a + b$.*

[illegible]

† See the reference to Moore and Menger in § 2.

ceding sections by the same method as used in my paper, *Continuous Curves Without Local Separating Points*,† to prove the same theorem for a compact space M. The argument required here differs only in some of the details which must be modified slightly on account of the greater generality of the space, but is essentially the same proof.

10. *Conclusion. Some unsolved problems.* The proofs for a number of the propositions in the preceding sections are considerably complicated by our lack of knowledge of the existence of a property of spaces M analogus to the property of continuous curves C to the effect that each subcontinuum K of C is, for each $\epsilon > 0$, contained in a sub-continuous curve K^* of C such that $\delta(K^*) < \delta(K) + \epsilon$. This suggests the following problem, a positive solution to which would make possible greatly simplified proofs for some of the propositions in the present paper.

(10.1). PROBLEM. *If K is any connected subset of a space M, then is it true that for each $\epsilon > 0$, K is contained in a closed, connected and locally connected subset K^* of M such that $\delta(K^*) < \delta(K) + \epsilon$?*

It seems likely that a space M without cut points may possess a certain higher connectivity between any two of its points, even though these points be local end points. In this connection a solution to the following problem would be of interest.

(10.2). PROBLEM. *If a and b are any two points of a space M having no cut point, then does there exist two mutually exclusive regions R_1 and R_2 in M such that $\bar{R}_1 \cdot \bar{R}_2 \supset a + b$?* ‡

It may be noted here that it is not true that such regions R_1 and R_2 always exist so that $\bar{R}_1 \cdot \bar{R}_2 = a + b$. A simple modification of the space $\mathfrak{M}$ in § 3 shows that this is not always possible.

Finally, the author wishes to mention the desirability of a thorough investigation of the properties of the local end points of a space M and of the subsets of the set E of all local end points of M. For example, *can M be disconnected by the omission of E or of any closed subset of E?*

THE JOHNS HOPKINS UNIVERSITY.

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† See *American Journal of Mathematics*, Vol. 53 (1931), pp. 163-166.

‡ Essentially this same problem was raised by the author in *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 943, problem (9.1). It is remarked that the results in the present paper greatly clarify the difficulties discussed in this reference.

GENERALIZED GREEN'S MATRICES FOR COMPATIBLE SYSTEMS OF DIFFERENTIAL EQUATIONS.*

By William T. Reid.†

1. *Introduction.* The existence and properties of the Green's matrix for an incompatible system of linear differential equations of the first order has been treated by Bounitzky, Birkhoff and Langer, Bliss and others.‡ In the systems considered by the above mentioned writers the coefficients of the system are supposed to be continuous functions of the independent variable. W. M. Whyburn § has demonstrated the existence of the Green's matrix for a system of equations whose coefficients are only Lebesgue summable functions of the independent variable.

For certain special types of compatible differential systems, where the differential equation is a single linear equation with continuous coefficients, the existence of generalized Green's functions has been shown by Hilbert, Bounitzky, Westfall and others.¶ Recently Elliott ‖ has treated generalized Green's functions for general compatible differential systems consisting of a single differential equation of the n-th order with continuous coefficients, together with boundary conditions involving the values of the solution and its first $n-1$ derivatives at two points.

It is the purpose of this paper to show that theorems corresponding to those obtained by Elliott are true for a system of n ordinary linear differential equations of the first order with Lebesgue summable coefficients, together with boundary conditions involving the values of the solution at two points.

Vector and matrix notation is used throughout the paper. A capital

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† National Research Fellow in Mathematics.

‡ E. Bounitzky, *Journal de Mathématiques*, (6), Vol. 5 (1909), p. 65; G. D. Birkhoff and R. E. Langer, *Proceedings of the American Academy of Arts and Sciences*, Vol. 58 (1923), p. 51; G. A. Bliss, *Transactions of the American Mathematical Society*, Vol. 28 (1926), p. 561.

§ W. M. Whyburn, *Annals of Mathematics*, (2), Vol. 28 (1927), p. 291.

¶ D. Hilbert, *Göttinger Nachrichten*, (1904), p. 213, and *Grundzüge einer allgemeinen Theorie der Integralgleichungen*, Berlin, (1912), p. 44; E. Bounitzky, *loc. cit.*; W. Westfall, [illegible] (Göttingen, 1905), and [illegible] 19 (1909), p. 177.

‖ W. W. Elliott, *American Journal of Mathematics*, Vol. 50 (1928), p. 243; also, *American Journal of Mathematics*, Vol. 51 (1929), p. 397.

letter will denote a square matrix with n rows and n columns whose element in the i-th row and j-th column is given by the same letter with the subscript ij. E is used to denote the unit matrix, that is, $E_{ij}=0$ if $i\neq j$, $E_{ii}=1$, and 0 is the matrix each of whose elements is zero. It may be mentioned here that in this paper the idea of the zero matrix and the unit matrix is used only in connection with matrices whose elements are constants or continuous functions. If A and B are two square matrices, then the matrix product AB is the matrix C, where $C_{ij}=A_{i\alpha}B_{\alpha j}$.* Similarly, all vectors are supposed to have n components and if η is a vector, then η_i is the i-th component of η. The product $A\eta$ of a matrix A by the vector η is given by the vector b, where $b_i=A_{i\alpha}\eta_\alpha$ $(i=1,2,\cdots,n)$. Similarly, the product ηA is given by the vector c, where $c_i=\eta_\alpha A_{\alpha i}$ $(i=1,2,\cdots,n)$. The conjugate imaginary of a complex quantity a is denoted by $\bar{a}$, and the conjugate of the matrix A, which will be denoted by $\bar{A}$, is the matrix each element of which is the conjugate imaginary of the corresponding element of A. $\tilde{A}\equiv\|\tilde{A}_{ij}\|$ is used to denote the adjoint, or transpose, of A, i.e., $\tilde{A}_{ij}=A_{ji}$ $(i,j=1,2,\cdots,n)$.

A solution of the vector differential equation

$$(1)\qquad y'=A(x)y+g(x),$$

where $A_{ij}(x)$ $(i,j=1,2,\cdots,n)$ and $g_i(x)$ $(i=1,2,\cdots,n)$ are Lebesgue summable functions, real or complex, of the real variable x on the interval $X: a\leqq x\leqq b$, we define as an absolutely continuous vector $y(x)\equiv[y_\alpha(x)]$ which satisfies (1) on X_0.† With the homogeneous vector equation

$$(2)\qquad y'=A(x)y$$

we associate boundary conditions

$$(3)\qquad My(a)+Ny(b)=0,$$

where M and N are square matrices such that the matrix $\|M,N\|$ ‡ is of rank n. The differential equation adjoint to (2) is

$$(4)\qquad z'=-zA(x)$$

* The repetition of a subscript in an expression will denote summation with respect to that subscript over the values from 1 to n.

† X_0 is used to denote 'almost everywhere' on X. The excepted null set is not constant and may vary. Throughout this paper primes are used to denote differentiation with respect to an independent variable x.

‡ If M and N are two matrices of n rows and n columns, then $\|M;N\|$ is used to denote the matrix U which has n rows and $2n$ columns and whose elements are defined as: $U_{i,2j-1}=M_{ij}$, $U_{i,2j}=N_{ij}$ $(i,j=1,2,\cdots,n)$.

and the adjoint boundary conditions are given by

$$(5) \qquad z(a)P + z(b)Q = 0,$$

where the matrix of coefficients of (5) is of rank n and $MP - NQ = 0$.*

A matrix $G(x;t)$ is said to be a Green's matrix for the system (2), (3) if for each value of t on $a < t < b$ we have:

(6) Each column of $G(x;t)$ as a function of x is a solution of (2) on $a \leqq x < t$ and $t < x \leqq b$,

$$(7) \qquad G(t+;t) - G(t-;t) = E,$$

$$(8) \qquad MG(a;t) + NG(b;t) = 0.$$

Whenever the system (2), (3) is incompatible there exists a unique Green's matrix for the system which is given by

$$(9) \qquad G(x:t) = (1/2)Y(x)[\,|x-t|/(x-t)E + D\Delta]Z(t),$$

where $Y(x)$ and $Z(x)$ are matrix solutions † of (2) and (4) respectively such that $Y(x)Z(x) = Z(x)Y(x) = E$, D is the reciprocal of the matrix $MY(a) + NY(b)$, and $\Delta = MY(a) - NY(b)$. We have

(A) When the system (2), (3) is incompatible the unique solution of (1), (3), where $g(x) \equiv [g_\alpha(x)]$ is any summable vector, is given by ‡

$$y(x) = \int_a^b G(x;t)g(t)\,dt.$$

(B) The functions $H_{ij}(x;t) = -G_{ji}(t;x)$ defined by (9) are the elements of the Green's matrix for the adjoint system (4), (5) and

(10) Each row of $G(x;t)$ as a function of t is a solution of (4) on $a \leqq t < x$ and $x < t \leqq b$,

$$(11) \qquad G(x;x-) - G(x;x+) = E,$$

$$(12) \qquad G(x;a)P + G(x;b)Q = 0.\S$$

2. *Existence of Generalized Green's Matrices.* We shall now assume that the system (2), (3) is compatible of index r. Then the adjoint system (4),

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* G. A. Bliss, *loc. cit.*, p. 564.

† A square matrix $Y(x)$ each column of which is a solution of (2) and such that the determinant $|Y(x)|$ is different from zero on X is called a matrix solution of (2). [illegible]

‡ [illegible]

§ G. A. Bliss, [illegible]

(5) is also compatible of index r.* Let the matrix ${}^rY(x) \equiv \| {}^rY_{ij}(x) \|$ be defined as follows: ${}^rY_1, {}^rY_2, \cdots, {}^rY_r$ $[{}^rY_j \equiv ({}^rY_{\alpha j})$ $(j = 1, 2, \cdots, r)]$ are r linearly independent solutions of the system (2), (3) and ${}^rY_{ij} = 0$ if $j > r$. Also by ${}^rZ(x)$ we denote a matrix whose first r rows are linearly independent solutions of the adjoint system (4), (5) and the elements of the remaining rows are all zero. The general solution of (2), (3) is then of the form

$$y(x) = \sum_{\alpha=1}^{r} {}^rY_\alpha(x) a_\alpha,$$

where the a_α's are arbitrary constants. Whenever (2), (3) is compatible the system (1), (3) has a solution if and only if the vector equation

$$\int_a^b {}^rZ(x) g(x)\, dx = 0 \tag{13}$$

is satisfied by the vector $g(x)$.

If condition (13) is satisfied we seek to determine a matrix $G(x;t)$ which is continuous in (x, t) at every point on $a \leqq {x \atop t} \leqq b$ except along the line $x = t$, which satisfies the condition

$$G(t+;t) - G(t-;t) = E, \tag{14}$$

and is such that every solution of (1), (3) may be written in the form

$$y(x) = \int_a^b G(x;t) g(t)\, dt + \sum_{\alpha=1}^{r} {}^rY_\alpha(x) a_\alpha. \tag{15}$$

If such a matrix $G(x;t)$ exists we will say that $G(x;t)$ is a *Generalized Green's Matrix* for the compatible system (2), (3).

Let $Y(x) \equiv \| Y_{ij}(x) \|$ be a matrix solution of (2) such that the first r columns of $Y(x)$ are ${}^rY_1, {}^rY_2, \cdots, {}^rY_r$. By $Z(x)$ we denote the matrix solution of (4) such that $Z(x)Y(x) = Y(x)Z(x) = E$ on X. Let $s_i(Y_j)$ denote $M_{i\alpha}Y_{\alpha j}(a) + N_{i\alpha}Y_{\alpha j}(b)$ $(i, j = 1, 2, \cdots, n)$. If the system (2), (3) has r linearly independent solutions, then the matrix

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* For the case in which the elements of the matrix $A(x)$ are real continuous functions of the variable x and the elements of M and N are real constants this result has been established by Bliss and his method of proof carries over wholly for the system (2), (3). See G. A. Bliss, *loc. cit.*, pp. 566-567. In a recent paper the author has also established this result for an infinite system of linear differential equations which includes as a special case the finite system (2), (3) when the coefficients of the system are real. See W. T. Reid, *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 284-318; in particular, pp. 306-311.

$$\left\| \begin{array}{cccc} s_1(Y_{r+1}) & s_1(Y_{r+2}) & \cdot\ \cdot & s_1(Y_n) \\ s_2(Y_{r+1}) & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ s_n(Y_{r+1}) & \cdot & \cdot\ \cdot & s_n(Y_n) \end{array} \right\|$$

is of rank $n-r$. The integers $k_1, k_2, \cdots, k_{n-r}$ may then be so chosen that the matrix

$$\left\| \begin{array}{ccc} s_{k_1}(Y_{r+1}) & \cdot\ \cdot & s_{k_1}(Y_n) \\ s_{k_2}(Y_{r+1}) & \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot & \cdot \\ s_{k_{n-r}}(Y_{r+1}) & \cdot\ \cdot & s_{k_{n-r}}(Y_n) \end{array} \right\|$$

has a unique reciprocal, which we will denote by

$$\left\| \begin{array}{cccc} s^{-1}{}_{11} & s^{-1}{}_{12} & \cdot\ \cdot & s^{-1}{}_{1,n-r} \\ s^{-1}{}_{21} & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ s^{-1}{}_{n-r,1} & \cdot & \cdot\ \cdot & s^{-1}{}_{n-r,n-r} \end{array} \right\| .$$

Now define the matrix $\mathfrak{D} \equiv \| \mathfrak{D}_{ij} \|$ as: $\mathfrak{D}_{r+i,k_j} = s_{ij}{}^{-1}$ $(i, j = 1, 2, \cdots n-r)$; $\mathfrak{D}_{ij} = 0$ if $i \leqq r$ or $j \neq k_\alpha$ $(\alpha = 1, 2, \cdots, n-r)$.

THEOREM 1. *A generalized Green's matrix for the compatible system* (2), (3) *exists and may be written as*

$$(16) \qquad G(x; t) = (1/2) Y(x) [\, |x-t| / (x-t) E + \mathfrak{D}\Delta] Z(t),$$

where $Y(x)$, $Z(x)$ *and* $\mathfrak{D}$ *are defined as above and* $\Delta = MY(a) - NY(b)$.

Let $K(x; t) = (1/2) Y(x) Z(t) [\, |x-t| / (x-t)]$. Then $K(x; t)$ as a function of x is a matrix solution of (2) on $a \leqq x < t$ and $t < x \leqq b$. Furthermore,

$$K(t+; t) - K(t-; t) = E.$$

Let
$$u(x) = \int_a^b K(x; t) g(t)\, dt,$$
$$= (1/2) Y(x) [\int_a^x Z(t) g(t)\, dt \quad \int_x^b Z(t) g(t)\, dt].$$

Since $u(x)$ is clearly absolutely continuous on X and

$$u'(x) = A(x) u(x) + g(x)$$

[illegible]

[illegible]

Since by hypothesis the system (1), (3) is compatible, there exist values of the a_α's so that

$$\sum_{\alpha=1}^{n} s_i(Y_\alpha)a_\alpha + s_i(u) = 0 \qquad (i = 1, 2, \cdots, n).$$

Since $s_i(Y_j) = 0$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, r)$, we have

$$\sum_{\alpha=1}^{n-r} s_{k_i}(Y_{r+\alpha})a_{r+\alpha} = -s_{k_i}(u) \qquad (i = 1, 2, \cdots, n-r),$$
$$= (1/2)\,[M_{k_i\alpha}Y_{\alpha\beta}(a) - N_{k_i\alpha}Y_{\alpha\beta}(b)] \int_a^b Z_{\beta\gamma}(t)g_\gamma(t)\,dt.$$

Then

$$a_{r+i} = (1/2) \sum_{\mu=1}^{n-r} \mathfrak{D}_{r+i,k_\mu}[M_{k_\mu\alpha}Y_{\alpha\beta}(a) - N_{k_\mu\alpha}Y_{\alpha\beta}(b)] \int_a^b Z_{\beta\gamma}(t)g_\gamma(t)\,dt \qquad (i = 1, 2, \cdots, n-r).$$

Therefore

$$y(x) = \sum_{\alpha=1}^{r} {}^rY_\alpha(x)a_\alpha + \int_a^b (1/2)Y(x)[\,|x-t|/(x-t)E + \mathfrak{D}\Delta]Z(t)g(t)\,dt,$$

and we have that a generalized Green's matrix for (2), (3) is given by (16).

It is to be noted that the choice of the integers $k_1, k_2, \cdots, k_{n-r}$ is in general not unique. When $r = 0$ we have from (16) the ordinary Green's matrix for the incompatible system as given by (9).

THEOREM 2. *The generalized Green's matrix for the compatible system* (2), (3) *is not unique. If* $G_1(x;t)$ *is one generalized Green's matrix, then every generalized Green's matrix is of the form*

$$G(x;t) = G_1(x;t) + {}^rY(x)U(t) + V(x)\,{}^rZ(t), \tag{17}$$

where $V(x)$ *and* $U(x)$ *are matrices each element of which is continuous on* X. *Furthermore, every function* $G(x;t)$ *of the form* (17) *is a generalized Green's matrix for the compatible system* (2), (3).

It is evident that there exist constant matrices T and T^* such that ${}^rY(x)T = \mathbb{Y}(x)$ and $T^{*r}Z(x) = \mathbb{Z}(x)$, where: (a) the first r columns of $\mathbb{Y}(x)$ and the first r rows of $\mathbb{Z}(x)$ are linearly independent solutions of (2), (3) and (4), (5) respectively; (b) $\mathbb{Y}_{ij}(x) = 0 = \mathbb{Z}_{ji}(x)$ if $j > r$;

(c) $$\int_a^b \sum_{\alpha=1}^{n} \bar{\mathbb{Y}}_{\alpha i}(x)\mathbb{Y}_{\alpha j}(x)\,dx = E_{ij} = \int_a^b \sum_{\alpha=1}^{n} \bar{\mathbb{Z}}_{i\alpha}(x)\mathbb{Z}_{j\alpha}(x)\,dx \qquad (i, j = 1, 2, \cdots, r);$$

(d) $$T_{ij} = 0 = T^*_{ij} \quad \text{if} \quad i > r \quad \text{or} \quad j > r.$$

Let

$$g_i(x) = \sum_{a=1}^{r} \bar{\mathbb{Z}}_{ai}(x) c_a + h_i(x) \qquad (i = 1, 2, \cdots, n),$$

where $h(x) = [h_a(x)]$ is any summable vector on X and

$$c_i = - \int_a^b \sum_{a=1}^{n} \mathbb{Z}_{ia}(v) h_a(v) dv \qquad (i = 1, 2, \cdots, r).$$

In view of (13) we have that for every vector $g(x) = [g_a(x)]$ so determined the system (1), (3) is compatible. If $G_1(x; t)$ and $G(x; t)$ are two generalized Green's matrices for (2), (3), let $G(x; t) - G_1(x; t) = D(x; t)$. $D(x; t)$ is continuous in (x, t) on $a \leq {x \atop t} \leq b$. Then

$$(18) \qquad \int_a^b D_{i\beta}(x; t) g_\beta(t) dt$$

$$= \int_a^b D_{i\beta}(x; t) [- \sum_{a=1}^{r} \bar{\mathbb{Z}}_{a\beta}(t) \{ \int_a^b \mathbb{Z}_{a\gamma}(v) h_\gamma(v) dv \} + h_\beta(t)] dt,$$

$$(19) \qquad = \int_a^b [- \sum_{a=1}^{r} \{ \int_a^b D_{i\beta}(x; v) \mathbb{Z}_{a\beta}(v) dv \} \bar{\mathbb{Z}}_{a\gamma}(t) + D_{i\gamma}(x; t)] h_\gamma(t) dt.\dagger$$

We also have

$$(20) \qquad \int_a^b D_{i\beta}(x; t) g_\beta(t) dt = \sum_{\mu=1}^{r} \mathbb{Y}_{i\mu}(x) d_\mu,$$

where the d_μ's are constants. Now

$$d_\mu = \int_a^b \bar{\mathbb{Y}}_{\sigma\mu}(u) \left(\sum_{\gamma=1}^{r} \mathbb{Y}_{\sigma\gamma}(u) d_\gamma \right) du \qquad (\mu = 1, 2, \cdots, r),$$

$$(21) \qquad = \int_a^b \bar{\mathbb{Y}}_{\sigma\mu}(u) \left(\int_a^b [- \sum_{a=1}^{r} \left\{ \int_a^b D_{\sigma\beta}(u; v) \bar{\mathbb{Z}}_{a\beta}(v) dv \right\} \mathbb{Z}_{a\gamma}(t) + D_{\sigma\gamma}(u; t)] h_\gamma(t) dt \right) du$$

in view of (19) and (20). From (19), (20) and (21) we then obtain

$$\int_a^b \Big[- \sum_{a=1}^{r} \left\{ \int_a^b D_{i\beta}(x; v) \bar{\mathbb{Z}}_{a\beta}(v) dv \right\} \mathbb{Z}_{a\gamma}(t) + D_{i\gamma}(x; t)$$

$$(22) \qquad + \sum_{\mu=1}^{r} \mathbb{Y}_{i\mu}(x) \left(\sum_{a=1}^{r} \Big[\int_a^b \bar{\mathbb{Y}}_{\sigma\mu}(u) \left\{ \int_a^b D_{\sigma\beta}(u; v) \bar{\mathbb{Z}}_{a\beta}(v) dv \right\} du \Big] \mathbb{Z}_{a\gamma}(t) \right.$$

$$\left. \int_a^b \bar{\mathbb{Y}} [\text{illegible}] \right) \Big] h [\text{illegible}] dt = 0.$$

† [illegible]

The change of the order of integration necessary to obtain relation (22) is, as before, permissible. Since the coefficient of $h_\gamma(t)$ $(\gamma=1,2,\cdots,n)$ in (22) is a continuous function on $a \leqq t \leqq b$ for each value of x on X, and the relation (22) is true for every summable vector $h(x)$, we have

$$D_{i\gamma}(x;t)=\sum_{\alpha=1}^{r} V^*_{i\alpha}(x)\mathbf{Z}_{\alpha\gamma}(t)+\sum_{\beta=1}^{r} \mathbf{Y}_{i\beta}(x)U^*_{\beta\gamma}(t) \qquad (i,\gamma=1,2,\cdots,n), \tag{23}$$

where

$$V^*_{ij}(x)=\int_a^b D_{i\beta}(x;v)\bar{\mathbf{Z}}_{j\beta}(v)\,dv \qquad (i=1,2,\cdots,n;\ j=1,2,\cdots,r),$$

$$U^*_{ij}(t)=-\sum_{\alpha=1}^{r}\left[\int_a^b \bar{\mathbf{Y}}_{\sigma i}(u)\left\{\int_a^b D_{\sigma\beta}(u;v)\bar{\mathbf{Z}}_{\alpha\beta}(v)\,dv\right\}du\right]\mathbf{Z}_{\alpha j}(t) + \int_a^b \bar{\mathbf{Y}}_{\sigma i}(u)D_{\sigma j}(u;t)\,du \qquad (i=1,2,\cdots,r;\ j=1,2,\cdots,n).$$

In view of (14) we have that each element of $D(x;t)$ is continuous in (x,t) on $a \leqq {x \atop t} \leqq b$ and therefore, since each element of $\mathbf{Z}(x)$ and $\mathbf{Y}(x)$ is continuous on X, it follows that $V^*_{ij}(x)$ and $U^*_{ij}(t)$ are continuous functions of their arguments. Let

$$U_{ij}(x)=\sum_{\alpha=1}^{r} T_{i\alpha}U^*_{\alpha j}(x) \quad \text{and} \quad V_{ij}(x)=\sum_{\alpha=1}^{r} V^*_{i\alpha}(x)T^*_{\alpha j} \qquad (i,j=1,2,\cdots,n).$$

Then

$$G(x;t)=G_1(x;t)+{}^rY(x)U(t)+V(x){}^rZ(t)$$

and clearly $U(x)$ and $V(x)$ are continuous on X. Also if $U(x)$ and $V(x)$ are any two matrices of continuous elements we have in view of (13) that if $G_1(x;t)$ is a generalized Green's matrix for (2), (3), then $G_1(x;t)+{}^rY(x)U(t)+V(x){}^rZ(t)$ is also a generalized Green's matrix for the compatible system (2), (3).

3. *Principal Generalized Green's Matrices.* We will now determine a generalized Green's matrix which shall possess for adjoint systems the same property as is given for the ordinary Green's matrix by (B) in the introduction.

Let M^*, N^*, P^* and Q^* be square matrices such that the matrices

$$\left\|\begin{matrix} M_{ij} & N_{ij} \\ M^*_{ij} & N^*_{ij}\end{matrix}\right\| \qquad \left\|\begin{matrix} -P^*_{ij} & -P_{ij} \\ Q^*_{ij} & Q_{ij}\end{matrix}\right\|$$

are unique reciprocals. For any pair of vectors $y(x)$ and $z(x)$ defined on X the vectors $s(y)$, $s^*(y)$, $t(z)$ and $t^*(z)$ are defined as

$$
(24)\qquad \begin{aligned} s(y) &= My(a) + Ny(b), & t(z) &= z(a)P + z(b)Q, \\ s^*(y) &= M^*y(a) + N^*y(b), & t^*(z) &= z(a)P^* + z(b)Q^*. \end{aligned}
$$

Then we have the identity †

$$
(25)\qquad s_\alpha(y)t^*_\alpha(z) + s^*_\alpha(y)t_\alpha(z) = y_\alpha(x)z_\alpha(x) \,\Big|_{x=a}^{x=b}
$$

If $y(x)$ and $z(x)$ are solutions of (2) and (4) respectively, then

$$
[z_\alpha(x)y_\alpha(x)]' = z_\alpha(x)y_\alpha'(x) + z_\alpha'(x)y_\alpha(x) = 0
$$

on X_0, and therefore $z_\alpha(x)y_\alpha(x)$ is a constant for x on X. Let $z(x)$ be any solution of (4) and $K(x;t) = (1/2)Y(x)Z(t)\,|x-t|/(x-t)$, where $Y(x)$ and $Z(x)$ are matrix solutions of (2) and (4) respectively which are reciprocals for x on X. Then

$$
\begin{aligned}
0 = \int_a^b \{z_\alpha(x)[(\partial/\partial x)K_{\alpha j}(x;t) - A_{\alpha\beta}(x)K_{\beta j}(x;t)] \\
+ [z_\alpha'(x) + z_\beta(x)A_{\beta\alpha}(x)]K_{\alpha j}(x;t)\}\,dx \\
(j = 1, 2, \cdots, n), \\
= z_\alpha(x)K_{\alpha j}(x;t)\,\Big|_{x=a}^{x=t-} + z_\alpha(x)K_{\alpha j}(x;t)\,\Big|_{x=t+}^{x=b}.
\end{aligned}
$$

Hence

$$
(26)\qquad \begin{aligned} z_\alpha(x)K_{\alpha j}(x;t)\,\Big|_{x=a}^{x=b} &= z_\alpha(x)K_{\alpha j}(x;t)\,\Big|_{x=t-}^{x=t+}, \\ &= z_j(t) \qquad\qquad (j = 1, 2, \cdots, n). \end{aligned}
$$

Let $K_j(x;t)$ denote the vector $[K_{\alpha j}(x;t)]$ which is the j-th column of $K(x;t)$. In view of relation (25) we have

LEMMA 1. *If $K(x;t) = (1/2)Y(x)Z(t)\,|x-t|/(x-t)$, where $Y(x)$ and $Z(x)$ are matrix solutions of* (2) *and* (4) *respectively which are unique reciprocals on X, then for every solution $z(x) \equiv [z_\alpha(x)]$ of* (4) *we have*

$$
t^*_\alpha(z)s_\alpha(K_j)_x + t_\alpha(z)s^*_\alpha(K_j)_x = z_j(t)\,‡ \qquad (j = 1, 2, \cdots, n).
$$

As before, let ${}^rZ(x)$ denote a matrix whose first r rows are linearly independent solutions of the system (4), (5) and the elements of the remaining $n-r$ rows are all zero. Let $\Psi(x)$ denote any matrix of Lebesgue summable functions of the form

$$
(27)\qquad \Psi(x) = \left\|\begin{matrix} \Psi_{11}(x) & \Psi_{12}(x) & \cdot & \cdot & \Psi_{1r}(x) & 0 & \cdot & 0 \\ \Psi_{21}(x) & \cdot & \cdot & \cdot & \Psi_{2r}(x) & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & 0 \\ \Psi_{n1}(x) & \cdot & \cdot & \cdot & \Psi_{nr}(x) & 0 & \cdot & 0 \end{matrix}\right\|,
$$

† See Bliss, *loc. cit.*, p. [illegible].

‡ The subscript x is used to denote that the vectors $s(K_j)$ and $s^*(K_j)$ are determined by (24) when K_j is considered as a function of x.

and such that the r-rowed determinant whose general element is

$$\int_a^b {}^rZ_{i\alpha}(x)\Psi_{\alpha j}(x)\,dx$$

is different from zero. Since the first r rows of ${}^rZ(x)$ are linearly independent solutions of (4), (5) it follows that there always exists a matrix $\Psi(x)$ satisfying the above condition. Let

$$R = \left\| \begin{matrix} R_{11} & R_{12} & & \cdot & R_{1r} & 0 & \cdot & 0 \\ R_{21} & \cdot & \cdot & \cdot & R_{2r} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & 0 \\ R_{r1} & \cdot & \cdot & \cdot & R_{rr} & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & 0 \end{matrix} \right\|,$$

where the r-rowed square matrix

$$\left\| \begin{matrix} R_{11} & R_{12} & \cdot & \cdot & R_{1r} \\ R_{21} & \cdot & \cdot & \cdot & R_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{r1} & \cdot & \cdot & \cdot & R_{rr} \end{matrix} \right\|$$

is the unique reciprocal of the matrix

$$\left\| \begin{matrix} \int_a^b {}^rZ_{1\alpha}(x)\Psi_{\alpha 1}(x)\,dx & \cdot & \cdot & \int_a^b {}^rZ_{1\alpha}(x)\Psi_{\alpha r}(x)\,dx \\ \int_a^b {}^rZ_{2\alpha}(x)\Psi_{\alpha 1}(x)\,dx & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int_a^b {}^rZ_{r\alpha}(x)\Psi_{\alpha 1}(x)\,dx & \cdot & \cdot & \int_a^b {}^rZ_{r\alpha}(x)\Psi_{\alpha r}(x)\,dx \end{matrix} \right\|. \tag{28}$$

We now define the matrix $N(x;t)$ as

$$N(x;t) = -\Psi(x)R\,{}^rZ(t).$$

Then

$$\begin{aligned} \int_a^b {}^rZ_{i\alpha}(x)N_{\alpha j}(x;t)\,dx &= -\int_a^b {}^rZ_{i\alpha}(x)\Psi_{\alpha\beta}(x)R_{\beta\gamma}\,{}^rZ_{\gamma j}(t)\,dx, \\ &= -{}^rZ_{ij}(t), \\ &= -t^*_\alpha({}^rZ_i)s_\alpha(K_j)_x \\ & \qquad (i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, n), \end{aligned}$$

in view of Lemma 1 if we denote by ${}^rZ_i$ the vector $[{}^rZ_{i\alpha}(x)]$.

Let $\mathfrak{Y}^*(x)$ be a matrix of absolutely continuous functions satisfying on X_0 the matrix differential equation

$$(d/dx)\mathfrak{Y}^*(x) = A(x)\mathfrak{Y}^*(x) - \Psi(x)R.$$

Then $\mathfrak{Y}(x) = \mathfrak{Y}^*(x)\,{}^rZ(t)$ is continuous in (x, t) and $\mathfrak{Y}_j(x; t) \equiv [\mathfrak{Y}_{aj}(x; t)]$ is a particular solution of the non-homogeneous equation

$$(29) \qquad (\partial/\partial x)\mathfrak{Y}_j = A(x)\mathfrak{Y}_j + N_j(x; t) \qquad (j = 1, 2, \cdots, n),$$

where $N_j(x; t)$ is the vector $[N_{aj}(x; t)]$. Then we have

$$\begin{aligned}\int_a^b {}^rZ_{ia}(x) N_{aj}(x; t)\, dx &= \int_a^b (\partial/\partial x)[{}^rZ_{ia}(x)\mathfrak{Y}_{aj}(x; t)]\, dx, \\ &= {}^rZ_{ia}(x)\mathfrak{Y}_{aj}(x; t) \Big|_{x=a}^{x=b}, \\ &= t^*_a({}^rZ_i)\, s_a(\mathfrak{Y}_j)_x \\ &\qquad (i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, n),\end{aligned}$$

in view of relation (25). Therefore we have

$$(30) \qquad t^*_a({}^rZ_i)\, s_a(\mathfrak{Y}_j)_x = -\, t^*_a({}^rZ_i)\, s_a(K_j)_x \qquad (i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, n).$$

Let $Y(x) \equiv \| Y_{ij}(x) \|$ be a matrix solution of (2) and Y_j denote the vector (Y_{aj}). From (25) we have

$$t^*_a({}^rZ_i)\, s_a(Y_j) = 0 \qquad (i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, n),$$

and furthermore the vectors $t^*({}^rZ_i)$ $(i = 1, 2, \cdots, r)$ are linearly independent.† Then in view of (30) we may determine quantities $C_{ij}(t)$ such that

$$s(Y_a)C_{aj} + s(\mathfrak{Y}_j)_x = -\, s(K_j)_x \qquad (i = 1, 2, \cdots, n).$$

Furthermore, since $s(\mathfrak{Y}_j)_x$ and $s(K_j)_x$ are continuous functions of t, the elements of the matrix $C(t) \equiv \| C_{ij}(t) \|$ may be chosen continuous functions of t. We have then established the following

LEMMA 2. *Each element of the matrix* $\mathrm{Y}(x; t) = \mathfrak{Y}(x; t) + Y(x)C(t)$ *is continuous in* (x, t) *on* $a \leqq x, t \leqq b$, *is absolutely continuous in* x *on* X *for each fixed value of* t *on* X, *and is such that on* X_0 *the matrix differential*

† See [illegible], p. [illegible].

is satisfied, and

$$s(\Upsilon_j)_x = -s(K_j)_x \qquad (j = 1, 2, \cdots, n). \tag{32}$$

The solution of (31), (32), is not unique but is determined except for added solutions of the homogeneous system (2), (3). Let $\Theta(x)$ be any matrix of Lebesgue summable functions of the form

$$\Theta(x) = \left\| \begin{matrix} \Theta_{11}(x) & \Theta_{12}(x) & \cdots & \Theta_{1n}(x) \\ \Theta_{21}(x) & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \Theta_{r1}(x) & \cdot & \cdots & \Theta_{rn}(x) \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{matrix} \right\|$$

and such that the r-rowed determinant whose general element is

$$\int_a^b \Theta_{i\alpha}(x)\, {}^rY_{\alpha j}(x)\, dx$$

is different from zero. Then there exists a matrix $L(t) \equiv \| L_{ij}(t) \|$ each element of which is continuous on $a \leqq t \leqq b$ and such that the matrix

$$G(x;t) = K(x;t) + \Upsilon(x;t) + {}^rY(x)L(t) \tag{33}$$

satisfies the relation

$$\int_a^b \Theta(x)\, G(x;t)\, dx = 0.$$

We have therefore established the following

THEOREM 3. *If the system (2), (3) is compatible of index r, then for each pair of matrices $\Psi(x)$ and $\Theta(x)$ defined as above there exists a matrix $G(x;t)$ which is continuous in (x, t) on $a \leqq {}^x_t \leqq b$ except at $x = t$, which is absolutely continuous in x on $a \leqq x < t$ and $t < x \leqq b$, and which is such that*

$$(\partial/\partial x)\, G(x;t) = A(x)\, G(x;t) + N(x;t) \quad \text{on } X_0, \tag{34}$$

$$G(t+;t) - G(t-;t) = E, \tag{35}$$

$$MG(a;t) + NG(b;t) = 0, \tag{36}$$

$$\int_a^b \Theta(x)\, G(x;t)\, dx = 0. \tag{37}$$

THEOREM 4. *The matrix $G(x;t)$ satisfying the conditions of Theorem 3 is a generalized Green's matrix for the compatible system (2), (3).*

Let

$$u(x)=\int_a^b G(x;t)g(t)\,dt,$$

where the form of $G(x;t)$ is given by (33). Then in view of (13),

$$u(x)=Y(x)\int_a^b[(1/2)Z(t)\mid x-t\mid/(x-t)$$
$$+C(t)]g(t)\,dt+{}^rY(x)\int_a^b L(t)g(t)\,dt.$$

Then $u(x)$ is absolutely continuous on X and on X_0 we have

$$u'(x)=A(x)u(x)+g(x).$$

From (36) we obtain $Mu(a)+Nu(b)=0$ and therefore $u(x)$ satisfies the compatible semi-homogeneous system (1), (3). Then every solution $y(x)$ of the system (1), (3) may be written in the form

$$y(x)=\sum_{\alpha=1}^{r}{}^rY_\alpha(x)a_\alpha+\int_a^b G(x;t)g(t)\,dt,$$

and therefore the matrix $G(x;t)$ defined by (33) is a generalized Green's matrix for (2), (3).

A generalized Green's matrix for (2), (3) which satisfies the conditions of Theorem 3 will be called a *principal generalized Green's matrix* for the compatible system (2), (3).

COROLLARY. *If* $u(x)=\int_a^b G(x;t)f(t)\,dt$, *where* $G(x;t)$ *is a principal generalized Green's matrix for the system* (2), (3) *and* $f(x)\equiv[f_\alpha(x)]$ *is any summable vector, then*

$$\begin{aligned}&u'(x)=A(x)u(x)+f(x)+\int_a^b N(x;t)f(t)\,dt \text{ on } X_0,\\ &Mu(a)+Nu(b)=0, \qquad\qquad (38)\\ &\int_a^b \Theta(x)u(x)\,dx=0.\end{aligned}$$

Now let

$$\mathfrak{N}=\left\|\begin{matrix}\mathfrak{N}_{11} & \mathfrak{N}_{12} & \cdot & \cdot & \mathfrak{N}_{1r} & 0 & \cdot & 0\\ \mathfrak{N}_{21} & \cdot & \cdot & \cdot & \mathfrak{N}_{1r} & 0 & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & 0\\ \mathfrak{N}_{r1} & \cdot & \cdot & \cdot & \mathfrak{N}_{rr} & 0 & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & 0\end{matrix}\right\|,$$

where the r-rowed matrix

$$\begin{Vmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} & \cdot & \cdot & \mathfrak{R}_{1r} \\ \mathfrak{R}_{21} & \cdot & \cdot & \cdot & \mathfrak{R}_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathfrak{R}_{r1} & \cdot & \cdot & \cdot & \mathfrak{R}_{rr} \end{Vmatrix}$$

is the unique reciprocal of

$$\begin{Vmatrix} \int_a^b \Theta_{1\alpha}(x)\,{}^rY_{\alpha 1}(x)\,dx & \cdot & \cdot & \int_a^b \Theta_{1\alpha}(x)\,{}^rY_{\alpha r}(x)\,dx \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int_a^b \Theta_{r\alpha}(x)\,{}^rY_{\alpha 1}(x)\,dx & \cdot & \cdot & \int_a^b \Theta_{r\alpha}(x)\,{}^rY_{\alpha r}(x)\,dx \end{Vmatrix}. \tag{39}$$

With respect to the matrices $\Psi(x)$ and $\Theta(x)$ a matrix $H(x;t)$ is said to be a *principal generalized Green's matrix* for the compatible adjoint system if each element is continuous in (x, t) on $a \leqq {x \atop t} \leqq b$ except at $x = t$, if each element is absolutely continuous in x on $a \leqq x < t$ and $t < x \leqq b$, and if furthermore:

$$(\partial/\partial x)\tilde{H}(x;t) = -\tilde{H}(x;t)A(x) - {}^rY(t)\mathfrak{R}\Theta(x) \text{ on } X_0, \tag{40}$$

$$H(t+;t) - H(t-;t) = E, \tag{41}$$

$$\tilde{H}(a;t)P + \tilde{H}(b;t)Q = 0, \tag{42}$$

$$\int_a^b \tilde{H}(x;t)\Psi(x)\,dx = 0. \tag{43}$$

The existence of a principal generalized Green's matrix for the adjoint system may be established by argument similar to that used in the proof of Theorem 3.

THEOREM 5. *If $G(x;t)$ and $H(x;t)$ are principal generalized Green's matrices for the system* (2), (3) *and the adjoint system* (4), (5) *respectively, with respect to a pair of matrices $\Psi(x)$ and $\Theta(x)$, then $G(x;t) = -\tilde{H}(t;x)$.*

For let ξ and η be any two distinct points of X and suppose $\xi < \eta$. Then we have the matrix equation

$$\begin{aligned} \int_a^b \{\tilde{H}(x;\xi)[(\partial/\partial x)G(x;\eta) - A(x)G(x;\eta)] \\ + [(\partial/\partial x)\tilde{H}(x;\xi) + \tilde{H}(x;\xi)A(x)]G(x;\eta)\}dx \\ = \tilde{H}(x;\xi)G(x;\eta)\Big|_{x=a}^{x=\xi-} + \tilde{H}(x;\xi)G(x;\eta)\Big|_{x=\xi+}^{x=\eta-} + \tilde{H}(x;\xi)G(x;\eta)\Big|_{x=\eta+}^{x=b} \end{aligned}$$

But

$$\int_a^b \tilde{H}(x;\xi)[(\partial/\partial x)G(x;\eta)-A(x)G(x;\eta)]dx$$
$$= -\int_a^b \tilde{H}(x;\xi)\Psi(x)R^r Z(\eta)\,dx,$$

and

$$= 0,$$

$$\int_b^b [(\partial/\partial x)\tilde{H}(x;\xi)+\tilde{H}(x;\xi)A(x)]G(x;\eta)\,dx$$
$$= -\int_a^b {}^r\Upsilon(\xi)\mathfrak{R}\Theta(x)G(x;\eta)\,dx,$$
$$= 0.$$

Therefore

$$\tilde{H}(x;\xi)G(x;\eta)\,\Big|_{x=a}^{x=b} = \tilde{H}(x;\xi)G(x;\eta)\,\Big|_{x=\xi-}^{x=\xi+} + \tilde{H}(x;\xi)G(x;\eta)\,\Big|_{x=\eta-}^{x=\eta+}.$$

We have in view of (25), (36) and (42) that

$$[\tilde{H}(\xi+;\xi)-\tilde{H}(\xi-;\xi)]G(\xi;\eta) = -\tilde{H}(\eta;\xi)[G(\eta+;\eta)-G(\eta-;\eta)]$$

and hence in view of (35) and (41) that

$$G(\xi;\eta) = -\tilde{H}(\eta;\xi).$$

Since this is true for each pair of distinct points ξ and η, we have established Theorem 5.

COROLLARY. *For any chosen pair of matrices* $\Psi(x)$ *and* $\Theta(x)$ *satisfying the conditions described above, the principal generalized Green's matrix for the compatible system* (2), (3), *and also for the compatible adjoint system* (4), (5), *is unique.*

The matrices $\Psi(x)$ and $\Theta(x)$ have been chosen so that the r-rowed square matrices (28) and (39) have non-vanishing determinants. In particular, $\Psi(x)$ and $\Theta(x)$ may be determined so that each of the matrices (28) and (39) is the identity matrix. If this is done some of the preceding formulas will simplify considerably.

4. *Examples.* Consider the system

$$y' = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \cdot y, \tag{44}$$

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \cdot y(-\pi) + \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix} \cdot y(\pi) = 0. \tag{45}$$

This system is compatible of order one, and has a solution $y(x) = (\cos x, \sin x)$. The adjoint system is

$$z' = -z \cdot \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \tag{46}$$

$$z(-\pi) \cdot \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} + z(\pi) \cdot \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix} = 0, \tag{47}$$

which has the solution $z(x) = (\cos x, -\sin x)$. According to the notation introduced above we have the matrices ${}^1Y(x)$ and ${}^1Z(x)$ defined by

$${}^1Y(x) = \begin{Vmatrix} \cos x & 0 \\ -\sin x & 0 \end{Vmatrix},$$

and

$${}^1Z(x) = \begin{Vmatrix} \cos x & -\sin x \\ 0 & 0 \end{Vmatrix}.$$

A matrix solution $Y(x)$ of (44) and its reciprocal $Z(x)$, which is a matrix solution of (46), is given by

$$Y(x) = \begin{Vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{Vmatrix}, \qquad Z(x) = \begin{Vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{Vmatrix}.$$

The generalized Green's matrix for the system (44), (45) which is given by the formula (16) is readily found to be equal to the matrix

$$K(x;t) = (1/2) Y(x) Z(t) [\, |x-t| / (x-t) \,].$$

From Theorem 2 it then follows that every generalized Green's matrix for (44), (45) is of the form

$$G(x;t) = K(x;t) + {}^1Y(x) U(t) + V(x) {}^1Z(t), \tag{48}$$

where the matrices $U(t)$ and $V(x)$ depend upon the choice of the matrices $\Psi(x)$ and $\Theta(x)$.

(a) If we choose $\Psi(x) = (1/2\pi)\, {}^1Y(x)$ and $\Theta(x) = (1/2\pi)\, {}^1Z(x)$, then the principal generalized Green's matrix for the system (44), (45) is given by the relation (48), where

$$U(t) = 1/2\pi \begin{Vmatrix} t\cos t & -t\sin t \\ 0 & 0 \end{Vmatrix}, \qquad V(x) = 1/2\pi \begin{Vmatrix} -x\cos x & 0 \\ x\sin x & 0 \end{Vmatrix} = -\bar{U}(x).$$

(b) If we choose

$$\Psi(x) = \Theta(x) = 1/\pi \begin{Vmatrix} \cos x & 0 \\ 0 & 0 \end{Vmatrix},$$

then the values of $U(t)$ and $V(x)$ become

$$U(t) = 1/2\pi \left\| \begin{matrix} t\cos t + \sin t & -t\sin t \\ 0 & 0 \end{matrix} \right\| = -\tilde{V}(t).$$

(c) If

$$\Psi(x) = 1/\pi \left\| \begin{matrix} \cos x & 0 \\ 0 & 0 \end{matrix} \right\|, \qquad \Theta(x) = -1/\pi \left\| \begin{matrix} 0 & \sin x \\ 0 & 0 \end{matrix} \right\|,$$

the corresponding values of $U(t)$ and $V(x)$ are given by

$$U(t) = 1/2\pi \left\| \begin{matrix} t\cos t - \sin t & -t\sin t \\ 0 & 0 \end{matrix} \right\|,$$

$$V(x) = 1/2\pi \left\| \begin{matrix} -x\cos x \quad \sin x & 0 \\ x\sin x & 0 \end{matrix} \right\|.$$

For each choice of the matrices $\Psi(x)$ and $\Theta(x)$ the corresponding value of $H(x;t)$, the principal generalized Green's matrix for the adjoint system (46), (47), is determined by the relation $\tilde{H}(x;t) = -G(t;x)$, in view of Theorem 5.

5. *Remark.* The author has recently shown that for certain types of infinite systems of ordinary linear differential equations of the first order with two-point boundary conditions an ordinary Green's matrix may be defined, whenever the system is incompatible, in a manner entirely analogous to that used in the finite case.* For such infinite linear systems we may establish by the above method the existence of the principal generalized Green's matrix.

UNIVERSITY OF CHICAGO,
CHICAGO, ILL.

A CREMONA GROUP OF ORDER THIRTY-TWO OF CUBIC TRANSFORMATIONS IN THREE-DIMENSIONAL SPACE.

By Ethel Isabel Moody.

1. *Introduction.* Since each four-dimensional quadric variety of the ∞^5 system $|H(z)| = 0$ having a common self-polar simplex Σ is transformed into itself by the G_{32} of linear transformations consisting of the six central harmonic homologies, defined by the vertices and opposite four-dimensional faces of Σ, and the different products of these, the intersections of these varieties by twos, threes, $\cdots$ are also invariant under G_{32}. By successive stereographic projections these varieties can be mapped in S_3. It is the purpose of this paper to derive the equations of the corresponding transformations in S_4 and in S_3 and to determine their characteristics and fundamental elements.

In a paper, published in the *Atti del R. Istituto Veneto,** Montesano has given a brief synthetic outline of this group of transformations, which he calls the group *II,* and also of the corresponding groups associated with two and three dimensional quadric varieties.

The corresponding G_4 in S_2 is expressed by the non-cyclic G_4 of harmonic homologies.

In S_3 the G_8 of central and axial involutions can be projected into a plane, where it is represented by four perspective quadratic inversions, and the three products of these in pairs. A cubic curve is left invariant; on it the three non-perspective inversions fix the three fundamental irrational involutions belonging to the curve.†

The G_{16} in S_4 can be mapped in S_3 by means of a group of sixteen quadratic transformations which transform into itself each surface of an ∞^3 system of $F_4: C_2{}^2$. Within this system there are ∞^2 which are composite consisting of the plane of the conic and a cubic surface.‡ By mapping an F_3

* D. Montesano, *Su alcuni gruppi chiusi di trasformazioni involutorie nel piano e nello spazio,* Ser. 6, Vol. 6 (1888), pp. 1425-1444.

† The equations and properties of this group (Montesano's group γ) were derived by Miss B. I. Hart in her thesis for the degree of Master of Arts, Cornell University, September, 1926.

‡ The equations and essential properties of this group in S_3 (Montesano's group G) of quadratic transformations were derived in my thesis for the degree of Master of Arts, Cornell University, June, 1927.

[illegible]

[illegible]ons, consisting of the perspective Jonquières involutions and their products [illegible]

[illegible] to perspective Jonquières transformations are known, since the cubic curve passing through the vertex O of the transformation J and touching the lines joining O, to the other four vertices at these vertices is uniquely determined. This cubic curve is point by point invariant under the transformation J_i.

2. *The Group of linear transformations in S_5.* The equation of any four-dimensional quadratic variety in S_5 referred to a self-polar simplex as simplex of reference can be written in the form

$$H(z) \equiv \sum_{i=1}^{6} h_i z_i^2 = 0.$$

This variety is invariant under the group of thirty-two linear transformations, consisting of the six central harmonic homologies

$$T_i: \quad z_i' = -z_i, \quad z_r' = z_r; \qquad (r = 1 \cdots 6,\ r \neq i),$$
$$i = 1 \cdots 6$$

the fifteen products of these in pairs

$$T_{ij}: \quad z_i' = -z_i, \quad z_j' = -z_j, \quad z_r' = z_r; \quad (r = 1 \cdots 6,\ r \neq i, j),$$
$$i = 1 \cdots 5,\ j = 2 \cdots 6,\ i \neq j,$$

the ten products of these taken three at a time

$$T_{1ij}: \quad z_1' = -z_1, \quad z_i' = -z_i, \quad z_j' = -z_j, \quad z_r' = z_r; \quad (r = 2 \cdots 6,\ r \neq i, j),$$
$$i = 2 \cdots 5,\ j = 3 \cdots 6,\ i \neq j,\ i, j \neq 1$$

and the identity. The products of the T_i taken one, two, and three at a time will be referred to as transformations of the first, second, and third species respectively.

A transformation of the first species T_i is defined by the center

$$O_i, \ (z_i = 1, \ z_r = 0), \ (r = 1 \cdots 6,\ r \neq i)$$

and the four-space S_i of invariant points

while those of a transformation of the third species are the two planes of invariant points

$$(O_1O_iO_j): \qquad z_{r_1}=0, \quad z_{r_2}=0, \quad z_{r_3}=0,$$

and

$$S_2: \qquad z_1=0, \quad z_i=0, \quad z_j=0.$$

The sections of $H(z)=0$ by the invariant S_4, S_3, and S_2 are the point by point invariant varieties Ω_i, Ω_{ij}, Ω_{1ij} of the transformations of the first, second, and third species respectively. These are all quadric varieties, Ω_i being three-dimensional, Ω_{ij} two-dimensional and Ω_{1ij} one-dimensional.

3. *The G'_{32} of quadratic transformations in S_4.* Let $(\bar{z})$ be a point of $H(z)=0$ and from it project $H(z)=0$ into $S_4: z_6=0$. The equations of this stereographic projection are

$$y_g=\bar{z}_6z_g-z_6\bar{z}_g; \qquad (g=1\cdots5), \tag{3.1}$$

where the y's represent the coördinates of points in $z_6=0$. Conversely, given a point (y) in $z_6=0$, the coördinates of the point corresponding to it on $H(z)=0$ are

$$\begin{aligned} z_g&=2y_gH'(y,\bar{z})-\bar{z}_gH'(y); \qquad (g=1\cdots5)\\ z_6&=-\bar{z}_6H'(y) \end{aligned} \tag{3.2}$$

where

$$H'(y)=\sum_{g=1}^{5}h_gy_g{}^2$$

and

$$H'(y,\bar{z})=\sum_{g=1}^{5}h_gy_g\bar{z}_g.$$

The projection T' of T in $z_6=0$ is the product $(3.2)T(3.1)$, and the equations of the transformations of G'_{32} are

$$\begin{array}{lll}
T_i': & y_i'=-y_iH'(y,\bar{z})+\bar{z}_iH'(y) & \\
(i=1\cdots5) & y_r'=y_rH'(y,\bar{z}); & (r=1\cdots5,\ r\neq i)\\[1ex]
T_6': & y_r'=y_rH'(y,\bar{z})-\bar{z}_rH'(y); & (r=1\cdots5)\\[1ex]
T'_{ij}: & y_i'=-y_iH'(y,\bar{z})+\bar{z}_iH'(y) & \\
(i=1\cdots5 & y_j'=-y_jH'(y,\bar{z})+\bar{z}_jH'(y) & \\
\ j=2\cdots5 & y_r'=y_rH'(y,\bar{z}); & (r=1\cdots5,\ r\neq i,j)\\
\ i\neq j) & & \\[1ex]
T'_{i6}: & y_i'=-y_iH'(y,\bar{z}) & \\
(i=1\cdots5) & y_r'=y_rH'(y,\bar{z})-\bar{z}_rH'(y); & (r=1\cdots5,\ r\neq i)
\end{array}$$

$$\begin{array}{lll} T'_{1ij}: & y_1' = -y_1 H'(y,\bar z) + \bar z_1 H'(y) & \\ (i = 2 \cdots 4 & y_i' = -y_i H'(y,\bar z) + \bar z_i H'(y) & \\ \ j = 3 \cdots 5 & y_j' = -y_j H'(y,\bar z) + \bar z_j H'(y) & \\ \ i \neq j) & y_r' = y_r H'(y,\bar z); & (r = 2 \cdots 5,\ r \neq i, j) \end{array}$$

and

$$\begin{array}{lll} T'_{1i6}: & y_1' = -y_1 H'(y,\bar z) & \\ (i = 2 \cdots 5) & y_i' = -y_i H'(y,\bar z) & \\ & y_r' = y_r H'(y,\bar z) - \bar z_r H'(y); & (r = 2 \cdots 5,\ r \neq i). \end{array}$$

The quadric surface

$$Q: \qquad H'(y,\bar z) = 0, \quad H'(y) = 0$$

is a common fundamental element of all the transformations of G'_{32}.

The fundamental points of the transformations of G'_{32} are

$$\begin{array}{ll} O_i': & (y_i = 1,\quad y_r = 0) \\ O_6': & (y_r = \bar z_r) \\ O'_{ij}: & (y_i = \bar z_i,\quad y_j = \bar z_j,\quad y_r = 0) \\ O'_{i6}: & (y_i = 0,\quad y_r = \bar z_r) \\ O'_{1ij}: & (y_1 = \bar z_1,\quad y_i = \bar z_i,\quad y_j = \bar z_j,\quad y_r = 0) \\ O'_{1i6}: & (y_1 = y_i = 0,\quad y_r = \bar z_r) \end{array}$$

For each of these O' the restrictions on the i, j, and r are the same as for the T with corresponding subscript.

The image of the fundamental point O' under the corresponding transformation T' is the $S_3: H'(y,\bar z) = 0$.

The image of Q under a transformation of the first species of G'_{32} is its three-dimensional projecting cone from O'. To prove this, let (p) be a point of Q and let the coördinates of O' be represented by Y. The coördinates of any point (y) on the line joining (p) to O' are of the form

$$y_g = \lambda Y_g + \mu p_g; \qquad (g = 1 \cdots 5).$$

By substituting these values for the y's in the equations of the transformations of the first species of G'_{32} and making use of the fact that $H'(p,\bar z) = 0$ and $H'(p) = 0$, the resulting point (y') is found to be (p). Hence each point of Q is imaged by a generator of the projecting cone of Q having its vertex at O'. This cone is of the second order. Its equation for T_i' is

$$[H'(y,\bar z)_i]^2 + h_i \bar z_i^2 H'(y)_i = 0;\text{*}$$

and for T_6'

* A subscript written after the parenthesis will be used throughout this paper to indicate that the function to which it applies contains no term in the variable of that subscript.

$$H'(\bar{z})H'(y) - [H'(y,\bar{z})]^2 = 0.$$

This cone with $[H'(y,\bar{z})]^3$ forms the complete Jacobian of the defining web.

The Jacobian of the defining web of a transformation of the second or third species of G'_{32} consists of $[H'(y,\bar{z})]^3$ and a three-dimensional quadric variety containing the quadric surface Q, of which it is the image, simply. The equation of this quadric for T'_{ij} is

$$[H'(y,\bar{z})]^2 - 2(h_i y_i \bar{z}_i + h_j \bar{z}_j y_j)H'(y,\bar{z}) \\ + (h_i \bar{z}_i^2 + h_j \bar{z}_j^2)H'(y) = 0;$$

that for T'_{i6},

$$H'(y,\bar{z})[H'(y,\bar{z}) - 2H'(y,\bar{z})_i] + H'(y)H'(\bar{z})_i = 0;$$

that for T'_{1ij},

$$H'(y,\bar{z})[H'(y,\bar{z}) - 2h_1 y_1 \bar{z}_1 - 2h_i y_i \bar{z}_i - 2h_j y_j \bar{z}_j] \\ + H'(y)[h_1 \bar{z}_1^2 + h_i \bar{z}_i^2 + h_j \bar{z}_j^2] = 0;$$

and that for T'_{1i6},

$$H'(y,\bar{z})[H'(y,\bar{z}) - 2h_{r_1} y_{r_1} \bar{z}_{r_1} - 2h_{r_2} y_{r_2} \bar{z}_{r_2} - 2h_{r_3} y_{r_3} \bar{z}_{r_3}] \\ + H'(y)[h_{r_1} \bar{z}_{r_1} + h_{r_2} \bar{z}_{r_2} + h_{r_3} \bar{z}_{r_3}] = 0.$$

The projection upon $z_6 = 0$ of the point by point invariant quadric variety Ω of a transformation T of the first species of G_{32} is the invariant variety Ω' of the corresponding transformation T' of the first species of G'_{32}. The equations of the Ω''s are therefore

$$\Omega_i': \quad 2y_i H'(y,\bar{z}) - \bar{z}_i H'(y) = 0; \qquad (i = 1 \cdots 5)$$

and

$$\Omega_6': \quad H'(y) = 0.$$

The result of eliminating y_5 between Ω_i' and Ω_j' shows that the complete intersection of these two varieties lies in the three-space $y_i \bar{z}_j - y_j \bar{z}_i = 0$, and the intersection of this three-space with each of the quadrics Ω_i' and Ω_j' or

$$\begin{array}{ll} \Omega'_{ij} & 2\bar{z}_j y_j H'(y,\bar{z})_i - \bar{z}_j^2 H'(y)_i + h_i y_j^2 \bar{z}_i^2 = 0 \\ (i = 1 \cdots 5 & y_i \bar{z}_j - y_j \bar{z}_i = 0 \\ \ j = 2 \cdots 6 \quad i \neq j) & \end{array}$$

is the point by point invariant quadric surface of T'_{ij}; and

$$\Omega'_{i6}: \quad H'(y)_i = 0, \quad y_i = 0; \qquad (i = 1 \cdots 5)$$

is the point by point invariant surface of T'_{i6}. In a similar manner, the point by point invariant conic of T'_{1ij} is

$$\begin{array}{ll} \Omega'_{1ij}: & 2\bar{z}_j y_j H'(y,\bar{z})_{i,1} - \bar{z}_j^2 H'(y)_{i,1} + y_j^2(h_1 \bar{z}_1^2 + h_i \bar{z}_i^2) = 0 \\ (i = 2 \cdots 5, & y_1 \bar{z}_i \bar{z}_j = y_i \bar{z}_1 \bar{z}_j = y_j \bar{z}_1 \bar{z}_i. \\ \ j = 3 \cdots 6, \quad i \neq j, \quad i,j \neq 1) & \end{array}$$

The corresponding conic for T'_{1i6} is

$$\begin{array}{ll} \Omega'_{1i6}: & H'(y) = 0 \\ (i = 2 \cdots 5) & y_1 = y_i = 0. \end{array}$$

Given a point common to $H(z) = 0$ and another quadric variety $G(z) = 0$ of the system $| H(z) | = 0$, the coördinates of its projection upon $z_6 = 0$ are

$$(3.4) \qquad \rho y_i = \bar{z}_6 z_i - z_6 \bar{z}_i; \qquad (i = 1 \cdots 5).$$

From (3.4) it follows that

$$(3.5) \qquad z_i / z_6 = (k y_i + \bar{z}_i) / \bar{z}_6.$$

If the values of the ratios in (3.5) are substituted in $H(z) = 0$ and $G(z) = 0$, the resulting equations are

$$(3.6) \qquad \begin{array}{c} G(\bar{z}) + 2kG'(y, \bar{z}) + k^2 G'(y) = 0 \\ 2H'(y, \bar{z}) + kH'(y) = 0. \end{array}$$

The result of eliminating k between the equations (3.6) is

$$G(\bar{z}) [H'(y)]^2 - 4G'(y, \bar{z}) H'(y) H'(y, \bar{z}) + 4G'(y) [H'(y, \bar{z})]^2 = 0.$$

This equation represents a three-dimensional quartic variety in S_4 having the quadric surface

$$F_2: \quad H'(y) = 0, \quad H'(y, \bar{z}) = 0$$

as double quadric. The three-space $H'(y, z) = 0$ of the double quadric is the polar three-space of O_i' with respect to Ω_i'.

Every three-dimensional quartic variety having a double quadric is invariant under a G'_{32}, for such a variety is the projection in S_4 of the intersection of two quadric varieties in S_5 which have a common self-polar simplex.

Let $A(z) = 0$ be a second quadric variety belonging to the system $| H(z) | = 0$ and containing the center of projection $(\bar{z})$. The coördinates of any point (z') on the line joining a point (z) to $(\bar{z})$ are

$$z_i' = \lambda \bar{z}_i + \mu z_i; \qquad (i = 1 \cdots 6).$$

The points of intersection of this line with $A(z) = 0$ and $H(z) = 0$ are given by

$$2\lambda H(z, \bar{z}) + \mu H(z) = 0, \qquad 2\lambda A(z, \bar{z}) + \mu A(z) = 0.$$

The condition that these two equations in λ and μ have a solution in common is

$$H(z, \bar{z}) A(z) - A(z, \bar{z}) H(z) = 0.$$

This is the equation of the projecting cone with vertex at $(\bar{z})$ of the intersection of these two varieties. The equation of the projection of this intersection upon $S_4: z_6 = 0$ is therefore

$$V_3: \quad H'(y,\bar{z})A'(y) - A'(y,\bar{z})H'(y) = 0,$$

which is the equation of a three-dimensional cubic variety containing a plane. The equations of this plane are

$$H'(y,\bar{z}) = 0, \qquad A'(y,\bar{z}) = 0.$$

But, in general, the projection upon S_4 of the intersection of two quadric varieties is a quartic variety. The residual part of this projection is the three-space corresponding to $(\bar{z})$ in the stereographic projection. This three-space is

$$H'(y,\bar{z}) = 0,$$

the intersection of the tangent S_4 to

$$H(z) = 0$$

at $(\bar{z})$ and the S_4 of projection. This is the three-space of the double quadric of V_4.

Therefore, among the ∞^4 quartic varieties which are invariant under G'_{32} there are ∞^3 which are composite, consisting of the cubic variety and the three-space of the double quadric.

4. G''_{32} *of cubic transformations in* S_3. The plane

$$A'(y,\bar{z}) = 0, \qquad H'(y,\bar{z}) = 0$$

meets the quartic surface

$$A'(y) = 0, \qquad H'(y) = 0$$

in four points which are double points of

$$V_3: \quad H'(y,\bar{z})A'(y) - A'(y,\bar{z})H'(y) = 0.^*$$

V_3 can be projected stereographically into $S_3: y_5 = 0$ from any one $(\bar{y})$ of these four double points. The equations of the projection are

$$x_g = y_g\bar{y}_5 - y_5\bar{y}_g; \qquad (g = 1 \cdots 4). \tag{4.1}$$

Conversely, given a point (x) in $y_5 = 0$, the point corresponding to it on V_3 is

$$\begin{aligned} y_g &= w\bar{y}_g + kx_g; \qquad (g = 1 \cdots 4) \\ y_5 &= w\bar{y}_5, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} k &= -2\,[H''(x,\bar{z})A''(x,\bar{y}) - A''(x,\bar{z})H''(x,\bar{y})] \\ w &= H''(x,\bar{z})A''(x) - A''(x,\bar{z})H''(x).\dagger \end{aligned}$$

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* The properties of a three-dimensional cubic variety containing a plane are discussed by Segre, "Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario." See sections 5-11. *Memorie della Reale Accademia delle Scienze di Torino*, Ser. 2, Vol. 39 (1889), pp. 3-48.

† The H'' and A'' are used to denote functions similar to the H''s of equations (3.2) but for which the summations are from $g = 1$ to $g = 4$.

The projection T'' of a transformation T' in [illegible] is the product (4.2) (T') (4.1). Let

$$m = A''(x)H''(x, y) - A''(x, y)H''(x).$$

The equations of the transformations T'' can then be written in the form

T_i'': $\quad x_i' = -kx_i - 2w\bar{y}_i + 2\bar{z}_i m$

$(i = 1 \cdots 4)$ $\quad x_r' = kx_r; \qquad (r = 1 \cdots 4, \quad r \neq i).$

T_5'': $\quad x_r' = -kx_r\bar{y}_5 + 2w\bar{y}_5\bar{y}_r - 2\bar{z}_5\bar{y}_r m; \qquad (r = 1 \cdots 4).$

T_6'': $\quad x_r' = x_r\bar{y}_5 k - 2(\bar{z}_r\bar{y}_5 - \bar{z}_5\bar{y}_r)m; \qquad (r = 1 \cdots 4).$

T''_{ij}: $\quad x_i' = -kx_i - 2w\bar{y}_i + 2\bar{z}_i m$

$(i = 1 \cdots 3$ $\quad x_j' = -kx_j - 2w\bar{y}_j + 2\bar{z}_j m$

$j = 2 \cdots 4$ $\quad x_r' = kx_r; \qquad (r = 1 \cdots 4, \quad r \neq i, j).$

$i \neq j)$

T''_{i5}: $\quad x_i' = -kx_i\bar{y}_5 + 2(\bar{z}_i\bar{y}_5 - \bar{z}_5\bar{y}_i)m$

$(i = 1 \cdots 4)$ $\quad x_r' = kx_r\bar{y}_5 + 2\bar{y}_r\bar{y}_5 w - 2\bar{z}_5\bar{y}_r m; \quad (r = 1 \cdots 4, \quad r \neq i).$

T''_{i6}: $\quad x_i' = -kx_i\bar{y}_5 - 2w\bar{y}_i\bar{y}_5 + 2\bar{z}_5\bar{y}_i m$

$(i = 1 \cdots 4)$ $\quad x_r' = kx_r\bar{y}_5 - 2(\bar{z}_r\bar{y}_5 - \bar{z}_5\bar{y}_r)m; \quad (r = 1 \cdots 4, \quad r \neq i).$

T''_{56}: $\quad x_r' = kx_r + 2w\bar{y}_r - 2\bar{z}_r m; \qquad (r = 1 \cdots 4).$

T''_{1ij}: $\quad x_1' = -kx_1 - 2w\bar{y}_1 + 2\bar{z}_1 m$

$(i = 2 \cdots 4$ $\quad x_i' = -kx_i - 2w\bar{y}_i + 2\bar{z}_i m$

$j = 3 \cdots 4$ $\quad x_j' = -kx_j - 2w\bar{y}_j + 2\bar{z}_j m$

$i \neq j$ $\quad x_r' = kx_r; \qquad (r = 2 \cdots 4, \quad r \neq i, j).$

$i, j \neq 1)$

T''_{1i5}: $\quad x_1' = -kx_1\bar{y}_5 + 2(\bar{z}_1\bar{y}_5 - \bar{z}_5\bar{y}_1)m$

$(i = 2 \cdots 4)$ $\quad x_i' = -kx_i\bar{y}_5 + 2(\bar{z}_i\bar{y}_5 - \bar{z}_5\bar{y}_i)m$

$\quad x_r' = kx_r\bar{y}_5 + 2w\bar{y}_r\bar{y}_5 - 2\bar{z}_5\bar{y}_r m; \quad (r = 2 \cdots 4, \quad r \neq i).$

T''_{1i6}: $\quad x_1' = -kx_1\bar{y}_5 - 2w\bar{y}_5\bar{y}_1 + 2\bar{y}_1\bar{z}_5 m$

$(i = 2 \cdots 4)$ $\quad x_i' = -kx_i\bar{y}_5 - 2w\bar{y}_5\bar{y}_i + 2\bar{y}_i\bar{z}_5 m$

$\quad x_r' = k\bar{y}_5 x_r - 2m(\bar{z}_r\bar{y}_5 - \bar{z}_5\bar{y}_r); \quad (r = 2 \cdots 4, \quad r \neq i).$

T''_{156}: $\quad x_1' = -kx_1$

$\quad x_r' = kx_r + 2w\bar{y}_r$ [illegible] $\qquad (r = 2 \cdots 4)$

respectively, have a solution in common. Therefore, if $k=0$ and $w=0$ when $H''(x,\bar{z})$ and $A''(x,\bar{z})$ are not both zero, $m=0$. But $k=0$ and $w=0$ intersect in a C_6, which is composite, consisting of the straight line

$$C_1: \quad H''(x,\bar{z})=0, \quad A''(x,\bar{z})=0$$

and a C_5. This C_5 is therefore a fundamental curve of all of the transformations of G''_{32}.

The fundamental points O'' of the transformations T'' of the first species are:

$$\begin{aligned} O_i'' &: \quad (x_i=1, \quad x_r=0) \\ O_5'' &: \quad (x_r=\bar{y}_r) \\ O_6'' &: \quad (x_r=\bar{z}_r\bar{y}_5-\bar{z}_5\bar{y}_r). \end{aligned}$$

The point O'' is a double point of each of the surfaces of the defining web of the corresponding T'', and hence the six transformations of the first species of G''_{32} are monoidal. The fundamental points O'' all lie on the common fundamental C_5.

If (p) represents a point on this C_5 and if the point O'' is represented by (x'), the coördinates of any point (x) on a line joining (p) to (x') are

$$(4.3) \qquad x_g=\lambda p_g+\mu x_g'; \qquad (g=1\cdot\cdot\cdot 4).$$

If the x's in the equations of the transformations of the first species of G''_{32} are replaced by their values from (4.3), the image of any point on this line is found to be the point (p). Therefore, the image of C_5 under a transformation of the first species of G''_{32} is the quartic cone having the corresponding O'' as vertex and containing C_5.

The image of O'' under its corresponding T'' is the quadric $k=0$. Both C_1 and C_5 lie on $k=0$. The two systems of generators of this quadric are

$$(1) \qquad \begin{aligned} A''(x,\bar{z})-\mu H''(x,\bar{z})&=0 \\ A''(x,\bar{y})-\mu H''(x,\bar{y})&=0 \end{aligned}$$

and

$$(2) \qquad \begin{aligned} A''(x,\bar{z})-\lambda A''(x,\bar{y})&=0 \\ H''(x,\bar{z})-\lambda H''(x,\bar{y})&=0. \end{aligned}$$

C_1 is a line of (2), and, therefore, its symbol on $k=0$ is [1, 0]. The symbol of the complete intersection of $k=0$ and $w=0$ is [3, 3], and therefore, C_5 must be of symbol [2, 3] on $k=0$. Hence, through each point O'', passes one line of the regulus (2) and therefore one trisecant of C_5. This line is a fundamental element of the transformation to which O'' belongs, each point

of the line having for its image the whole line. The equations of these fundamental lines are:

L_i:

$$\bar{y}_i H''(x,\bar{z}) - \bar{z}_i H''(x,\bar{y}) = 0$$
$$\bar{y}_i A''(x,\bar{z}) - \bar{z}_i A''(x,\bar{y}) = 0$$

L_5:

$$H''(x,\bar{z}) H''(\bar{y}) - H''(x,\bar{y}) H''(\bar{y},\bar{z}) = 0$$
$$A''(x,\bar{z}) H''(\bar{y}) - A''(x,\bar{y}) H''(\bar{y},\bar{z}) = 0$$

L_6:

$$H''(x,\bar{z})\,[\bar{y}_5 H''(\bar{y},\bar{z}) - \bar{z}_5 H''(\bar{y})] - H''(x,\bar{y})\,[\bar{y}_5 H''(\bar{z}) - \bar{z}_5 H''(\bar{y},\bar{z})] = 0$$
$$A''(x,\bar{z})\,[\bar{y}_5 H''(\bar{y},\bar{z}) - \bar{z}_5 H''(\bar{y})] - A''(x,\bar{y})\,[\bar{y}_5 H''(\bar{z}) - \bar{z}_5 H''(\bar{y},\bar{z})] = 0.$$

The complete Jacobian of a transformation of the first species consists of k^2 and the quartic cone.

The invariant surfaces of the transformations of the first species of G''_{32} are found by projecting the Ω''s into $y_5 = 0$. Their equations are:

$$\Omega_i'': \quad w\bar{y}_i + kx_i - \bar{z}_i m = 0$$
$$\Omega_5'': \quad w\bar{y}_5 - \bar{z}_5 m = 0$$
$$\Omega_6'': \quad m = 0.$$

The general plane of the system through O_i'' is

$$\lambda x_{r_1} + \mu x_{r_2} + \nu x_{r_3} = 0.$$

Under T_i'' this becomes

$$k(x_{r_1}\lambda + \mu x_{r_2} + \nu x_{r_3}) = 0.$$

But $k = 0$ is the image of O'', and, therefore, the plane is transformed into itself. Any line l in such a plane, π, goes into a cubic curve under T_i''. Since the line l meets k in two points this C_3 must have a double point at O_i''. But, if a line l meets the fundamental trisecant L_i, the trisecant itself is a component of the image, and the line goes into a conic. But l has one other point S in common with $k = 0$ and meets each of the lines joining the point O'' to the two residual intersections R_1, R_2, of C_5 and π. Therefore, the image conic passes through O_i'', R_1 and R_2. A similar argument holds for T_5'' and T_6''. Therefore, in any plane of the pencil through L, the corresponding T'' of the first species of G''_{32} is quadratic, the image of any straight line l being a conic through R_1, R_2, and O''.

For transformations of the second species, the fundamental elements which are transformed into $k = 0$ are the straight lines:

$$
\begin{array}{lll}
O_i''O_j'': & x_{r_1}=0; & (r_1=1\cdots 4) \\
& x_{r_2}=0; & (r_2=1\cdots 4,\quad r_1, r_2\neq i, j,\quad r_1\neq r_2). \\
O_i''O_5'': & x_{r_1}\bar{y}_{r_3}-x_{r_3}\bar{y}_{r_1}=0 & \\
& x_{r_1}\bar{y}_{r_2}-x_{r_2}\bar{y}_{r_1}=0 & \\
& \quad (r_1=1\cdots 4,\quad r_2=1\cdots 4,\quad r_3=1\cdots 4) & \\
& \qquad\qquad (r_1\neq r_2\neq r_3;\quad r_1, r_2, r_3\neq i). & \\
O_i''O_6'': & x_{r_1}(\bar{z}_{r_2}\bar{y}_5-\bar{z}_5\bar{y}_{r_2})-x_{r_2}(\bar{z}_{r_1}\bar{y}_5-\bar{z}_5\bar{y}_{r_1})=0 & \\
& x_{r_1}(\bar{z}_{r_3}\bar{y}_5-\bar{z}_5\bar{y}_{r_3})-x_{r_3}(\bar{z}_{r_1}\bar{y}_5-\bar{z}_5\bar{y}_{r_1})=0 & \\
& \quad (r_1=1\cdots 4,\quad r_2=1\cdots 4,\quad r_3=1\cdots 4) & \\
& \qquad\qquad (r_1\neq r_2\neq r_3;\quad r_1, r_2, r_3\neq i). & \\
O_5''O_6'': & x_1(\bar{y}_2\bar{z}_3-\bar{z}_2\bar{y}_3)+x_2(\bar{y}_3\bar{z}_1-\bar{z}_3\bar{y}_1)+x_3(\bar{y}_1\bar{z}_2-\bar{z}_1\bar{y}_2)=0 & \\
& x_2(\bar{y}_3\bar{z}_4-\bar{z}_3\bar{y}_4)+(\bar{y}_4\bar{z}_2-\bar{z}_4\bar{y}_2)x_3+x_4(\bar{y}_2\bar{z}_3-\bar{z}_2\bar{y}_3)=0. &
\end{array}
$$

The cubic surface

$$\bar{z}_j(w\bar{y}_i+kx_i)-\bar{z}_i(w\bar{y}_j+kx_j)=0$$

meets each of the cubics Ω_i' and Ω_j' in the

$$C_9:\quad \bar{z}_j(w\bar{y}_i+kx_i)-\bar{z}_i(w\bar{y}_j+kx_j)=0,\qquad m=0.$$

This C_9 is the complete intersection of the two cubics Ω_i'' and Ω_j'', but it is composite, consisting of the C_5 and a residual C_4. This C_4 is therefore point by point invariant under the transformations T''_{ij}. The equations of these invariant C_4 of the transformations of the second species of G''_{32} are

$$
\begin{array}{ll}
C_{4_{ij}}: & (\bar{z}_j\bar{y}_i-\bar{z}_i\bar{y}_j)A''(x)-2(\bar{z}_jx_i-x_j\bar{z}_i)A''(x,\bar{y})=0 \\
& (\bar{z}_j\bar{y}_i-\bar{z}_i\bar{y}_j)H''(x)-2(\bar{z}_jx_i-x_j\bar{z}_i)H''(x,\bar{y})=0 \\
C_{4_{i5}}: & (\bar{y}_i\bar{z}_5-\bar{z}_i\bar{y}_5)A''(x)-2x_i\bar{z}_5A''(x,\bar{y})=0 \\
& (\bar{y}_i\bar{z}_5-\bar{z}_i\bar{y}_5)H''(x)-2x_i\bar{z}_5H''(x,\bar{y})=0 \\
C_{4_{i6}}: & \bar{y}_iA''(x)-2x_iA''(x,\bar{y})=0 \\
& \bar{y}_iH''(x)-2x_iH''(x,\bar{y})=0 \\
C_{4_{56}}: & A''(x)=0 \\
& H''(x)=0.
\end{array}
$$

If r is the rank of a space curve C_m, and C_m and $C_{m'}$ form the complete intersection of two surfaces, F_μ and F_ν, then

$$m(\mu+\nu-2)=r+t,$$

where t is the number of intersections of C_m and $C_{m'}$. C_4 and C_5 form the complete intersection of two cubics. The C_5 was of symbol $[2, 3]$ on $k=0$, and the number of apparent double points of a curve of symbol $[k_1, k_2]$ is $\frac{1}{2}(k_1^2+k_2^2-k_1-k_2)$. Therefore C_5 has four apparent double points and is of genus 2. $r=m(m-1)-2h$, and therefore C_5, $p=2$, is of rank 12, and $t=8$. Therefore C_5 and C_4 have 8 points in common.

The fundamental lines of the transformations of the third species are:

L_{1ij}: $\quad x_1(\bar{y}_i\bar{z}_j - \bar{z}_i\bar{y}_j) + x_i(\bar{y}_j\bar{z}_1 - \bar{y}_1\bar{z}_j) + x_j(\bar{y}_1\bar{z}_i - \bar{y}_i\bar{z}_1) = 0,$
$\quad x_r = 0$

L_{1i5}: $\quad x_1(\bar{z}_i\bar{y}_5 - \bar{z}_5\bar{y}_i) - x_i(\bar{z}_1 y_5 - \bar{z}_5\bar{y}_1) = 0$
$\quad x_{r_1}\bar{y}_{r_2} - x_{r_2}\bar{y}_{r_1} = 0$

L_{1i6}: $\quad x_{r_1}(\bar{z}_{r_2}\bar{y}_5 - \bar{z}_5\bar{y}_{r_2}) - x_{r_2}(\bar{z}_{r_1}\bar{y}_5 - \bar{z}_5\bar{y}_{r_1}) = 0$
$\quad x_1\bar{y}_i - x_i\bar{y}_1 = 0$

L_{156}: $\quad x_1 = 0$
$\quad x_{r_1}\bar{y}_{r_2}\bar{z}_{r_3} + x_{r_3}\bar{y}_{r_1}\bar{z}_{r_2} + x_{r_2}\bar{y}_{r_3}\bar{z}_{r_1} - x_{r_3}\bar{y}_{r_2}\bar{z}_{r_1} - x_{r_1}\bar{y}_{r_3}\bar{z}_{r_2} - x_{r_2}\bar{y}_{r_1}\bar{z}_{r_3} = 0.$

The invariant elements of the transformations of the third species are isolated points, for the

$$C_9: \quad \bar{z}_1(w\bar{y}_1 + kx_1) = \bar{z}_i(w\bar{y}_i + kx_i) = (w\bar{y}_j + kx_j)\bar{z}_j$$

meets each of the cubic surfaces Ω_1'', Ω_i'', Ω_j'' in the configuration of points represented by the equations

$$\bar{z}_1(w\bar{y}_1 + kx_1) = \bar{z}_i(w\bar{y}_i + kx_i) = \bar{z}_j(w\bar{y}_j + kx_j), \qquad m = 0.$$

These would in general represent 27 points, but the equivalence of C_5, $p = 2$, is 23, and, therefore, there are four residual points of intersection which are invariant under T''_{1ij}. The four points which with C_5 form the complete intersection of

$$\bar{z}_r(w\bar{y}_r + kx_r) = w\bar{z}_5\bar{y}_5 = 0, \qquad m = 0$$

are also invariant under T''_{1ij}. The invariant points of T''_{1i5} are those which with C_5 form the complete intersections of

$$\bar{z}_1(w\bar{y}_1 + kx_1) = \bar{z}_i(w\bar{y}_i + kx_i) = \bar{z}_5\bar{y}_5 w, \qquad m = 0$$

and of

$$\bar{z}_{r_1}(w\bar{y}_{r_1} + kx_{r_1}) = \bar{z}_{r_2}(w\bar{y}_{r_2} + kx_{r_2}) = 0, \qquad m = 0.$$

Those of T''_{1i6} are those which with C_5 form the complete intersection of

$$\bar{z}_1(w\bar{y}_1 + kx_1) = \bar{z}_i(w\bar{y}_i + kx_i) = 0, \qquad m = 0$$

and of

$$\bar{z}_{r_1}(w\bar{y}_{r_1} + kx_{r_1}) = \bar{z}_{r_2}(w\bar{y}_{r_2} + kx_{r_2}) = w\bar{z}_5\bar{y}_5, \qquad m = 0,$$

and those of T''_{156} are those which with C_5 form the complete intersection of

$$\bar{z}_1(w\bar{y}_1 + kx_1) = \bar{z}_5\bar{y}_5 w = 0, \qquad m = 0,$$

and of

$$\bar{z}_{r_1}(w\bar{y}_{r_1} + kx_{r_1}) = \bar{z}_{r_2}(w\bar{y}_{r_2} + kx_{r_2}) = \bar{z}_{r_3}(w\bar{y}_{r_3} + kx_{r_3}), \qquad m = 0.$$

The transformations of the first species of G''_{32} are monoidal, while those of the second and third species are not. The various types of non-

monoidal cubic involutorial transformations have been discussed by F. R. Morris.* In the work which follows, frequent reference has been made to his paper.

If a non-monoidal cubic transformation is to be involutorial, all of the cubics of the defining web must have a C_6, $p=3$ in common, and the image of a point P may be defined as the point of intersection of the polar planes of P with respect to all the quadrics of a bundle having C_6 as the locus of the vertices of the cones of the bundle. If this bundle of quadrics contains a composite quadric, the line of intersection of the component planes of this quadric must be a component of the vertex locus, and the whole locus consists of C_5, $p=2$, and one of its bisecants. This composite sextic is the fundamental curve of the corresponding non-monoidal cubic transformation.

The transformations of the second and third species of G''_{32} are of this type, and, from the properties of the general non-monoidal cubic transformations in S_3 having for their fundamental curves C_6's consisting of C_5, $p=2$, and one of its bisecants, the nature of the principal systems of these transformations can be determined. For the case in which C_6 is non-composite, its image consists of the ∞^1 trisecants of C_6, which constitute a ruled surface $R_8: C_6^3$. If, however, the C_6 is composite, consisting of a C_5 and a line, this R_8 breaks up into two parts, one consisting of the bisecants of C_5 which meet the fundamental line and the other of the trisecants of C_5. The trisecants of C_5 form one regulus of a quadric containing C_5. This quadric is the image of the fundamental line. The bisecants of C_5 which meet the fundamental line generate a ruled surface of order six, containing C_5 as a double curve and the fundamental bisecant as triple line. This sextic surface is the image of C_5. Therefore, for a transformation of either the second or third species, the principal system consists of the quadric surface, $k=0$, image of the fundamental line, and the image of C_5 which is a ruled sextic surface, containing the fundamental line as triple line and the C_5 as double curve.

Any plane through $O_i''O_j''$ has an equation of the form

$$\lambda x_{r_1} + \mu x_{r_2} = 0.$$

Its image under T''_{ij} is

$$k(\lambda x_{r_1} + \mu x_{r_2}) = 0.$$

The image of $O_i''O_j''$ is $k=0$, and therefore under T''_{ij} every plane π of the pencil through $O_i''O_j''$ is transformed into itself. Under T_i'' the image of any straight line l of π is a cubic curve having a double point at O_i''. Since l meets each of the lines joining O_i'' to the three residual points of intersection R_1, R_2, R_3 of C_5 and π, the image cubic must pass through these

* Morris, "Classification of Involutory Cubic Space Transformations," *University of California Publications in Mathematics*, Vol. 1, No. 11 (1920), pp. 223-240.

points. Under T_j'' this cubic goes into a C_9. But, since C_3 contains O_j'', k_π (the section of $k=0$ by π) must be a component of this C_9, as must also the lines joining O_j to the three residual intersections of C_5 and π; and the line $O_i''O_j''$, counted twice, is also a component. Therefore, the image of l is a conic containing R_1, R_2, R_3. The same argument holds for planes through the lines $O_i''O_5''$, $O_i''O_6''$, and $O_5''O_6''$. Therefore, in any plane π of the pencil through a fundamental line, the corresponding transformation of the second species is quadratic, the image of any line l being a conic containing R_1, R_2, and R_3.

The equation of any plane through the line L_{1ij} is

$$\lambda x_1(\bar{y}_i\bar{z}_j-\bar{y}_j\bar{z}_i)+\lambda x_i(\bar{y}_j\bar{z}_1-\bar{z}_j\bar{y}_1)+\lambda x_j(\bar{y}_1\bar{z}_i-\bar{y}_i\bar{z}_1)+\mu x_r=0.$$

Under T''_{1ij} this goes into

$$k[\lambda x_1(\bar{y}_i\bar{z}_j-\bar{y}_j\bar{z}_i)+\lambda x_i(\bar{y}_j\bar{z}_1-\bar{y}_1\bar{z}_j)+\lambda x_j(\bar{y}_1\bar{z}_i-\bar{y}_i\bar{z}_1)-\mu x_r]=0.$$

The component $k=0$ is the image of L_{1ij}, and any plane through L_{1ij} is transformed into a plane through L_{1ij}. The particular plane of the system through L_{1ij} which contains O_1'', O_i'', O_j'' is $x_r=0$, and this is transformed into itself, as is also the plane containing O_r'', O_5'', O_6'' or

$$x_1(\bar{y}_i\bar{z}_j-\bar{y}_j\bar{z}_i)+x_i(\bar{y}_j\bar{z}_1-\bar{y}_1\bar{z}_j)+x_j(\bar{y}_1\bar{z}_i-\bar{y}_i\bar{z}_1)=0.$$

Let the two points common to L_{1ij} and C_5 be R_1 and R_2. Then in the plane determined by O_1'', O_i'', O_j'', any straight line l goes into $C_3: O_iO_jR_1R_2O_1^2$ under T_1''. Under T_i''

$$C_3:\ O_iO_jR_1R_2O_1^2 \sim C_2:\ R_1R_2O_j.$$

Under T_j''

$$C_2:\ R_1R_2O_j \sim C_2:\ O_1O_iO_j.$$

Therefore, in this plane, the transformation T''_{1ij} is quadratic, any line being conjugate to a conic through O_1, O_i, and O_j. A similar argument holds for the plane through L_{1ij} and O_r'', O_5'', O_6'', and also for the corresponding planes through the other fundamental lines L and their corresponding transformations. Hence, associated with each transformation of the third species of G''_{32} are two planes which are transformed each into itself and in which the transformation is quadratic.

The projection in $y_5=0$ of the intersection of the given V_3 and another V_3 of the system

$$H'(y,\bar{z})B'(y)-B'(y,\bar{z})H'(y)=0$$

$$\begin{aligned}F_i:\quad & w^2[H''(x,\bar{z})B''(\bar{y})-2B''(y,\bar{z})H''(x,\bar{y})]\\ &+kw[2H''(x,z)B''(\bar{y},x)-B''(\bar{y},\bar{z})H''(x)-2B''(x,\bar{z})H''(x,\bar{y})]\\ &+k^2[H''(\text{[illegible]})B''(\text{[illegible]})-B''(\text{[illegible]})H''(\text{[illegible]})]=0\end{aligned}$$

on which
$$k = 0, \qquad w = 0,$$
is double. But $H''(x, \bar{z}) = 0$, the image of $(\bar{y})$, is a component of F_7. The other component is

$$F_6: \quad B''(\bar{y})w^2 - 2B''(\bar{y}, \bar{z})wm + 2kwB''(\bar{y}, x) - 2B''(x, \bar{z})km + B''(x)k^2 = 0.$$

The double line
$$C_1: \quad H''(x, \bar{z}) = 0, \quad A''(x, \bar{z}) = 0$$
lies in the plane $H''(x, \bar{z}) = 0$, but the double C_5 does not, and hence there are ∞ $F_6: C_5^2, C_1$, which are invariant under G''_{32}.

Every cubic variety of the ∞^3 system
$$H'(y, \bar{z})B'(y) - B'(y, \bar{z})H'(y) = 0$$
is invariant under G'_{32}. The cubic variety
$$H'(y, \bar{z})A'(y) - A'(y, \bar{z})H'(y) = 0$$
has $(\bar{y})$ as double point, and every variety of the system contains $(\bar{y})$ simply. A variety of the system will have $(\bar{y})$ as a double point provided $B'(\bar{y}) = 0$ and $B'(\bar{y}, \bar{z}) = 0$. Therefore, there are ∞^1 cubics of the system which have $(\bar{y})$ as a double point. Let
$$H'(y, \bar{z})C'(y) - C'(y, \bar{z})H'(y) = 0$$
represent one such variety. The projection in $S_3: y_5 = 0$ of the intersection of this variety and
$$H'(y, \bar{z})A'(y) - A'(y, \bar{z})H'(y) = 0$$
is composite, consisting of
$$H''(x, \bar{z}) = 0$$
and the
$$F_4: \quad kC''(x) - 2mC''(x, \bar{z}) + 2wC''(x, \bar{y}) = 0.$$
Therefore, there are ∞^0 $F_4: C_5$ which are invariant under G''_{32}. The F_4 have all six of the points O'' as double points and contain the fifteen lines joining these in pairs. The quadric cone K_2, determined by a vertex O'' and the five lines of F_4 which pass through it, meets F_4 in a C_8 which is composite, consisting of the five lines and the C_3 determined by the six points O''. The F_4 must therefore contain this C_3. An F_4 having these properties is a Weddle surface, and, since the Weddle surface is known to be irrational, and F_4 and $[H''(x, \bar{z})]^2 = 0$ form a composite F_6 of the system of surfaces invariant under G''_{32}, the general sextic of this system cannot be mapped upon a plane.

CORNELL UNIVERSITY.

ON THE CAPACITY OF SETS OF CANTOR TYPE.*

By Oliver D. Kellogg.

1. *Introduction.* If E denotes any bounded set of points, then E, together with its limit points, contains the boundary of an infinite domain T. To this domain and to the boundary values 1, may be assigned, by the method of sequences,† a harmonic function, called the *conductor potential* of the set E. This conductor potential, V, is unique, in the sense that it is the only one yielded by the method of sequences, and in particular, is independent of the sequence of regions with T as limit. The points of E at which V approaches 1 (and also points of E not boundary points of T), if such exist, are called *regular* points of E. All other points of E are called *exceptional.* The total mass, as given by Gauss' integral, producing the potential V, is called the capacity of E.

The question of the unique determination of a harmonic function by continuous boundary values, not yet generally settled for boundaries containing exceptional points, would be definitely and affirmatively answered if the following lemma were established:

Any bounded closed set of points of positive capacity contains regular points.

This lemma was formulated in 1926,‡ and established in the case of the logarithmic potential in 1928,§ but it has not yet been proved in the case of the Newtonian potential. In view of this fact, and of the central position of the lemma with respect to the Dirichlet problem for general domains, the consideration of the lemma in special cases appears to be a task worth while.

In particular, the sets of Cantor type, studied in the following pages, have a peculiar interest, for the following reason. Some of them have 0 capacity, and some positive capacity. The latter owe their positive capacity to the relatively great separation of the points of the set, a quality which ordinarily reduces the likelihood of regular points. It would accordingly

* Read before the American Mathematical Society, September 11, 1930.

† See Kellogg and Vasilesco, "A Contribution to the Theory of Capacity," *American Journal of Mathematics*, Vol. 51 (1929), pp. 515-526. References to the literature are there given.

‡ Kellogg, "On the Classical Dirichlet Problem for General Domains," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), p. 106, foot note 11.

§ Kellogg, "Unicité des fonctions harmoniques," *Comptes Rendus*, Vol. 13 (1928), pp. 526 27.

seem probable that if the lemma were to fail, it would fail for a set of this sort. In the general cases studied, it is true.

2. *Construction of the Sets of Cantor Type.* Let C denote a closed cube, of unit side, and let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be an infinite sequence of positive proper fractions. We remove from C all points whose distances from any of the three planes through the center, and parallel to the faces, are less than $\alpha_1/2$. There remain eight closed cubes of side $\delta_1 = (1 - \alpha_1)/2$. From each of these, we remove all points whose distances from any of the three planes through the centers and parallel to the faces are less than $\alpha_2 \delta_1/2$. There remain sixty-four cubes C_2 of side $\delta_2 = (1 - \alpha_2)\delta_1/2$. Continuing in this way, there are defined 8^n cubes C_n, of side $\delta_n = (1 - \alpha_n)\delta_{n-1}/2$, $n = 1, 2, 3, \cdots$. The set of points E common to all the cubes of the infinite set thus determined, is the *set of Cantor type corresponding to the sequence* $[\alpha_i]$.

3. *A Lemma on Capacity.* Using the notation of section 1, and denoting the capacity of E by $c(E)$, we have

THEOREM I. *If U is harmonic in T (and this includes regularity at infinity), and never negative, and if its lower and upper limits on the boundary of T lie between a and b, $0 < a \leqq b$, then*

$$m/b \leqq c(E) \leqq m/a,$$

where m is the mass producing U, as given by Gauss' integral.

Proof. Given ϵ, $0 < \epsilon < a$, we consider the domain $U < a - \epsilon$. This is an infinite domain, lying, with its boundary, in T. Its conductor potential is $U/(a - \epsilon)$, and hence the capacity of its boundary is $m/(a - \epsilon)$. But the capacity of a set is never greater than that of an including set,* and hence $c(E) \leqq m/(a - \epsilon)$. As $c(E)$ is independent of ϵ, the second inequality of the theorem is established.

For the first inequality, we note that U/b never exceeds 1 near the boundary of T, and that it is therefore less than 1 in T. Accordingly, the conductor potentials of the sequence of domains approximating to T all exceed U/b in these domains. Hence their limit, V, the conductor potential of T, is nowhere less than U/b. If $V - U/b$ vanishes at any point of T, it is identically 0, by Gauss' theorem of the arithmetic mean, and then $c(E) = m/b$. Otherwise, there is an equipotential surface, $V - U/b = k$, $k > 0$, enclosing E, on which the derivative of $V - U/b$ in the direction of the normal,

* See Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 331. Here further properties of capacity are treated, and the method of sequences described, pp. 322-338.

(counted as positive in the sense in which it points into the infinite region bounded by the equipotential surface), is never positive, and somewhere negative. Gauss' integral, extended over this surface, yields

$$c(E) - m/b = -\frac{1}{4\pi}\int\!\!\int_S \frac{\partial}{\partial n}(V - U/b)\,dS > 0,$$

and the first inequality of the theorem is established.

4. *Inequalities between the Capacities of Certain Sets and Subsets.* Let C denote a closed cube of side δ, and C' any one of the eight cubes formed from C by discarding the points whose distances from any of the three planes through the center and parallel to the faces is less than $\delta\alpha/2$ $(0 < \alpha < 1)$. Let e denote any set of points in one of the cubes C', and E the set consisting of e and the seven congruent and symmetrically placed sets in the remaining cubes C'. We desire inequalities on $c(E)$ in terms of $c(e)$.

We consider first the sum U of the conductor potentials of the eight sets e, at the points of E. Let P be such a point. The conductor potential of the subset e to which P belongs, does not exceed 1. The value at P of the conductor potential of any of the other sets e does not exceed $c(e)$ divided by the distance between the two cubes C', one containing P, and the other the subset in question.* Hence

$$U(P) \leqq 1 + \frac{c(e)}{\alpha\delta}\left\{3 \times 1 + 3 \times \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}}\right\} = 1 + \frac{c(e)\mu''}{\alpha\delta},$$

where μ'' is a number, about 5.701. As the mass producing U is $8c(e)$, it follows from theorem I that

$$c(E) \geqq 8c(e)/[1 + c(e)\mu''/\alpha\delta].$$

To obtain an upper bound for $c(E)$, we first replace each set e by an including set $\bar{e}$, bounded by one or more surfaces such that the Dirichlet problem is possible for the infinite complement of $\bar{e}$ and any continuous boundary values. This can be done in such a way that the sets $\bar{e}$ lie in the cubes C', for instance by enclosing the set e, with its limit points, in a finite number of spheres, and discarding the portions of the spheres outside of the corresponding cube C'. Distinguishing by a bar the quantities for the sets $\bar{e}$ from the corresponding quantities for the sets e, we may reason as follows. The value at a point P of $\bar{E}$ of the conductor potential of the set $\bar{e}$ to which P belongs, is 1, because the Dirichlet problem is possible for the infinite domain of the complement of $\bar{e}$. The value at P of each of the [illegible]

* See *Foundations of Potential Theory*, l. c., p. 331, exercise 4.

potentials of the seven remaining sets $\bar{e}$ is not greater than $c(\bar{e})$ divided by the greatest distance from P to any of its points (or any greater distance). Hence

$$\bar{U}(P) \geqq 1 + c(\bar{e})\,\mu'/\delta,$$

where μ' is a number, about 5.026. It follows from theorem I that

$$c(\bar{E}) \leqq 8c(\bar{e})/[1 + c(\bar{e})\mu'/\delta].$$

In the limit, as the enveloping surfaces shrink down on the sets e, we have the same inequality for the sets E and e.

We shall find it convenient to use the inequalities derived in the form

$$(1) \qquad 1/8c(e) + \mu'/8\delta \leqq 1/c(E) \leqq 1/8c(e) + \mu''/8\alpha\delta.$$

Since $c(e) = 0$ implies $c(E) = 0$, and conversely, vanishing capacities need not be excluded from the inequalities in this form, provided they are understood in this sense.

5. *A Necessary Condition that a Set of Cantor Type Have Positive Capacity.* Let E denote a set of Cantor Type, and let γ_n denote the capacity of that portion of E in one of the cubes C_n, *i. e.* $\gamma_n = c(E \cdot C_n)$. The first of the inequalities (1) then yields

$$1/\gamma_n \geqq 1/8\gamma_{n+1} + \mu'/8\delta_n.$$

Replacing n successively by $0, 1, 2, \cdots n$, and adding the resulting inequalities, we have

$$1/\gamma_0 = 1/c(E) \geqq (\mu'/8)\,[1 + 1/8\delta_1 + 1/8^2\delta_2 + \cdots + 1/8^n\delta_n] + 1/8^{n+1}\gamma_{n+1}, \quad (n = 1, 2, 3, \cdots).$$

The two terms on the right are never negative, and we therefore infer

THEOREM II. *Necessary conditions that the capacity of the Cantor set E be positive, are* (a) *that the series*

$$(2) \qquad \sum_1^{\infty} 1/8^n\delta_n = \sum_1^{\infty} 1/[4^n \prod_1^n (1 - \alpha_i)],$$

converge, and (b) *that $8^n\gamma_n$ have a positive lower bound.*

Incidentally, we remark that the series (2) will converge if $\limsup \alpha_n < 3/4$. It will diverge if $\liminf \alpha_n > 3/4$.

6. *A Sufficient Condition that a Set of Cantor Type Contain Regular Points.* From the criterion of Wiener for the regularity of points may be derived the following*:

* See Kellogg and Vasilesco, *loc. cit.*, p. 519.

The point P of E is regular or exceptional, according as the integral

$$\int^1 \frac{c(\rho)}{\rho^2}\, d\rho$$

is divergent or convergent, $c(\rho)$ being the capacity of the portion of E in the sphere of radius ρ about P. It will be convenient to know that in this integral criterion, the function $c(\rho)$ may be replaced by the capacity $C(\rho)$ of the portion of E in a cube of fixed orientation, of half-side ρ, and with center at P. But this fact is made evident by the inequalities

$$\int^1 \frac{C(\rho/3^{1/2})}{\rho^2} d\rho = \frac{1}{3^{1/2}} \int_{\epsilon/3^{1/2}}^{1/3^{1/2}} \frac{C(\rho)}{\rho^2}\, d\rho \leqq \int_\epsilon^1 \frac{c(\rho)}{\rho^2}\, d\rho \leqq \int_\epsilon^1 \frac{C(\rho)}{\rho^2}\, d\rho,$$

which show that the integrals involving $c(\rho)$ and $C(\rho)$ converge or diverge together.

If P be taken at a corner of the unit cube C, then $C(\delta_n) = \gamma_n$, and the second inequality (1) will give us information about the function $C(\rho)$. Applied to the subsets in C_n and C_{n+1}, it yields

$$1/\gamma_n \leqq 1/8\gamma_{n+1} + \mu''/8\alpha_{n+1}\delta_n.$$

Combining this inequality with those obtained from it by replacing n by $n+1,\ n+2, \cdots n+p-1$, we find

$$\frac{1}{\gamma_n} \leqq \frac{\mu''}{8}\left[\frac{1}{\alpha_{n+1}\delta_n} + \frac{1}{8\alpha_{n+2}\delta_{n+1}} + \cdots + \frac{1}{8^{p-1}\alpha_{n+p}\delta_{n+p-1}}\right] + \frac{1}{8^p\gamma_{n+p}}.$$

We should now like to allow p to become infinite, discarding the remainder term outside the bracket. But it is not clear that its limit is 0. The difficulty can be turned by replacing E, for the moment, by the set of *all* points in all the cubes C_{n+p}; if we then denote by $\gamma_{n,p}$ the capacity of the portion of this set in a cube C_n, we have $\gamma_n = \lim\limits_{p\to\infty} \gamma_{n,p}$, and $c(C_{n+p}) = K\delta_{n+p}$, where K is the capacity of the unit cube. The above inequality then becomes

$$\frac{1}{\gamma_{n,p}} \leqq \frac{\mu''}{8}\left[\frac{1}{\alpha_{n+1}\delta_n} + \frac{1}{8\alpha_{n+2}\delta_{n+1}} + \cdots + \frac{1}{8^{p-1}\alpha_{n+p}\delta_{n+p-1}}\right] + \frac{1}{8^pK\delta_{n+p}}.$$

If we confine ourselves to the case which alone interests us, namely that in which the capacity of E is positive, the series (2) converges, and the term outside the bracket approaches 0 as p becomes infinite. The result is

$$(3) \qquad 1/\gamma_n \leqq (\mu''/8)\, 8^n R_{n-1},$$

where R_n is the remainder, after n terms, of the series

$$(4) \qquad \sum_0^\infty 1/8^i\alpha_{i+1}\delta_i, \qquad (\delta_0 = 1).$$

Returning, with the inequality (3), to the integral criterion for regularity, we note that

$$\int_{\delta_{n+1}}^{\delta_n} [C(\rho)/\rho^2]\,d\rho \geqq \gamma_{n+1} \int_{\delta_{n+1}}^{\delta_n} d\rho/\rho^2 = \gamma_{n+1}(1/\delta_{n+1} - 1/\delta_n),$$

and hence that

$$(5) \qquad \int_0^1 [C(\rho)/\rho^2]\,d\rho \geqq \sum_1^\infty \gamma_{n+1}(1/\delta_{n+1} - 1/\delta_n)$$

$$\geqq (8/\mu'')\sum_1^\infty (1/8^{n+1}R_n)(1/\delta_{n+1} - 1/\delta_n) = (8/\mu'')\sum_1^\infty [1/8^{n+1}\delta_{n+1}R_n]\,[1 + \alpha_{n+1})/2].$$

As α_n lies between 0 and 1, we arrive at

THEOREM III. *A sufficient condition that the set E of Cantor type have a regular point, is that $8^n\delta_n$ become infinite with n, and that the series*

$$(6) \qquad \sum_1^n 1/8^{n+1}\delta_{n+1}R_n$$

diverge.

In order to apply this condition, we shall have need of a theorem on infinite series with positive terms, which is a special case of one due to Dini,* and which may be stated as follows:

If $\sum_1^\infty c_n$ is a convergent series of positive terms and if $r_{n-1} = c_n + c_{n+1} + \cdots$ denotes its remainders, then the series

$$\sum_1^\infty c_n/r_{n-1}$$

is divergent.

7. *Establishment of the Fundamental Lemma in Two General Cases.* We proceed to establish the lemma in the two cases (a) in which the terms of the sequence $[\alpha_i]$ have a positive lower bound, and (b) in which this sequence has the unique limit 0.

In the first case, on the hypothesis that E has positive capacity, the series (2) converges, by theorem II. Since the α_i have a positive lower bound, the series (4) also converges. It then follows from the theorem of Dini that the series

$$\sum_1^\infty 1/8^{n+1}\delta_{n+1}\alpha_{n+2}R_n$$

diverges, and from this follows the divergence of the series (6). The con-

* See Knopp, *Theorie und Andwendung der unendlichen Reihen,* Berlin 1922, p. 285, 2nd ed., (1924), p. 294, or Fort, *Infinite Series,* Oxford 1930, pp. 42-43.

vergence of (2) implies that $8^n\delta_n$ becomes infinite with n, and hence by theorem III, E has the regular point P.

The second case can be reduced to the first by the following device. We form a subset E' of E by omitting those points of E which are in those cubes C_2 which are not at the corners of C, omitting the points in those cubes C_4 which are not at the corners of cubes C_2, and so on. Those points of E are omitted which are in those cubes C_{2n+2} which are not at the corners of cubes C_{2n}. It is at once clear that E' is a set of Cantor type, and a little reckoning shows that the sequence $[\alpha_i']$, which characterizes it, is as follows

$$\begin{aligned} \alpha_1' &= \alpha_1 + (1-\alpha_1)(1+\alpha_2)/2 \\ &= 1/2 + [\alpha_2 + \alpha_1(1-\alpha_2)]/2, \\ \alpha_2' &= 1/2 + [\alpha_4 + \alpha_3(1-\alpha_4)]/2, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \alpha_n' &= 1/2 + [\alpha_{2n} + \alpha_{2n-1}(1-\alpha_{2n})]/2, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad. \end{aligned}$$

The expression for α_n' shows that $\lim \alpha_n' = 1/2$, and it follows from the remark following theorem II that the series (2) for E' converges. The same expression shows that $\alpha_n' > 1/2$, and it follows then from case (a) that the series (6) for E' diverges. Theorem III then shows that E' has regular points. Hence, E, which contains E', also has regular points.

We remark that the reasoning employed above shows that in the cases studied the vertices of all the cubes C_n are regular points. Thus the regular points, in these cases, are everywhere dense in E. If the fundamental lemma is true, this property is possessed by all reduced * sets of positive capacity.

The fundamental lemma is thus established for sets of Cantor type, except when the set $[\alpha_n]$ has two or more limit points, one of them 0. The series set up enable one to study further cases, but I have not settled them all. On the other hand, no cases have come to light in which the lemma fails.

8. *Two Remarks on Capacity in General.* We formulate the first remark as

THEOREM IV. *Any closed bounded set of positive Jordan outer content contains regular points.* On the other hand, the set of Cantor type formed with $\alpha_i = 1/2$ shows that a set may have 0 outer content, and still have regular points, by theorem III.

If the set is a plane set, we may understand the content referred to as [illegible] three dimensional content. Regularity refers to behavior with respect to the Newtonian potential.

We give the proof of the theorem for the case of a plane set. Only

[illegible]

formal modifications are necessary for the case in which three dimensional content is involved. Suppose then, that E is a closed set, lying in the unit square, with the outer content $a > 0$. If we divide the square into four equal squares, the outer content of the portion of E in at least one of them will be not less than $a/4$. Call such a square S_1. If S_1 be similarly divided into four equal squares, the part of E in at least one of them will have an outer content not less than $a/4^2$. Continuing this process, we construct an infinite sequence of squares $S_1, S_2, S_3, \cdots$, such that the outer content of the part of E in S_n is at least $a/4^n$. If P is the limit of this sequence, then P is a regular point of E, as we now show.

Let γ_n denote the capacity of the portion E_n of E in S_n. Then E_n can be enclosed in a set of squares of a quadratic mesh so fine that the capacity of this set of squares differs arbitrarily little from γ_n, while the area A of the set of squares is not less than $a/4^n$. Let U denote the potential of a spread of unit surface density on these squares. Then U cannot exceed, at any point P, the potential at P of a spread of unit surface density on a circle of area A with center at P.* It follows that $U \leqq 2(\pi A)^{1/2}$. As the mass producing U is A, theorem I assures us that the capacity of the set of squares is not less than $\frac{1}{2}(A/\pi)^{1/2} \geqq \frac{1}{2^{n+1}}(a/\pi)^{1/2}$. Since this last expression is independent of the fineness of the mesh, it follows that it is also a lower bound for γ_n. Using this lower bound in the integral criterion of section 6, we see that P is indeed a regular point of E. In fact, since $C(\rho)$ is a monotonically increasing function of ρ, $C(\rho)/\rho$ has a positive lower bound, and the integral is hence divergent.

The second remark concerns the question of the possible topological character of the capacity of a set. Formulating it as broadly as possible, we ask: does a continuous one-to-one transformation, not only of the set E, but of the whole of space, necessarily carry E into a set E' which has positive capacity when, and only when, E has?

The sets of Cantor type furnish an immediate negative answer. In fact, let the set E be characterized by the sequence in which α_i has the constant value $3/5$, and E' by the sequence in which α_i has the constant value $4/5$. Then E has regular points, and hence positive capacity, by theorem III, while E' has 0 capacity, by theorem II. And there is no difficulty in setting up a continuous one-to-one transformation of space, carrying E into E'. These sets therefore constitute an example in point.

CAMBRIDGE, MASS.,
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* See *Foundations of Potential Theory, loc. cit.*, p. 149, lemma III (b).

NOTE ON FRACTIONAL OPERATORS AND THE THEORY OF COMPOSITION.

By Leonard M. Blumenthal.

In this paper the methods and notation of the theory of composition of the first kind, developed by V. Volterra and J. Pérès,* are applied to certain fractional operators.† The concept of the "finite part" of an integral developed almost simultaneously by J. Hadamard and R. D'Adhemar ‡ is employed to give a new definition of negative fractional orders of integration that is amenable to treatment as a very special case of composition. The symbolism already in use in the theory of composition is utilized with obvious advantage in the theory dealt with in this paper. Finally, definitions are given for the operators in the case of the complex variable, the concept of "finite part" being extended to contour integrals.

1. If a function $F(x, y)$ can be put in the form

$$F(x,y) = \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}\,\phi(x,y),$$

where α is different from zero or a negative integer, and the function $\phi(x, y)$, finite and continuous, is not zero for $y = x$, then $F(x, y)$ is said to be of regular order α.

The resultant (Volterra product)

$$FG = \int_x^y F(x,\xi)\,G(\xi,y)\,d\xi$$

of two functions of regular orders α and β is, by a fundamental theorem, of order $\alpha + \beta$.

For $\alpha > 0$, the integral of order α has been defined by the expression §

* A systematic development of the theory is to be found in Volterra et Pérès, *Leçons sur la composition*, Paris (1924).

† For a historical sketch as well as a bibliography see "A Survey of Methods for the Inversion of Integrals of Volterra type," H. T. Davis, *Indiana University Studies*, No. 76, 77; also two papers by the same author, *American Journal of Mathematics*, Vol. 46 (1924), pp. 95-109; Vol. 49 (1927), pp. 123-142.

‡ J. Hadamard, *Annales de l'École Normale Supérieure* (1905), p. 122; R. D'Adhemar, *Exercices et leçons d'analyse*, Paris (1908), pp. 150-180.

§ G. H. Hardy, "Notes on Some Points in the Integral Calculus," *Messenger of Mathematics*, Vol. 47 (1917-18), p. 145.

$$ {}_0F_\alpha[f(y)] = [1/\Gamma(\alpha)] \int_0^y (y-t)^{\alpha-1} f(t)\,dt, $$

where the function f is summable throughout any interval of positive values of y, $y=0$ included. Hardy, in the paper referred to below, shows that the expression exists for almost all values of y and is summable. It is to be noted that the lower limit of the integral is an essential part of the definition.

Now *

$$ 1^\alpha = (y-x)^{\alpha-1}/\Gamma(\alpha). $$

Hence we may write, putting $F = f(y)$,

$$ {}_xF_\alpha[f(y)] = f1^\alpha, $$

the lower limit x being a parameter which hereafter we shall omit exhibiting. The whole theory of integrals of positive order may thus be treated by means of the theory of composition. It is immediate, for example, that the operator $F_\alpha[f(y)]$ obeys the index law

$$ F_\mu F_\nu[f(y)] = F_{\mu+\nu}[f(y)], \qquad \mu > 0,\ \nu > 0 $$

for

$$ F_\nu[f(y)] = f1^\nu, $$

and

$$ F_\mu F_\nu[f(y)] = (f1^\nu)1^\mu. $$

But composition is associative, and hence

$$ F_\mu F_\nu[f(y)] = f(1^\nu 1^\mu), $$

which, by means of the theorem cited above on the order of the resultant, yields

$$ F_\mu F_\nu[f(y)] = f1^{\mu+\nu} = F_{\mu+\nu}[f(y)]. $$

Again, making use of the concept of fractions of composition it is immediate that

$$ \lim_{\nu\to 0} F_\nu[f(y)] = f(y). $$

2. When two functions are of negative orders, the integral defining their composition may not exist. It has been found, however, that all the theorems valid for ordinary composition hold if we agree, that whenever the integral FG does not exist, we are to take the *finite part* of the integral.† It is upon this that we base our definition of negative fractional order of

* It is usual to place an asterisk above the 1 to indicate the Volterra product.

† J. Pérès, "Sur la composition 1ère espèce: Les fonctions d'ordre quelconque et leur composition," *Rendiconti della Realle Accademia dei Lincei*, (1917), p. 45, 104.

[illegible]
the finite part of all non-convergent integrals is to be taken.

Let now $\alpha = -\eta$, $0 < \eta < 1$. We define the integral of negative order η by the expression

$$F_{-\eta}[f(y)] = 1/[\Gamma(-\eta)] \int_{x}^{y} (y-t)^{-\eta-1} f(t)\,dt.$$

3. *Finite part of* $\int_x^y (y-t)^{-\eta-1} f(t)\,dt$. Let the function f satisfy a Lipschitz condition throughout any interval of positive values of y, and consider the expression

$$I = \lim_{\epsilon\to 0} \left[\int_x^{y-\epsilon} (y-t)^{-\eta-1} f(t)\,dt + \phi(y-\epsilon)\epsilon^{-\eta} \right],$$

where

(1) $$f(y) = -\eta\phi(y).$$

THEOREM. *The limit I exists.*

We note first that I differs from

$$I' = \lim_{\epsilon\to 0} \left[\int_x^{y-\epsilon} \frac{f(t)-f(y)}{(y-t)^{1+\eta}}\,dt + \frac{\phi(y-\epsilon)-\phi(y)}{\epsilon^{\eta}} \right]$$

by a finite term $f(y)/\eta(y-x)^{\eta}$; for

$$I' = \lim_{\epsilon\to 0} \left[\int_x^{y} (y-t)^{-\eta-1} f(t)\,dt + \epsilon^{-\eta}\phi(y-\epsilon) - \epsilon^{-\eta}\left\{ \frac{f(y)}{\eta} + \phi(y) \right\} + \frac{f(y)}{\eta(y-x)^{\eta}} \right]$$

whence, using (1)

(2) $$I' = I + f(y)/\eta(y-x)^{\eta}.$$

Now the limit I' is readily seen to exist, for

$$I' = \lim_{\epsilon\to 0} \left[\int_x^{y-\epsilon} \frac{f(t)-f(y)}{(y-t)^{1+\eta}}\,dt + \frac{\phi(y-\epsilon)-\phi(y)}{\epsilon^{\eta}} \right],$$

and using the Lipschitz condition

[illegible]

$$\int_x^y (y-t)^{-\eta-1} f(t)\,dt,$$

and hence, by our definition

$$F_{-\eta}[f(y)] = \lim_{\epsilon\to 0} \int_x^{y-\epsilon} \frac{(y-t)^{-\eta-1}f(t)}{\Gamma(-\eta)}\,dt + \frac{\phi(y-\epsilon)}{\Gamma(-\eta)\epsilon^{\eta}}\Big],$$

If the function f is differentiable, this expression can be put in a more useful form. First we write $F_{-\eta}[f(y)]$ in the equivalent form

$$(3)\quad F_{-\eta}[f(y)] = \lim_{y_1\to y}\Big[\int_x^{y_1} \frac{(y-t)^{-\eta-1}f(t)}{\Gamma(-\eta)}\,dt + \frac{\phi(y_1)}{\Gamma(-\eta)(y-y_1)^{\eta}}\Big],\quad y_1 < y.$$

Now

$$\frac{d}{dt}\frac{\phi(t)}{(y-t)^{\eta}} = \frac{\eta\phi(t)+\phi'(t)(y-t)}{(y-t)^{1+\eta}}$$

whence

$$\frac{\phi(y_1)}{(y-y_1)^{\eta}} = \int_x^{y_1}\frac{\eta\phi(t)+\phi'(t)(y-t)}{(y-t)^{1+\eta}}\,dt + \frac{\phi(x)}{(y-x)^{\eta}}.$$

Substituting in (3), and taking account of (1), we have

$$(4)\qquad F_{-\eta}[f(y)] = \frac{\phi(x)}{\Gamma(-\eta)(y-x)^{\eta}} + \int_x^y \frac{\phi'(t)}{\Gamma(-\eta)(y-t)^{\eta}}\,dt = \frac{1}{\Gamma(-\eta)}\ \frac{d}{dy}\int_x^y \frac{\phi(t)}{(y-t)^{\eta}}\,dt.$$

In general, if f is differentiable $(p-1)$-times and its $(p-1)$-st derivative satisfies a Lipschitz condition,

$$(5)\qquad \int_x^y (y-t)^{-p-\mu}f(t)\,dt = \int_x^y (y-t)^{-p-\mu}f_1(t)\,dt + \int_x^y \frac{f(t)-f_1(t)}{(y-t)^{p+\mu}}\,dt,\quad 0<\mu<1$$

where

$$f_1(t) = f(y) - f'(y)(y-t) + f''(y)(y-t)^2/2! + \cdots + (-1)^{p-1}f^{(p-1)}(y)(y-t)^{p-1}/(p-1)!\,.$$

Now

$$\int_x^y (y-t)^{-p-\mu}dt = 11^{-p-\mu+1}\Gamma(-p-\mu+1) = (y-x)^{1-p-\mu}/(1-p-\mu).$$

The first member of the right hand side of (5) is a sum of terms of the form

$$[f^{(i)}(y)/i!\,]\int_x^y (y-t)^{-p-\mu+i}dt$$

and hence may be written

$$\sum_{i=0}^{p-1} (-1)^i [f^{(i)}(y)/i!] \cdot (y-x)^{1+i-p-\mu}/(1+i-p-\mu).$$

Whence

$$\int_x^y \frac{f(t)}{(y-t)^{p+\mu}}\,dt = \sum_{i=0}^{p-1} (-1)^i \frac{f^{(i)}(y)\,(y-x)^{1+i-p-\mu}}{(1+i-p-\mu)\,i!} + \int_x^y \frac{f(t)-f_1(t)}{(y-t)^{p+\mu}}\,dt.$$

3. A few examples of functions obtained by the use of the definitions used in this paper are given. In this table we have set the parameter x equal to zero. The functions are obtained with a minimum of calculation.

$$F_{-\eta}[k] = k \cdot 1 1^{-\eta} = k \cdot 1^{1-\eta} = [k/\Gamma(1-\eta)]y^{-\eta},$$
$$F_{-\eta}[y^k] = 1^{k+1}1^{-\eta}\Gamma(1+k) = [\Gamma(1+k)/\Gamma(k-\eta+1)]y^{k-\eta},$$
$$F_{-\eta}[e^y] = \sum_{i=1}^{\infty} 1^i 1^{-\eta} = \sum_{i=1}^{\infty} y^{i-\eta-1}/\Gamma(i-\eta), \text{ also, from (4),}$$
$$F_{-\eta}[e^y] = \frac{y^{-\eta}}{\Gamma(1-\eta)} + e^y \int_0^y \frac{e^{-t}t^{-\eta}}{\Gamma(1-\eta)}\,dt,$$
$$F_{-\eta}[\sin y] = \sum_{p=1}^{\infty} (-1)^{p-1} 1^{2p} 1^{-\eta} = \sum_{p=1}^{\infty} (-1)^{p-1} [y^{2p-\eta-1}/\Gamma(2p-\eta)].$$

For $f(t)$ representable in a series

$$f(t) = a_0 + a_1 t + a_2 t^2/2! + \cdots + a_n t^n/n! + \cdots,$$

convergent for $0 \leqq t \leqq y$, we have

$$F_{-\eta}[f(y)] = \sum_{i=0}^{\infty} [a_i/\Gamma(i-\eta)] y^{i-\eta-1}.$$

4. *Differentiation.* The finite part of the integral

$$\int_x^y (y-t)^{-p-\mu} f(t)\,dt$$

admits directly of differentiation with respect to y. It is performed by differentiating under the integral sign as though the integrand were continuous at y and ignoring the fact that the upper limit of the integral is a function of the parameter. Thus

$$F_{-\eta}[f(y)] = [1/\Gamma(-\eta)] \int_x^y (y-t)^{-\eta-1} f(t)\,dt;$$
$$dF_{-\eta}/dy = [(-1-\eta)/\Gamma(-\eta)] \int_x^y (y-t)^{-\eta-2} f(t)\,dt,$$

and since

$$[(-1-\eta)/\Gamma(-\eta)] = [1/\Gamma(-1-\eta)],$$

we have

$$dF_{-\eta}/dy = [1/\Gamma(-1-\eta)] \int_x^y (y-t)^{-\eta-2} f(t)\,dt = F_{-\eta-1}[f(y)].$$

It is readily seen that the operator satisfies the condition of linearity and the index law. The latter is immediate upon using the classical properties of the finite part, or may be even more readily seen by use of the theorems valid for composition of functions of negative orders. Thus

$$F_{-\mu}F_{-\nu}[f(y)]=F_{-\mu}(f1^{-\nu})=(f1^{-\nu})1^{-\mu}=f(1^{-\nu}1^{-\mu})=f1^{-\mu-\nu}=F_{-\mu-\nu}[f(y)].$$

Example. Let us apply our definition of fractional operator to obtain the solution of Abel's equation:

$$\phi(y)=\int_0^y (y-t)^{-\eta}u(t)\,dt=u1^{1-\eta}\Gamma(1-\eta),\ \ 0<\eta<1,$$

where the function $\phi(y)$ satisfies a Lipschitz condition (and is therefore absolutely continuous) and is not necessarily zero for $y=0$. Operating with the symbol $1^{\eta-1}$, we have

$$u(y)=[1/\Gamma(1-\eta)]\,\phi 1^{\eta-1}=[1/\Gamma(1-\eta)]\,F_{\eta-1}[\phi(y)],$$

where, in accordance with our agreement we are to take the *finite part* of the *non-convergent integral* appearing in the right-hand member. But by our definition, the finite part of $[1/\Gamma(1-\eta)]\,F_{\eta-1}[\phi(y)]$ is readily given, almost everywhere, by the expression

$$\frac{\sin\pi\eta}{\pi}\left[\frac{\phi(0)}{y^{1-\eta}}+\int_0^y \frac{\phi'(t)}{(y-t)^{1-\eta}}\,dt\right].$$

Hence

$$u(y)=\frac{\sin\pi\eta}{\pi}\left[\frac{\phi(0)}{y^{1-\eta}}+\int_0^y \frac{\phi'(t)}{(y-t)^{1-\eta}}\,dt\right]$$

almost everywhere. But L. Tonelli has shown that the function $u(y)$ determined for almost all values of t by this expression satisfies Abel's equation for all values of the variable y. Thus, substituting, we have

$$\begin{aligned}\int_0^y \frac{u(t)}{(y-t)^{\eta}}\,dt&=\frac{\sin\pi\eta}{\pi}\,\phi(0)\int_0^y \frac{dt}{t^{1-\eta}(y-t)^{\eta}}\\&\quad+\frac{\sin\pi\eta}{\pi}\int_0^y \frac{dt}{(y-t)^{\eta}}\int_0^t \frac{\phi'(z)}{(t-z)^{1-\eta}}\,dz\\&=\phi(0)+\frac{\sin\pi\eta}{\pi}\int_0^y \phi'(z)\int_z^y (y-t)^{-\eta}(t-z)^{\eta-1}dt\,dz\\&=\phi(0)+\int_0^y \phi'(z)\,dz\\&=\phi(y).\end{aligned}$$

* L. Tonelli, "Su un Problema di Abel," *Mathematische Annalen*, Vol. 99 (1928), p. 191.

5. [illegible] integer and $f(t)$ is holomorphic within a region S whose boundary C consists [illegible] along C, then the n-th derivative of f at the point z lying within C is given by

$$f^{n}(z) = (n!/2\pi i)\int_C [illegible]$$

Consider the expression

$$(6) \quad f^{-y}(z) = [\Gamma(1-y)/2\pi i]\int_C f(t)\exp[(y-1)\{\log(t-z)+2k\pi i\}]\,dt,$$

where k is an integer and the point z is a point of S or on C. We prove the following theorem:

Theorem. *$f^{-p}(z)=\lim_{y\to p} f^{-y}(z)$, p, a positive integer, exists and differs from the iterated integral of order p by a polynomial $\phi(z)$ such that $f^{-p}(z)-\phi(z)$ is zero at the point $z=a$, together with its $(p-1)$ derivatives, where a is a point within or on C.*

For

$$\lim_{y\to p} f^{-y}(z) = \lim_{y\to p}\frac{\Gamma(1+p-y)}{(1-y)(2-y)\cdots(p-1-y)} \cdot \frac{(1/2\pi i)\int_C f(t)\exp[(y-1)\{\log(t-z)+2k\pi i\}]\,dt}{y-p}$$

$$= \frac{(1/2\pi i)}{(-1)^p(p-1)!}\int_C f(t)(t-z)^{p-1}\{\log(t-z)+2k\pi i\}\,dt,$$

which is independent of k.

Thus

$$f^{-p}(z) = \lim_{y\to p} f^{-y}(z) = [(1/2\pi i)/(-1)^p(p-1)!]\int_C f(t)(t-z)^{p-1}\log(t-z)\,dt,$$

of which the p-th derivative is $f(z)$. Whence

$$f^{-p}(z)-\phi(z) = \int_a^z dt\int_a^{t_1} dt\cdots\int_a^{t_{p-1}} f(t_p)\,dt_p, \quad a \text{ in } S.$$

The operator $f^{-y}(z)$ defined in (6) [illegible]

For a fractional index the operator is a mixed linear polydromic functional with polydromic indicatrix. Now it is natural to define the positive fractional order of integration in the complex domain by the expression

$$f_0^{-\mu}(z) = [\Gamma(1-\mu)/2\pi i] \int_C (t-z)^{-1+\mu} f(t)\,dt, \quad 0 < \mu < 1$$

where the curve C has the origin as its initial and final point and incloses the branch point z, and an assigned branch of the multiform integrand is holomorphic in the region bounded by C. The choice of the curve C to pass through the origin is an essential part of the definition, since integration along two closed curves around z not both starting and ending at the same point will give different values to the operator.

6. *Development of $f_0^{-\mu}(z)$*. The function $f(t)$ being expressible in the form $f(t) = \sum_{n=0}^{\infty} a_n t^n$, the series $(t-z)^{\mu-1} \sum_{n=0}^{\infty} a_n t^n$ can be integrated termwise, and we have

$$f_0^{-\mu}(z) = [\Gamma(1-\mu)/2\pi i] \sum_{n=0}^{\infty} a_n \int_C t^n (t-z)^{\mu-1} dt.$$

Now for C we chose a loop from the origin around the branch-point z and back to the origin. Whence

$$\int_C t^n (t-z)^{\mu-1} dt = (1 - e^{2\pi i\mu}) \int_0^z t^n (t-z)^{\mu-1} dt,$$

where the integral may be considered taken along the straight line from the origin to the point z. If in this integral we put $t = \xi z$, we obtain

$$\int_C t^n (t-z)^{\mu-1} dt = (1 - e^{2\pi i\mu}) (-1)^{\mu-1} z^n z^\mu \int_0^1 \xi^n (1-\xi)^{\mu-1} d\xi.$$

Whence

$$f_0^{-\mu}(z) = (1/2\pi i)\Gamma(1-\mu)(1-e^{2\pi i\mu}) e^{-\pi i(\mu-1)} z^\mu \sum_{n=0}^{\infty} a_n \beta(n+1,\mu) z^n$$
$$= z^\mu \sum_{n=0}^{\infty} \frac{a_n \Gamma(n+1)}{\Gamma(n+\mu+1)} z^n.$$

Thus this definition of the positive fractional order of integration in the complex domain as a contour or loop integral is seen to yield the same Taylor development as the definition for this operator adopted by Hadamard.* The

* The Taylor development for the Hadamard definition is given in Mandelbrojt, "Modern Researches on the Singularities of Functions Defined by Taylor's Series," *The Rice Institute Pamphlet*, Vol. 14 (1927), No. 4, p. 291.

ment and serves to make the theory of this operator a part of the already existing, developed theory of complex functions of V. Volterra and L. Fantappiè.

In defining the negative fractional order of integration we use a device entirely similar to that of the finite part of an integral. Some such device is essential, since in integrating a function $f(t)(t-z)^{-\mu-1}$ with $0<\mu<1$ along C, a loop from the origin around the point z and back to the origin, the integral around the circle with z as center *does not approach zero* with the radius of the circle. We write, therefore

$$_0F_{-\mu}[f]=(1/2\pi i)\Gamma(1+\mu)\overline{\int_C (t-z)^{-\mu-1}f(t)\,dt},$$

where, by definition

$$\overline{\int_C (t-z)^{-\mu-1}f(t)\,dt}=(1-e^{-2\pi i\mu})\int_0^z (t-z)^{-\mu-1}f(t)\,dt,$$

with the *finite part* of the integral in the second member being taken. Thus

$$_0F_{-\mu}[f]=(1/2\pi i)\Gamma(1+\mu)(1-e^{-2\pi i\mu})e^{\pi i(\mu+1)}\int_0^z (z-t)^{-\mu-1}f(t)\,dt$$

$$=\frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{-\mu-1}f(t)\,dt=\frac{1}{\Gamma(1-\mu)}\left[\frac{f(0)}{z^{\mu}}+\int_0^z\frac{f'(t)}{(z-t)^{\mu}}\,dt\right]$$

7. *The Index Law for $f_0^{-\mu}(z)$.*

THEOREM. $f_0^{-\nu}f_0^{-\mu}(z)=f_0^{-\mu-\nu}(z),\quad 0<\mu,\ \nu<1,\quad \mu+\nu\neq 1.$

By definition we have

$$f_0^{-\nu}f_0^{-\mu}(z)=[\Gamma(1-\nu)\Gamma(1-\mu)/(2\pi i)^2]\int_{C_1}(t-z)^{-1+\nu}dt\int_{C_2}(u-t)^{-1+\mu}f(u)\,du,$$

where both C_1 and C_2 have their initial and end-points at the origin and inclose the points z and t respectively.

Now choosing C_2 as a loop from the origin around the point t in the positive direction and back to the origin, we have

$$f_0^{-\nu} f_0^{-\mu}(z) = (\tfrac{1}{2}\pi i)^2 \Gamma(1-\nu)\Gamma(1-\mu)(1-e^{2\pi i\mu})(1-e^{2\pi i\nu}) \int_0^z f(u)\,du \int_u^z (t-z)^{\nu-1}(u-t)^{\mu-1}dt.$$

A change of variable in the second integral enables us to write

$$f_0^{-\nu} f_0^{-\mu}(z) = \frac{-(1-e^{2\pi i\nu})(1-e^{2\pi i\mu})\Gamma(1-\nu)\Gamma(1-\mu)\Gamma(\mu)\Gamma(\nu)}{(2\pi i)^2\Gamma(\mu+\nu)} \int_0^z (u-z)^{\mu+\nu-1} f(u)\,du.$$

Writing the expression as a loop integral around z the above expression becomes, after some reductions,

$$f_0^{-\nu} f_0^{-\mu}(z) = \frac{\Gamma(1-\mu-\nu)}{2\pi i} \int_{c_1} (u-z)^{\mu+\nu-1} f(u)\,du,$$

and hence

$$f_0^{-\nu} f_0^{-\mu}(z) = f_0^{-\mu-\nu}(z).$$

THE RICE INSTITUTE.

POTENTIALS OF GENERAL MASSES IN SINGLE AND DOUBLE LAYERS. THE RELATIVE BOUNDARY VALUE PROBLEMS.

By G. C. Evans and E. R. C. Miles,

1. *Introduction.* The potential due to the most general distribution of finite positive and negative mass deposited in a single layer on a closed surface S may be written in the form

$$v(M)=\int_S \frac{1}{MP}\, d\mu(e_P) \tag{1}$$

where the mass function $\mu(e)$ is a completely additive function of point sets e on S. The most general distribution of mass in a double layer on S yields similarly the potential

$$u(M)=\int_S \frac{\cos(MP, n_P)}{MP^2}\, d\nu(e_P) \tag{2}$$

where $\nu(e)$ is likewise a completely additive function; here n_P denotes the direction of the interior normal to S at P. In fact, for all the closed surfaces to be discussed the direction n is that of the interior normal to the surface, whether or not it may be interior to the region in question.

In terms of these potentials, by means of Stieltjes integral equations, one can solve generalized boundary value problems of the first and second kinds. The first boundary value problem is solved by (2) when the limiting values are given of the quantity $\int u d\omega$, extended over an arbitrary portion ω of S', which is a surface neighboring S, as S' approaches S. In the second boundary value problem, limiting values of the flux $\int dv/dn\, d\omega$ are similarly given, and the problem is solved in terms of (1). Special cases of these problems are the Dirichlet and Neumann problems, respectively, with boundary values summable on S.*

2. *Differential geometry of S and of its neighborhood.* We assume that S is a simple closed surface with a tangent plane at every point, whose orientation changes continuously with respect to displacement of its point of tan-

* A summary of this paper appeared in the *Proceedings of the National Academy of Sciences,* Vol. 15 (February, 1929), pp. 102-108. Subsequently, the special case of the Neumann problem, the $\nu(e)$ being subject to the restriction of absolute continuity, was treated by M. Gunther, "Sur une application des intégrales de Stieltjes au problème de Neumann," *Comptes Rendus de l'Académie des Sciences* Vol. 189 (September, 1929), pp. 447-450.

gency on S. A further restriction which is of importance for the potentials (1) and (2) is that there shall be a constant Γ such that

$$(2.1) \qquad \int_S \frac{|\cos(MP, n_P)|}{MP^2} d\omega_P < \Gamma$$

irrespective of the position of the point M, and

$$(2.2) \qquad \int_S \frac{|\cos(QP, n_Q)|}{QP^2} d\omega_P < \Gamma$$

irrespective of the position of the point Q on S, $d\omega$ being the element of surface area of S. A general type of such surface is furnished by the following theorem.*

Lemma. *Let $QP = s$ be the arc length of the curve of section of S made by the plane which is determined by n_Q and P, and $f(s)$ a positive, continuous, non-decreasing function of s such that $\int_0^s f(s)/s\, ds$ is a convergent integral. If there is a number δ' such that the inequality*

$$(2.3) \qquad |\sphericalangle(n_Q, n_P)| \leq f(s)$$

holds for all Q, P on S for which $s \leq \delta'$, then there is a constant Γ such that (2.1) and (2.2) are satisfied.

There is no loss of generality if we take δ' small enough so that $f(s) < 1$, $s \leq \delta'$.

We take ρ, θ, z cylindrical coördinates of S in the neighborhood of Q, where ρ, θ are polar coördinates, referred to Q, in the tangent plane at Q. Let $\omega(\delta, Q)$ denote the portion of S containing Q which is bounded by the curve on S determined by $\rho = \delta$.

For a point P_1 in $\omega(\delta, Q)$, where δ is small enough, we have $\rho_1 > \frac{1}{2}s_1$. In fact, since $s = \int_0^\rho \sec(s, \rho)\, d\rho$ and $\sec(s, \rho) \leq \sec(n_Q, n_s)$, where n_s is the normal to S at the point whose parameter is s, this relation holds for $s_1 \leq \delta'$. If, then, we let $\delta = \delta'/2$, $g(\rho) = f(2\rho)$, we have $f(s) \leq g(\rho)$ for $\rho \leq \delta$, where $g(\rho)$ is a continuous non-decreasing function of ρ such that

$$\int_0^\delta \frac{g(\rho)}{\rho} d\rho = m(\delta)$$

converges. Then (2.3) may be rewritten

* Curves in the plane that satisfy the relation analogous to (2.1) have been studied as "curves of class Γ." See G. C. Evans, "Fundamental Points of Potential Theory," *The Rice Institute Pamphlet*, Vol. 7 (1920), No. 4, pp. 252-329. See p. 261.

$$(2.4) \qquad |\sphericalangle(M_Q, M_P)| \leq g(\rho), \qquad (\rho \leq \delta = \delta'/2),$$

and we have also the relations

$$(2.5) \qquad \left.\begin{array}{r} z \leq 2\rho g(\rho) \\ |\cos(QP, n_Q)| \leq 2g(\rho) \\ |\cos(n_Q, n_P)| > 1/2 \\ |d\omega| < 2\rho d\rho d\theta \end{array}\right\} (\rho \leq \delta).$$

With respect to the integral (2.2), then, we have

$$(2.6) \qquad \int_{\omega(\delta,Q)} \frac{|\cos(QP, n_Q)|}{QP^2}\, d\omega_P < 8\pi m(\delta).$$

But the point Q is distant by a not zero amount Δ from the set of points comprising $S - \omega(\delta, Q)$. Hence

$$\int_S \frac{|\cos(QP, n_Q)|}{QP^2}\, d\omega_P = \int_\omega + \int_{S-\omega} < 8\pi m(\delta) + (1/\Delta^2)\,\text{meas.}\, S,$$

and the theorem is proved with reference to the integral (2.2).

The integral (2.1) may be considered first in the particular case where M is a point Q of S:

$$I = \int_S \frac{|\cos(QP, n_P)|}{QP^2}\, d\omega_P = I_\delta + I_\delta',$$

where I_δ is extended over $\omega(\delta, Q)$, and $I_\delta' \leq \text{meas.}\, S/\Delta^2$.

If we let $\psi = \sphericalangle(QP, n_P) - \sphericalangle(QP, n_Q)$, we shall have $|\psi| \leq g(\rho)$, and therefore

$$|\cos(QP, n_P)| \leq |\cos(QP, n_Q)| + g(\rho).$$

Hence $I_\delta < 12\pi m(\delta)$, so that

$$(2.7) \qquad I < 12\pi m(\delta) + (1/\Delta^2)\,\text{meas.}\, S.$$

For the general case, M not on S, let $\Delta(M)$ denote the distance of M from S, and $\Gamma(M)$ the corresponding value of the integral (2.1). Then

$$\Gamma(M) \leq \text{meas.}\, S/[\Delta(M)]^2.$$

If the $\Gamma(M)$ are not bounded, for all M, there will be a sequence of points $\{M_i\}$, with $\lim \Gamma(M_i) = \infty$. Then, since $\lim \Delta(M_i) = 0$, the points M_i will have a limit point Q on S. Without loss of generality we may suppose [illegible]

sequence.

On account of the continuity in the orientation of the normal, we know

that there is at least one normal to S in the neighborhood of Q which passes through M_k.* We may therefore, writing Q_k for the foot of one such normal, suppose that $\lim Q_k = Q$, and accordingly $\lim \overline{M_k Q_k} = 0$. We have

$$\Gamma(M_k) = \Gamma_\delta(M_k) + \Gamma_\delta'(M_k), \tag{2.8}$$

where the integrations are extended over $\omega(\delta, Q_k)$ and $S - \omega(\delta, Q_k)$, respectively. The second term on the right is evidently bounded, $< K$, for all M_k with k sufficiently large.

As for $\Gamma_\delta(M_k)$, we may write

$$\measuredangle(M_kP, n_P) = \measuredangle(M_kP, n_{Q_k}) + \psi',$$

where

$$|\psi'| \leq |\measuredangle(n_{Q_k}, n_P)| \leq g(\rho), \qquad M_kP \geq \rho,$$

ρ being measured in the tangent plane at Q_k. Hence as before

$$\Gamma_\delta(M_k) \leq \int_{\omega(\delta, Q_k)} \frac{|\cos(M_kP, n_Q)|}{M_kP^2}\, d\omega_P + 4\pi m(\delta).$$

But if we write $\measuredangle(M_kP, n_Q) = \measuredangle(M_kP', n_Q) + \psi''$, where P' is the projection of P on the tangent plane at Q_k, we have

$$|\cos(M_kP, n_Q)| \leq |\cos(M_kP', n_{Q_k})| + |\sin\psi''|,$$
$$\sin\psi'' \leq z/\rho.$$

Hence, writing ω' for the projection of $\omega(\delta, Q_k)$ on the tangent plane,

$$\Gamma_\delta(M_k) < 2\int_{\omega'} \frac{|\cos(M_kP', n_{Q_k})|}{M_kP'^2}\, d\omega' + 2\int_{\omega'} (z/\rho^3)\, d\omega' + 4\pi m(\delta).$$

But

$$\left|\frac{1}{M_kP^2} - \frac{1}{M_kP'^2}\right| = \frac{|M_kP + M_kP'|\;|M_kP' - M_kP|}{M_kP^2 M_kP'^2} \leq (2z/\rho^3),$$

and therefore

$$\Gamma_\delta(M_k) < 2\int_{\omega'} \frac{|\cos(M_kP', n_{Q_k})|}{M_kP'^2}\, d\omega' + 4\int_{\omega'} (z/\rho^3)\, d\omega' + 4\pi m(\delta)$$
$$< 4\pi + 8\pi m(\delta) + 4\pi m(\delta).$$

Hence, referring to (2.8), we see that $\Gamma(M_k)$ is bounded, contrary to our assumption.

It follows that $\Gamma(M)$ is bounded for all M not on S, and since by (2.7) it is bounded for all M on S, inequality (2.1) is established.

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* In fact we may draw a small sphere with center Q and then choose M_k close enough to Q so that it is nearer to Q than to any point of the surface of the sphere. There will then be a shortest distance from M_k to points of S within the sphere, and this distance will fix a normal from M_k to S whose foot is the desired point Q_k.

A particular form of $f(s)$ which satisfies the conditions of the lemma is $f(s) = C s^{\alpha}$, $0 < \alpha$. *Hereafter it will be assumed, except in* § 6, *that* $f(s) < Cs$, *that is*

$$(2.9) \qquad | \measuredangle (n_Q, n_P) | < Cs < 2C\rho, \qquad (\rho \leq \delta = \delta'/2).$$

Let now S' be a surface neighboring to S, simple and closed, with a tangent plane at every point, whose orientation changes continuously with respect to displacement of its point of tangency on S'.* Let it be defined by normal displacement of the point P of S in amount $n(P)$, $| n(P) | \leq \tau$, where $n(P)$ is a continuous non-vanishing function of P. If τ is small enough there will be a one to one correspondence between the points of S' and those of S. In fact, we see first that if P is a point of $\omega(\delta, Q)$ and n_P and n_Q intersect at M, we cannot have MP and $MQ < 1/C$. We refer S momentarily to rectangular coördinates at Q, the z-axis being in the direction of n_Q through M, and consider the plane section determined by n_Q and P. We take x along the intersection of this plane with the tangent plane at Q, in the direction of P, and assume that n_P and n_Q intersect at M.

For $\rho < \delta$, so that $s < \delta'$, we have the angle between n_Q and the normal to s at a point P' of s, in the plane section, $\leq$ the angle n_Q, n_P, and therefore $< Cs$, by (2.9). Hence

$$x = \int_0^s \cos(x, s)\, ds > \int_0^s \cos Cs\, ds = (1/C) \sin Cs$$

$$z = \int_0^s \sin(x, s) ds, \qquad | z | < \int_0^s \sin Cs\, ds = (1/C)(1 - \cos Cs).$$

Now for P in $\omega(\delta, Q)$ the angle (x, s) is less than 1 in numerical value, and accordingly $z_M > | z |$. Hence

$$| z_M - z | > | z_M - (1/C)(1 - \cos Cs) |$$

and

$$MP^2 = (z_M - z)^2 + x^2$$
$$MP^2 > z_M^2 + (2/C)[(1/C) - z_M](1 - \cos Cs).$$

From this inequality it follows that $MP > MQ$ if $z_M = MQ < 1/C$.

But similarly taking n_P as the z-axis, if $MP < 1/C$, we have $MQ > MP$. Hence both MP and MQ cannot be $< 1/C$.

Moreover the distance Δ from Q to the set of points $S - \omega(\delta, Q)$ has a lower bound Δ_0 for all Q, where $\Delta_0 > 0$. Hence, if τ is less than Δ_0

* The requirements on S' are not stated with sufficient precision in Evans and Miles, *loc. cit.*

no normal to a point of $S - \omega(\delta, Q)$ can cut n_Q in such a way that both normals are in length $< \tau$. Consequently if $\tau < \Delta_1$ and $< 1/C$ at the same time, the correspondence is one-one.

The orientation of the tangent plane to S' has been defined, according to hypothesis, by functions which are continuous at every point of S', and hence are uniformly continuous. The projection on the tangent plane at M of any element of arc ds on S', through R in the neighborhood of M is at least as great as $ds \cdot \cos(n_M, n_R)$; hence there is a small neighborhood of M of radius greater than some constant, independent of M, such that the projection on the tangent plane at M of any rectifiable arc s on S', in this neighborhood, will be a rectifiable arc on the tangent plane of length $\geqq s/2$.

For the analysis of the boundary value problem we consider a family of surfaces $S' = S_\tau'$ where $|n(P)| < \tau$, and τ approaches zero. In order to complete the description of the relation of the family S_τ' to S we assume that there are positive constants C, α, δ'' such that if n_M denotes the normal to S' at a point M on S', and n_Q the normal to S at Q on S, the inequality

$$(2.10) \qquad | \sphericalangle(n_M, n_Q) | < C\overline{MQ}^{\alpha}$$

will be valid, whenever $MQ < \delta''$, irrespective of the particular surface S', of the family, involved.* As in the case of δ' and (2.9), the quantity δ'' is assumed to be small enough so that the angle described in (2.10) is < 1. It follows then that the projection on the tangent plane to S at Q of an element of arc ds' at M on S' is $\geqq ds'/2$ if $MQ < \delta''$.

In particular, the requirements for S_τ' are satisfied if S_τ' is parallel to S internally or externally, and in its neighborhood, that is, if $n(P) = \tau$. Also, if S' is a surface with continuous curvatures, and if $n_0(P)$ is a continuous non-vanishing function on S, in absolute value $\leqq 1$, with continuous derivatives with respect to displacement on S, the family of surfaces S_τ', where $n(P) = \tau n_0(P)$, satisfies the above requirements.

A *regular curve* on S or on S', for the purposes of this article, will be a simple closed curve on the surface, made of a finite number of arcs, each with a continuously turning tangent, the two branches which come together making a not zero angle with each other. Since any part of such a curve, contained in a neighborhood of diameter sufficiently small, say equal to some constant δ''', when projected on the tangent plane at any point of this neighborhood, is itself a portion of a regular curve, and the projection of the

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* This condition, rather than a slightly less restrictive one, is introduced in conformity to (2.9). The condition (2.9) was introduced in order to generate a one to one correspondence between the points of S' and those of S.

[illegible]
on S or on S' can be subdivided into regular cells of diameter uniformly small. Vice versa, given a small regular [illegible] corresponds to it a small regular arc on the surface. A simple closed curve w on the surface may then be described as a regular curve if it can be subdivided into a finite number of cells of diameter arbitrarily small, each of which has as its image, on the tangent plane at one of its points, a closed regular curve.

We define a *regular net* G on the surface, as a system of lattices G_n, corresponding to positive numbers δ_n successively decreasing and approaching zero as a limit. Each lattice represents a partition of the preceding lattice into a finite number of cells w_{i_n}, each of which is a regular closed curve of diameter $< \delta_n$, and G_0 is the surface itself. In a similar manner we may define a net G for a regular closed curve w on the surface, with reference to its interior region ω, and given such a net, we may extend it through the complement of ω so that it becomes a regular net for the surface itself.*

3. *Stieltjes Integrals.* We may form the Stieltjes integral over S of any continuous function $h(P)$ with respect to $\nu(e)$ or $\mu(e)$, and by means of the postulates (C), (A), (L), (M) of Daniell, generalize these integrals so that they apply to integrands $h(P)$ which are not necessarily continuous, but may in particular be merely bounded and measurable Borel.† We may then define the two functions of regular curves

$$\nu(w) = \int_S q(P, w)\, d\nu(e_P), \tag{3.1}$$

$$\mu(w) = \int_S q(P, w)\, d\mu(e_P),$$

where $q(P, w)$ is the symmetric surface density at P of the region ω bounded by w.‡ The functions $\nu(w)$ and $\mu(w)$ are additive and bounded functions of regular curves on S.

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* Bray and Evans, "A Class of Functions Harmonic within the Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180. The definitions of lattice there given, on pp. 156 and 169 should be completed by making each lattice a partition of

Moreover we may form Stieltjes integrals of the form

$$\int_S h(P)\,d\nu(w_P), \tag{3.2}$$

where $h(P)$ is continuous, and evaluate the integral in terms of a Riemann sum based on arbitrary modes of division of S into regular cells of diameter $< \delta$, where $\lim \delta = 0$, and the integral will be identical with $\int h(P)\,d\nu(e_P)$.* The integral (3.2) may also be extended to functions $h(P)$ bounded and measurable Borel, and in particular to $h(P) = q(P, w)$. Since then

$$\int_S q(P, w)\,d\nu(e_P) = \int_S q(P, w)\,d\nu(w_P)$$

with a similar identity for the function μ, we have

$$\left\{\begin{aligned} \nu(w) &= \int_S q(P, w)\,d\nu(w_P) \\ \mu(w) &= \int_S q(P, w)\,d\mu(w_P). \end{aligned}\right. \tag{3.3}$$

Functions of curves which satisfy (3.3) are said to have *regular* discontinuities.† The functions defined by (3.1) are then not only bounded and additive, but have regular discontinuities.

In particular, therefore, we may rewrite the integrals (1) and (2) in the forms

$$v(M) = \int_S (1/MP)\,d\mu(w_P), \tag{3.4}$$

$$u(M) = \int_S \frac{\cos(MP, n_P)}{MP^2}\,d\nu(w_P), \tag{3.5}$$

where the $\mu(w)$ and $\nu(w)$ satisfy (3.3). We need also to consider the integral

(3.6)

$$\begin{aligned} v'(Q) &= \int_S \frac{\cos(QP, n_Q)}{QP^2}\,d\mu(w_P) \\ &= \int_S \frac{\cos(QP, n_Q)}{QP^2}\,d\mu(e_P) \end{aligned}$$

where Q is on S.

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* Bray and Evans, *loc. cit.*, p. 159.

† In analogy with regular discontinuities of functions of a single variable. See Bray and Evans, *loc. cit.*, pp. 157, 170. Stieltjes integrals may also be formed with respect to functions of curves whose discontinuities are not regular, and evaluated by means of an arbitrary mode of division. See R. N. Haskell, "A Note on Stieltjes Integrals," *Annals of Mathematics*, Vol. 29 (1928), pp. 543-548.

When M is not on S these integrals offer no difficulty. But (3. 6), and (3. 4) and (3. 5), when $M = Q$ on S need special treatment. Consider the case of (3. 5). In this case the integral is not defined as the limit of a sum, but is regarded as a generalized or improper integral.* That this integral exists for almost all Q on S and represents a summable function on S, and that the identity

$$(3.7) \qquad \int_\omega d\omega_Q \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) = \int_S d\nu(w_P) \int_\omega \frac{\cos(QP, n_P)}{QP^2}\, d\omega_Q$$

is valid, follow from the fact that the generalized integral

$$\int_\omega dt(w_P) \int_S \frac{|\cos(QP, n_P)|}{QP^2}\, d\omega_Q$$

is convergent, where $t(w)$ is a bounded additive function of positive type, with regular discontinuities, say the total variation function of $\nu(w)$.† Similarly,

$$(3.8) \qquad \int_\omega d\omega_Q \int_S \frac{\cos(QP, n_Q)}{QP^2}\, d\mu(w_P) = \int_S d\mu(w_P) \int_\omega \frac{\cos(QP, n_Q)}{QP^2}\, d\omega_Q$$

is valid. These results may be summarized in the following theorem.

THEOREM 1. *The integrals* (3. 4), (3. 5), *when M is a point Q on S, and the integral* (3. 6) *converge for almost all Q, and the identities* (3. 7), (3. 8) *are valid. In these formulae $\nu(w)$ and $\mu(w)$ are given in terms of $\nu(e)$ and $\mu(e)$ by* (3. 1), *and have regular discontinuities, but the analogous formulae may be written directly in terms of $\nu(e)$ and $\mu(e)$.*

4. *Boundary values of integrals.* With reference to the correspondence between points of S and points of S' with $|n(P)| < \tau$, as described in § 2, let w be a regular closed curve on S, and $w' = w'(\tau)$, $\omega' = \omega'(\tau)$ be the corresponding sets of points on S'. We indicate that S' approaches S from the interior or from the exterior, respectively, by the symbols $\lim\limits_{\tau=0+}$ and $\lim\limits_{\tau=0-}$ and consider the quantities $\lim\limits_{\tau=0+} U(\tau, w)$, $\lim\limits_{\tau=0-} U(\tau, w)$, $\lim\limits_{\tau=0+} V(\tau, w)$, $\lim\limits_{\tau=0-} V(\tau, w)$, where

$$(4.1) \qquad \begin{aligned} U(\tau, w) &= \int_{\omega'} u(M)\, d\omega_M, \\ V(\tau, w) &= \int_{\omega'} \frac{d\nu(M)}{d\omega_M}\, d\omega_M. \end{aligned}$$

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* Daniell, *loc. cit.* Also Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329, in particular pp. 257-260.

† Evans, *loc. cit.*, p. 258. That the total variation function has regular discon

the integrations being extended over the surface S', and n_M being the normal to S' at M. Of course the quantity $V(\tau, w)$ depends merely on the curve w' and not on the particular surface on which it happens to lie, and represents merely the flux of $v(M)$ through that curve.

THEOREM 2. *As S' approaches S entirely from the inside or entirely from the outside of S the following equations hold:*

$$(4.2)\qquad \lim_{\tau=0\pm} U(\tau, w) = \mp 2\pi\nu(w) + \int_S d\nu(e_P) \int_\omega \frac{\cos(QP, n_P)}{QP^2}\, d\omega_Q,$$

$$(4.3)\qquad \lim_{\tau=0\pm} V(\tau, w) = \mp 2\pi\mu(w) + \int_S d\mu(e_P) \int_\omega \frac{\cos(QP, n_Q)}{QP^2}\, d\omega_Q.$$

Let $\omega'(\tau, \delta, P)$ be the portion of $\omega'(\tau)$ which is cut out by the normals to S in a small neighborhood $\omega(\delta, P)$ of P, where the projection of $\omega(\delta, P)$ on the tangent plane at P is a circle with center P and radius δ. Indicate the respective boundaries of these regions by $w'(\tau, \delta, P)$ and $w(\delta, P)$, and their complementary regions in $\omega'(\tau)$ and ω respectively by $\Omega'(\tau, \delta, P)$ and $\Omega(\delta, P)$.

With P on S and M on S', let ϕ be the angle (MP, n_P), and ϕ' the angle (MP, n_M), n_M being the normal to S' at M. If δ is sufficiently small, we shall have, for M in $\omega'(\tau, \delta, P)$,

$$\frac{|\cos\phi - \cos\phi'|}{r^2} \leq \frac{2}{r^2}\sin\frac{|\phi - \phi'|}{2} \leq \frac{\sphericalangle(n_M, n_P)}{r^2}$$
$$< Cr^{\alpha-2} < C\rho^{\alpha-2},$$
$$d\omega_M < 2\rho\, d\rho\, d\theta,$$

where ρ, the projection of $r = MP$, and θ are polar coördinates in the tangent plane at P. Hence

$$(4.4)\qquad \int_{\omega'(\tau,\delta,P)} \frac{|\cos\phi - \cos\phi'|}{r^2}\, d\omega_M < \frac{4\pi C\delta^\alpha}{\alpha}.$$

Hence, given ϵ, we may take $\delta(\epsilon)$, and then $\tau(\delta)$ so small that, by (4.4),

$$\left| \int_{\omega'(\tau,\delta,P)} \frac{\cos\phi}{r^2} d\omega_M \pm 2\pi q(P, w) \right| < \epsilon,$$

$q(P, w)$ being the density function, and the $+$ or $-$ sign being used according as S' is interior or exterior to S. In fact,

$$\int_{\omega'(\tau,\delta,P)} \frac{\cos\phi'}{r^2}\, d\omega_M$$

is the measure of the solid angle subtended by $\omega'(\tau, \delta, P)$ at P, prefixed by

the algebraic sign opposite to that of τ, and on account of the inequalities given for z in (2.5) and (2.9) differs from $2\pi q(P, w)$ by less than $4\pi C\rho$ as τ approaches zero.

Hence

$$(4.5) \qquad \left| \int_S d\nu(e_P) \int_{\omega'(\tau,\delta,P)} \frac{\cos\phi}{r^2}\, d\omega_M \pm 2\pi \int_S q(P, w)\, d\nu(e_P) \right| < \epsilon N(S),$$

where $N(S)$ is the total absolute mass on S.

On the other hand, given δ, the quantity $\int_{\Omega'(\tau,\delta,P)} \cos\phi/r^2 d\omega_M$ is continuous as τ approaches and becomes zero. Hence, given ϵ' and δ, we may take τ so small that

$$(4.6) \qquad \left| \int_S d\nu(e_P) \int_{\Omega'(\tau,\delta,P)} \frac{\cos\phi}{r^2}\, d\omega_M - \int_S d\nu(e_P) \int_{\Omega(\delta,P)} \frac{\cos\phi}{r^2}\, d\omega_Q \right| < \epsilon'$$

But by (2.5) and (2.9) we have $|\cos(QP, n_P)| < K\rho$ uniformly in P for Q on S in the neighborhood of P. Consequently

$$\left| \int_{\omega(\delta,P)} \frac{\cos\phi}{r^2}\, d\omega_Q \right| \leq 4\pi K\delta,$$

and

$$(4.7) \qquad \left| \int_S d\nu(e_P) \int_{\Omega(\delta,P)} \frac{\cos\phi}{r^2}\, d\omega_Q - \int_S d\nu(e_P) \int_\omega \frac{\cos\phi}{r^2}\, d\omega_Q \right| < \epsilon'$$

if $\delta \leq \epsilon'/[4\pi K N(S)]$.

Hence, given $\bar{\epsilon}$, we can, by combining the inequalities (4.5), (4.6) and (4.7), choose δ small enough and then τ small enough so that

$$\left| \int_S d\nu(e_P) \int_{\omega'(\tau)} \frac{\cos\phi}{r^2}\, d\omega_M - \int_S d\nu(e_P) \int_\omega \frac{\cos\phi}{r^2}\, d\omega_Q \pm 2\pi\nu(w) \right| < \bar{\epsilon},$$

since $\nu(w) = \int_S q(P, w)\, d\nu(e_P)$. We have then finally

$$\lim_{\tau=0\pm} \int_S d\nu(e_P) \int_{\omega'(\tau)} \frac{\cos\phi}{r^2}\, d\omega_M = \mp 2\pi\nu(w) + \int_S d\nu(e_P) \int_\omega \frac{\cos\phi}{r^2}\, d\omega_Q,$$

which is the equation (4.2), which was to be proved.

Equation (4.3) is established in a similar way. We have

$$(4.8) \qquad \Gamma(\tau, w) = -\int_{\omega'} d\omega_M \int_S \frac{\cos(MP, n_M)}{MP^2}\, d\mu(e_P)$$

$$= -\int_S d\mu(e_P) \int_{\omega'} \frac{\cos\phi'}{r^2}\, d\omega_M$$

and if we write $\omega' = \omega'(\tau, \delta, P) + \Omega'(\tau, \delta, P)$, the proof proceeds as before.

Moreover, if n_Q denotes the normal to S at the point Q which, on S, corresponds to M on S' and we form $V_1(\tau, w)$:

$$V_1(\tau, w) = \int_{\omega'(\tau)} \frac{dv(M)}{dn_Q}\, d\omega_M \tag{4.9}$$

$$= \int_S d\mu(e_P) \int_{\omega'} \frac{\cos(MP, n_Q)}{MP^2}\, d\omega_M$$

and again write $\omega' = \omega'(\tau, \delta, P) + \Omega'(\tau, \delta, P)$, the quantity

$$\int_{\Omega'(\tau,\delta,P)} [\cos(MP, n_Q)/MP^2]\, d\omega_M$$

is continuous as τ approaches and becomes zero. But, denoting $\sphericalangle(MP, n_Q)$ by ϕ_1, we have for M in $\omega'(\tau, \delta, P)$

$$|\cos\phi_1 - \cos\phi'| \leq 2 \left| \sin\frac{\phi_1 - \phi'}{2} \right| \leq |\phi_1 - \phi'| \leq |\sphericalangle(n_Q, n_M)|$$
$$< C\overline{MQ}^a \leq C\overline{MP}^a$$

by (2.10), if δ has been chosen small enough. Hence we may substitute ϕ' for ϕ_1, as we substituted ϕ' for ϕ in the consideration of $U(\tau, w)$, and we have the following corollary.

COROLLARY. *The function $V_1(\tau, w)$ behaves in the same way as $V(\tau, w)$ when τ approaches zero,* i. e.,

$$\lim_{\tau=0\pm} V_1(\tau, w) = \lim_{\tau=0\pm} V(\tau, w).$$

5. *Generalized boundary value problems, and Stieltjes integral equations.* Let $F(w)$, $G(w)$ be arbitrary bounded additive functions of regular curves on S, with regular discontinuities. We speak of functions, given by potentials of a single layer (1), as of class (1), and of those given by potentials of a double layer (2), as of class (2).

THEOREM 3. *There is one and only one function $u(M)$ of class* (2) *for which*

$$\frac{1+\lambda}{2\lambda} U(0+, w) - \frac{1-\lambda}{2\lambda} U(0-, w) = F(w), \qquad (\lambda \neq 0), \tag{5.1}$$

and there is one and only one $v(M)$ of class (1) *for which*

$$\frac{1+\lambda}{2\lambda} V(0-, w) - \frac{1-\lambda}{2\lambda} V(0+, w) = G(w), \qquad (\lambda \neq 0) \tag{5.2}$$

unless λ *is a characteristic value of the kernel*

$$\frac{\lambda}{2\pi}\ \frac{\cos(QP, n_P)}{QP^2} .$$

THEOREM 4. *The value* $\lambda = +1$ *is not a characteristic value. Hence the solutions of the particular problems*

$$U(0+, w) = F(w), \qquad V(0-, w) = G(w)$$

are unique in the respective classes (2) *and* (1).

THEOREM 5. *The value* $\lambda = -1$ *is a characteristic value. For this value of* λ, *condition* (5.1) *can be satisfied if and only if*

$$\int_S \phi_2(P)\, dF(e_P) = 0,$$

where $\phi_2(P)$ *is a solution of the homogeneous equation with kernel*

$$\frac{\lambda}{2\pi}\ \frac{\cos(QP, n_Q)}{QP^2}, \qquad (\lambda = -1),$$

and is determined except for an arbitrary multiplicative constant.

The mass function $\nu(w)$ *of* (2) *is then determined except for an arbitrary additive term of the form* $C\omega$, *which does not change the value of* $u(M)$, M *exterior to* S.

Condition (5.2) *can be satisfied for* $\lambda = -1$, *if and only if*

$$G(S) = 0.$$

The mass function $\mu(w)$ *of* (1) *is then determined except for an arbitrary additive term of the form* $C\int_\omega \phi_2(P)\, d\omega_P$; *the corresponding* $v(M)$ *contains an arbitrary additive constant if* M *is interior to* S.

The value $\lambda = -1$ in (5.1) corresponds to the exterior Dirichlet problem, in (5.2) to the interior Neumann problem.

On account of (4.2), (4.3), the conditions (5.1) and (5.2) may be rewritten as Stieltjes integral equations:

$$(5.3) \qquad \nu(w) = \frac{\lambda}{2\pi} F(w) + \frac{\lambda}{2\pi}\int_S d\nu(e_P)\int_\omega \frac{\cos(QP, n_Q)}{QP^2}\, d\omega_Q .$$

$$(5.4) \qquad \mu(w) = -\frac{\lambda}{2\pi} G(w) - \frac{\lambda}{2\pi}\int_S d\mu(e_P)\int_\omega \frac{\cos(QP, n_P)}{QP^2}\, d\omega_Q .$$

If $v(M)$ is a function of class (1) which satisfies (5.2), the function $\mu(w)$, bounded and additive, with regular discontinuities, must be a solution of (5.4). Conversely, if $\mu(w)$ is a solution of (5.4), bounded and additive, and having regular discontinuities, the function $v(M)$ given by (1) will satisfy (5.2). Similarly, for the functions $u(M)$ of class (2).

Each of the equations (5.3), (5.4) has a unique solution which is a bounded additive function of curves, with regular discontinuities, unless λ is a characteristic value of the kernel

$$\frac{\lambda}{2\pi}\,\frac{\cos(QP,n_P)}{QP^2}.$$

Equations (5.3), (5.4) are solved by means of Fredholm equations, by means of the substitution

$$\nu(w)+\frac{\lambda}{2\pi}\,F(w)=R_1(w), \tag{5.5}$$

$$\mu(w)-\frac{\lambda}{2\pi}\,G(w)=R_2(w). \tag{5.6}$$

The possibility of reducing (5.3), (5.4) to Fredholm equations depends on the fact that $R_1(w)$ and $R_2(w)$ are absolutely continuous, on account of (3.7). Consider (5.3). Substituting the value of $\nu(w)$ from (5.5) into (5.3), we have

$$R_1(w)=-\frac{\lambda^2}{4\pi^2}\int_S dF(w_P)\int_\omega \frac{\cos(QP,n_P)}{QP^2}\,d\omega_Q + \frac{\lambda}{2\pi}\int_S dR_1(w_P)\int_\omega \frac{\cos(QP,n_P)}{QP^2}\,d\omega_Q.$$

The derivative nearly everywhere of $R_1(w)$ is

$$r_1(Q)=-\frac{\lambda^2}{4\pi^2}\int_S \frac{\cos(QP,n_P)}{QP^2}\,dF(w_P) + \frac{\lambda}{2\pi}\int_S \frac{\cos(QP,n_P)}{QP^2}\,dR_1(w_P),$$

or

$$r_1(Q)=f(Q)+\frac{\lambda}{2\pi}\int_S \frac{\cos(QP,n_P)}{QP^2}\,r_1(P)\,d\omega_P, \tag{5.7}$$

since $R_1(w)$ is absolutely continuous, where we have placed

$$f(Q)=-\frac{\lambda^2}{4\pi^2}\int_S \frac{\cos(QP,n_P)}{QP^2}\,dF(w_P).$$

In the same way, we have

$$(5.8) \qquad r_2(Q) = g(Q) - \frac{\lambda}{2\pi} \int_S \frac{\cos(QP, n_Q)}{QP^2} r_2(P)\, d\omega_P,$$

where

$$g(Q) = -\frac{\lambda^2}{4\pi^2} \int_S \frac{\cos(QP, n_Q)}{QP^2} dG(w_P).$$

Hence if $\nu(w)$ is a solution of (5.3), bounded and additive, with regular discontinuities, $r_1(Q)$ will be a solution of (5.7). That is, (5.7) is necessary for (5.3); and (5.4) implies (5.8). Conversely, if $r_1(Q)$ is a summable solution of (5.7), then the $\nu(w)$ given by (5.5) will be bounded and additive, and will satisfy (5.3). Similar statements may be made with regard to (5.8), (5.6) and (5.4).

Lemma. *Unless* λ *is a characteristic value,* i. e., *unless for this value of* λ *there is a continuous solution of the homogeneous equation with kernel*

$$\frac{\lambda}{2\pi} \frac{\cos(QP, n_P)}{QP^2},$$

each of the equations (5.7), (5.8) *has a unique summable solution.*

Suppose λ not to be a characteristic value. Then (5.7) has a summable solution. In fact, this equation is equivalent to that which is obtained by twice repeated substitution of (5.7) into itself, the resulting equation having a bounded continuous kernel.* Since the known function in this equation is summable, the solution, given in terms of the resolvent kernel is summable. The solution moreover is unique. For, suppose there were two different summable solutions of (5.7). Then their difference, say $\phi_1(Q)$, would be a summable solution of the homogeneous equation

$$(5.9) \qquad \phi_1(Q) = \frac{\lambda}{2\pi} \int_S \frac{\cos(QP, n_P)}{QP^2} \phi_1(P)\, d\omega_P.$$

But every solution of this equation is a solution of the homogeneous equation with twice iterated (and continuous) kernel, namely, the equation

$$(5.10) \qquad \phi_1(Q) = \frac{\lambda^3}{8\pi^3} \int_S K_3(Q, P) \phi_1(P)\, d\omega_P.$$

If $\phi_1(Q)$ is summable, it is continuous in virtue of (5.10). But this means

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* Goursat, *Cours d'Analyse*, Vol. 3, Paris (1915), pp. 355, 364; Kellogg *Foundations of Potential Theory*, Berlin [illegible] [illegible] that it is continuous on the basis of proper assumptions on S. The assumptions on S in the proof cited are slightly [illegible] restrictive than (2.9), but the proof remains valid merely under (2.9).

that (5.9) has a continuous non-zero solution. Consequently λ would be a characteristic value, contrary to the assumption. The lemma is proved.

Since the kernels

$$\frac{\lambda}{2\pi}\ \frac{\cos(QP, n_P)}{QP^2}, \qquad -\frac{\lambda}{2\pi}\ \frac{\cos(QP, n_Q)}{QP^2}$$

of (5.7) and (5.8) respectively are associated, and have the same characteristic values, we may consider both equations at the same time. It is known that $\lambda = -1$ is a characteristic value for these kernels, while $\lambda = 1$ is not.*

* These statements are usually proved on the basis of slightly more restrictive assumptions on S, as in Kellogg, *loc. cit.*, p. 311. Accordingly we indicate a brief proof on the basis of our hypotheses with regard to S, of which the characteristic assumption is (2.9).

Suppose $\lambda = 1$ were a characteristic value. There would then be a continuous, not identically vanishing solution $\phi(Q)$, of the homogeneous integral equation corresponding to (5.8), namely

$$(\alpha) \qquad 0 = \phi(Q) + \frac{\lambda}{2\pi}\int_S \frac{\cos(QP, n_Q)}{QP^2}\ \phi(P)\, d\omega_P, \qquad (\lambda = 1).$$

The function $v_0(M)$, given in the infinite region exterior to S as a potential of a single layer (1) on S,

$$v_0(M) = \int_S \frac{1}{MP}\ \phi(P)\, d\omega_P,$$

would then satisfy the boundary condition (4.3) with

$$\mu(w) = \int_\omega \phi(P)\, d\omega_P,$$

that is,

$$\lim_{\tau=0-} V_0(\tau, w) = \int_\omega \{2\pi\phi(Q) + \int_S \frac{\cos(QP, n_Q)}{QP^2}\ \phi(P)\, d\omega_P\} d\omega_Q = 0.$$

If now we apply Green's theorem to the infinite region T', exterior to S', we have

$$\begin{aligned}\int_{T'}\{(\partial v_0/\partial x)^2 + (\partial v_0/\partial y)^2 + (\partial v_0/\partial z)^2\} dT &= -\int_{S'} v_0(\partial v_0/\partial n)\, d\omega' \\ &= \int_{S'} v_0(M)\, dV_0(\tau, \omega_M),\end{aligned}$$

since $v_0(M)$ vanishes canonically at ∞. But since $v_0(M)$ is a continuous function of Q and τ as M approaches Q along n_Q, and the total variation of $V_0(\tau, \omega)$ is bounded, we have by a well known theorem

$$(\beta) \qquad \lim_{\tau=0-}\int_{S'} v_0(M)\, dV_0(\tau, \omega_M) = \int_{S'} v_0(M)\, d[\lim_{\tau=0-} V_0(\tau, \omega)] = 0.$$

In fact, the total variation of $V_0(\tau, \omega)$ does not exceed the quantity

$$\int_S d\omega_P\ |\ \phi(P)\ |\int_{S'} \left|\frac{\cos(MP, n_M)}{MP^2}\right|\ d\omega_M,$$

[illegible] λ [illegible] 1. (5.9) [illegible] 1. [illegible] homogeneous equation corresponding to (5.8) has a not identically vanishing solution $\varphi_2(Q)$. Both of these homogeneous equations have the same number of linearly independent solutions. It is known that for $\lambda = -1$ there cannot be two linearly independent solutions of (5.9) or of the homogeneous equation corresponding to (5.8).[8]

For the value $\lambda = -1$, a necessary and sufficient condition that (5.7) have a solution is that

$$\int_S \varphi_2(P) f(P)\, dP = 0$$

which, on replacing $f(P)$ by its equivalent in terms of $F(w)$, and reversing the order of integration, takes the form

$$\int_S \phi_2(P)\, dF(w_P) = 0, \tag{5.11}$$

$\phi_2(P)$ denoting a solution of the homogeneous equation with kernel

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and if we write

$$\int_{S'} = \int_{\omega'(\tau,\delta,P)} + \int_{\Omega'(\tau,\delta,P)},$$

the first of these integrals is $< 4\pi$ and the second is $<$ (const.) (meas. S') $<$ (const.) $\times$ (2 meas. S).

Hence if T is the region exterior to S, we have, from (β),

$$\int_T \{(\partial v_0/\partial x)^2 + (\partial v_0/\partial y)^2 + (\partial v_0/\partial z)^2\}\, dT = 0.$$

But this means that v_0 is constant outside S, and since it vanishes at ∞ and is continuous across S, it vanishes outside and on S. Hence it vanishes identically also inside S. The condition (4.3) yields now the fact that

$$\int_\omega \left\{ -2\pi\phi(Q) + \int_S \frac{\cos(QP, n_Q)}{QP^2}\, \phi(P)\, d\omega_P \right\} d\omega_Q = 0,$$

which with (a) and the continuity of $\phi(Q)$ implies that $\phi(Q) = 0$. This is contrary to the hypothesis. Hence $\lambda = 1$ is not a characteristic value.

[8] Suppose in fact that we apply Green's theorem to the region interior to S, in the same way as we applied it in the previous footnote to the region exterior to S, taking as $v(M)$ the potential of a single layer associated with a continuous solution $\varphi_2(Q)$. The potential $v(M)$ will be constant within and on S and will vanish [illegible]
[illegible]

$$-\frac{\lambda}{2\pi}\;\frac{\cos(QP, n_Q)}{QP^2}\,, \qquad\qquad (\lambda = -1).$$

Suppose the condition (5. 11) is satisfied. Then the general solution of (5. 7) is

$$r_1(Q) = \bar{r}_1(Q) + \bar{C},$$

since $\phi_1(Q) = \bar{C}$ is a solution of the homogeneous equation

$$\phi_1(Q) = -\frac{1}{2\pi}\int_S \frac{\cos(QP, n_P)}{QP^2}\;\phi_1(P)\,d\omega_P.$$

Hence by (5. 5), we have

$$\begin{aligned}\nu(w) &= \frac{1}{2\pi}F(w) + R_1(w)\\ &= \frac{1}{2\pi}F(w) + \int_\omega \bar{r}_1(Q)\,d\omega_Q + \int_\omega \bar{C}\,d\omega_Q\\ &= \nu_1(w) + \bar{C}\omega.\end{aligned}$$

If we substitute this value of $\nu(w)$ in (2), dealing as we are here with the exterior Dirichlet problem, the contribution due to the term $\bar{C}\omega$ is zero, and $u(M)$ is given simply by the formula (2) with $\nu(e) = \nu_1(e)$.

For the value $\lambda = -1$, and the interior Neumann problem, a necessary and sufficient condition that (5. 8) have a solution is that

$$\int_S g(Q)\,\phi_1(Q)\,d\omega_Q = \bar{C}\int_S g(Q)\,d\omega_Q = 0.$$

But this is equivalent to the condition

$$0 = -\frac{1}{4\pi^2}\int_S dG(w_P)\int_S \frac{\cos(QP, n_Q)}{QP^2}\;d\omega_Q = \frac{1}{2\pi}\int_S dG(w) = G(S).$$

Hence we have the condition

(5. 12) $$G(S) = 0.$$

But, since the steps may be retraced, this condition also is both necessary and sufficient in order that the interior Neumann problem be solved by means of a potential of a single layer.

If (5. 12) is satisfied, the general solution of (5. 8) is in the form

$$r_2(Q) = \bar{r}_2(Q) + C\phi_2(Q)$$

where C is an arbitrary constant, and $\phi_2(Q)$ is a solution of the homogeneous equation

$$\phi_2(Q) = \frac{1}{2\pi}\int_S \frac{\cos(QP, n_Q)}{QP^2}\;\phi_2(P)\,d\omega_P,$$

and is a continuous function. But as we have seen [in the preceding footnote] the potential (1) where $\nu(w) = \int_\omega \phi_2(P)\, d\omega_P$ reduces inside S to a constant. Hence the effect of the term $C\phi_2(Q)$ is merely to change $v(M)$ inside S by an arbitrary constant.

6. *Boundary values of $u(M)$ and $v(M)$.* For simplicity, we consider merely approach in the narrow sense, letting M approach Q on S along the normal n_Q ($\lim M = Q \pm$). We shall show that $u(M)$ and $dv(M)/dn_Q$ approach definite boundary values wherever $\nu(w)$ and $\mu(w)$ have derivatives, that is, nearly everywhere on S; and for S we may take a surface which satisfies merely the conditions of the Lemma of § 2, with no assumptions about the curvature. We say that $\nu(w)$ [or $\nu(e)$] has the derivative A at Q if $\lim_{\omega=0} \nu(w)/\omega = A$, where the regions ω form a regular family about Q.

THEOREM 6. *Let S satisfy the conditions of the Lemma of § 2, and let Q be a point on S where $\nu(w)$ has a derivative $\nu'(Q) = A$; then*

$$(6.1) \qquad \lim_{M=Q\pm} u(M) = \mp 2\pi A + \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(e_P).$$

If $\mu'(Q)$ exists, $= A$, then

$$(6.2) \qquad \lim_{M=Q\pm} \frac{dv(M)}{dn_Q} = \mp 2\pi A + \int_S \frac{\cos(QP, n_Q)}{QP^2}\, d\mu(e_P).$$

So far, in dealing with integrals over the whole of S it has made no difference whether we used the differential $d\nu(e)$ or the differential $d\nu(w)$. For integrals over a part of S, however, the symbols may involve some ambiguity. Hence for the integral $\int_w f(P)\, d\nu(w_P)$, extended over a region ω bounded by w, where $f(P)$ is integrable in the Borel sense, we introduce the definition

$$\int_w f(P)\, d\nu(w_P) = \int_S q(P, w) f(P)\, d\nu(w_P) = \int_S q(P, w) f(P)\, d\nu(e_P),$$

in which $q(P, w)$ is the density function for ω bounded by w. The integral is thus an additive function of curves w.

We indicate integration over the region Ω complementary to ω by the symbol $\int f(P)\, d\nu(w_P)$, W being the same curve as w, regarded, if we like, as having opposite sense. We have

$$\int_w f(P)\, d\nu(w_P) + \int_W f(P)\, d\nu(w_P) = \int_S f(P)\, d\nu(w_P).$$

It may be noted that given $\nu(w)$ there is one and only one $\nu(e)$ such that $\nu(w) = \nu(\omega)$ for curves w where $\nu(w)$ is continuous in the sense of Volterra; conversely, given $\nu(e)$ there is one and only one $\nu(w)$ corresponding to it in this sense, provided $\nu(w)$ has regular discontinuities.*

Since $\nu(w)$ and $\mu(w)$ are independent, we may as well assume that the hypotheses for each of them are satisfied at some point Q, and prove both identities at once. We must first examine the integral

$$I = \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) = \int_S \frac{\cos(QP, n_P)}{QP^2} \cdot d\nu(e_P)$$

and the similar integral J, with n_Q instead of n_P. We may write

$$\nu(w) = A\omega + \omega\eta(w)$$

where $\omega\eta(w)$ is a bounded additive function of regular curves on S, with regular discontinuities, such that $\lim_{\omega \to 0} \eta(w) = 0$ when the regions ω form a regular family about Q. Without loss of generality we may assume that $\eta(w)$ is of positive type.†

Let I_1 be that part of I referring to $A\omega$, and I_2 to $\omega\eta(w)$; the convergence of I_1 is a consequence of the fact that S is of class Γ. The convergence of I_2 will be established as a generalized integral, if it can be done when $\cos(QP, n_P)$ is replaced by $|\cos(QP, n_P)|$, and the convergence of the latter integral will be established if

$$\lim_{m \to \infty} \int_S h(m, P)\, d[\omega\eta(w)]$$

exists, where

$$h(m, P) = \frac{|\cos(QP, n_P)|}{QP^2}, \qquad (QP > 1/m),$$

$$= m^2\, |\cos(QP, n_P)|, \qquad \text{otherwise.}$$

In fact the $h(m, P)$ form an increasing sequence of functions of P, with $\lim_{m \to \infty} h(m, P) = |\cos(QP, n_P)|/QP^2$. Finally we need only consider the neighborhood $\omega(\delta, Q)$ of Q, so that $|\cos(QP, n_P)| < 3g(\rho)$, ρ being the projection of QP on the tangent plane at Q, and $g(\rho)$ being the continuous function of § 2 such that $m(\delta) = \int_0^\delta [g(\rho)/\rho]\, d\rho$ converges. In fact,

* Bray and Evans, *loc. cit.*, p. 159.

† Bray and Evans, *loc. cit.*, p. 173. In the analysis there given, the case where one or the other of the two families p_i', p_j'' may contain merely a finite number of elements should be considered; but in this special case the proof is immediate.

$$
\begin{aligned}
&| \cos(QP, n_P) - \cos(QP, n_Q) | \\
&\qquad \leq 2 \, | \sin\{(QP, n_P) + (QP, n_Q)\}/2 \, | \; | \sin(n_P, n_Q)/2 \, | \leq g(\rho).
\end{aligned}
$$

We may therefore substitute for the $h(m, P)$ the dominating increasing sequence

$$
\begin{aligned}
h_1(m, P) &= 3g(\rho)/\rho^2 && \text{if } \rho > 1/m, \\
&= 3m^2 g(1/m) && \text{otherwise.}
\end{aligned}
$$

The function $\omega\eta(w)$ being of positive type is a non-decreasing function of ρ if $w = w(\rho, Q)$. If we represent this function by $\beta(\rho)$ and remember that $h_1(m, P)$ is a function of ρ alone, we have

$$
\int_{w(\delta,Q)} h_1(m, P) \, d[\omega\eta(w)] = \int_0^\delta h_1(m, P) \, d\beta(\rho)
$$

and this by an integration by parts is equal in absolute value to

$$
\begin{aligned}
&\Big| \Big[\beta(\rho) h_1(m, P)\Big]_0^\delta - \int_0^\delta \beta(\rho) \, dh_1(m, P) \Big| \\
&\leq 3 \left\{ \frac{\beta(\delta) g(\delta)}{\delta^2} + \left| \int_{1/m}^\delta \beta(\rho) \, d \, \frac{g(\rho)}{\rho^2} \right| \right\} \\
&\leq 3g(\delta)\eta[w(\delta, Q)] + 3\pi\eta[w(\delta, Q)] \left| \int_{1/m}^\delta \rho^2 d \, \frac{g(\rho)}{\rho^2} \right|
\end{aligned}
$$

The Stieltjes integral in the last expression is however, by a second integration by parts, the same as

$$
\Big| \, g(\delta) - g(1/m) - 2 \int_{1/m}^\delta \frac{g(\rho)}{\rho} \, d\rho \, \Big| < g(\delta) + 2m(\delta).
$$

Hence

$$
\int_{w(\delta,Q)} h_1(m, P) \, d[\omega\eta(w)] < 6\pi\eta(w(\delta, Q))[g(\delta) + m(\delta)].
$$

But this bound is independent of m, so that the point is proved. Similar considerations apply to the integral J. Incidentally, this bound may be made as small as we please by taking δ small enough.

It is well to emphasize the following results, implied by the analysis which has just been completed.

LEMMA. *The integrals*

$$
\int_{\omega(\delta,Q)} [g(\rho)/\rho^2] \, d\omega, \quad \int_{w(\delta,Q)} [g(\rho)/\rho^2] \, d\,[\omega\eta(w)]
$$

are convergent, and may be made arbitrarily small with δ, if $g(\rho)$ is a positive continuous increasing function of ρ such that $\int_0^\delta [g(\rho)/\rho] \, d\rho$ is convergent.

COROLLARY. *We have, under the hypotheses of Theorem 6,*

$$(6.3)\qquad \lim_{\delta=0}\int_{\Omega(\delta,Q)}\frac{\cos(QP,n_P)}{QP^2}\,d\omega=\int_S\frac{\cos(QP,n_P)}{QP^2}\,d\omega,$$

$$(6.4)\qquad \lim_{\delta=0}\int_{W(\delta,Q)}\frac{\cos(QP,n_P)}{QP^2}\,d\nu(w_P)=\int_S\frac{\cos(QP,n_P)}{QP^2}\,d\nu(w_P),$$

with similar relations if n_P is replaced by n_Q and $\nu(w)$ by $\mu(w)$.

If we let ϕ be the angle (MP,n_P), ϕ' the angle (MP,n_Q), $\psi=\phi-\phi'$, $r=MP$, we have

$$(6.5)\qquad u(M)=\int_S\frac{\cos\phi}{r^2}\,d\nu(w_P),\qquad \frac{d\nu(M)}{dn_Q}=\int_S\frac{\cos\phi'}{r^2}\,d\mu(w_P).$$

We evaluate the limits of these integrals first by comparing them for the portion $\omega(\delta,Q)$ of S, then with corresponding integrals over the projection of $\omega(\delta,Q)$ on the tangent plane at Q, and finally by noticing that the integrals over the portion $\Omega(\delta,Q)$ are continuous as M approaches Q.

Accordingly, we show first that given ϵ we can choose δ small enough so that for all smaller values of δ we have

$$(6.6)\qquad \int_{\omega(\delta,Q)}\left|\frac{\cos\phi-\cos\phi'}{r^2}\right|d\omega\leq\epsilon$$

$$(6.7)\qquad \int_{w(\delta,Q)}\left|\frac{\cos\phi-\cos\phi'}{r^2}\right|d[\omega\eta(w)]\leq\epsilon,$$

independently of the position of M. We have in fact, if P is in $\omega(\delta,Q)$,

$$\left|\frac{\cos\phi-\cos\phi'}{r^2}\right|=\frac{2}{r^2}\left|\sin\frac{\phi+\phi'}{2}\sin\psi/2\right|\leq\psi/r^2\leq g(\rho)/\rho^2,$$

applying (2.4) and making use of the fact that $|\psi|\leq|n_P,n_Q|$. The desired inequalities follow then immediately from the lemma.

Let r_1 be MP_1 and ϕ_1 be the angle MP_1,n_Q where P_1 is the point, on the tangent plane at Q, at the foot of the perpendicular from P. We show now that, given ϵ, we can choose δ small enough so that for all smaller δ we have

$$(6.8)\qquad \int_{w(\delta,Q)}\left|\frac{\cos\phi'}{r^2}-\frac{\cos\phi_1}{r_1^2}\right|d([\omega\eta(\omega)]\leq\epsilon/2,$$

independently of the position of M. In fact, using the symbol z of § 2,

$$\left|\frac{\cos\phi'}{r^2}-\frac{\cos\phi_1}{r_1^2}\right|=\left|\frac{\cos\phi'-\cos\phi_1}{r^2}+\frac{\cos\phi_1(r_1+r)(r_1-r)}{r^2r_1^2}\right|$$
$$\leq 2g(\rho)/\rho^2+2z/\rho^3\leq 2g(\rho)/\rho^2+4g(\rho)/\rho^2=6g(\rho)/\rho^2,$$

the last inequality being a consequence of (2.5). Consequently, here again, the desired inequality follows immediately from the lemma.

Moreover if δ is small enough we can show that

$$\left| \int_{w(\delta,Q)} (\cos\phi_1/r_1^2)\, d[\omega\eta(w)] \right| < \epsilon/2,$$

whence, from (6.8),

$$\left| \int_{w(\delta,Q)} (\cos\phi'/r^2)\, d[\omega\eta(w)] \right| \leq \epsilon, \tag{6.9}$$

independently of M.

For this purpose we introduce again the notation $\beta(\rho)$, and write, integrating by parts,

$$\left| \int \frac{\cos\phi_1}{r^2}\, d[\omega\eta(w)] \right| = \left| \int \frac{\cos\phi_1}{r_1^2}\, d\beta(\rho) \right|$$
$$\leq \beta(\delta)/\delta^2 + \left| \int \beta(\rho)\, d(\cos\phi_1/r_1^2) \right|,$$

of which the first term of the last member may be made as small as we please, with δ. The function $\cos\phi_1/r_1^2$ is a monotonic function of ρ, since for a given M the numerator decreases in numerical value, without changing sign, and the denominator increases, as ρ increases. In the second term, then, we introduce η_δ as the upper bound of $\eta(w)$ when w is the circle of radius ρ, $0 < \rho \leq \delta$, and obtain

$$\left| \int \beta(\rho)\, d(\cos\phi_1/r_1^2) \right| < 2\pi\eta_\delta \left| \int \rho^2 d(\cos\phi_1/r_1^2) \right|,$$

for, since $|(n_Q, n_P)| < 1$, $\omega(\rho) < 2\pi\rho^2$. But by performing again the integration by parts, the right hand member of this inequality is

$$\leq 2\pi\eta_\delta + 2\eta_\delta \left| \int (\cos\phi_1/r_1^2)\, d(\pi\rho^2) \right|$$
$$\leq 2\pi\eta_\delta + 4\pi\eta_\delta = 6\pi\eta_\delta,$$

and this also may be made as small as we please with δ, independently of M, since η_δ approaches zero with δ. Thus the desired inequality is established.

The rest of the proof is not difficult. In accordance with (6.4) we can take δ small enough so that, given ϵ,

$$\left| \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) - \int_{W(\delta,Q)} \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) \right| \leq \epsilon. \tag{6.10}$$

Also, if we denote by γ the limit of the absolute value of the solid angle subtended at M by $\omega(\delta, Q)$ as M approaches Q, we can take δ small enough so that, given ϵ,

$$|A\gamma - 2\pi A| \leq \epsilon. \tag{6.11}$$

[illegible]

is fixed, however, we may choose M close enough to Q on n_Q so that the following inequalities hold, when ϵ_1 is given:

$$\left| \int_{W(\delta,Q)} \frac{\cos\phi}{r^2}\, d\nu(w_P) - \int_{W(\delta,Q)} \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) \right| \leq \epsilon_1, \tag{6.12}$$

$$\left| \int_{\omega(\delta,P)} \frac{\cos\phi}{r^2} A\, d\omega \pm A\gamma \right| \leq \epsilon_1, \qquad M \text{ on } \pm n_Q. \tag{6.13}$$

Consider now the inequalities:

$$\begin{aligned} R = & \left| \int_S \frac{\cos\phi}{r^2}\, d\nu(w_P) - \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) \pm 2\pi A \right| \\ \leq & \left| \int_S \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) - \int_{W(\delta,Q)} \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) \right| \\ & + \left| \int_{w(\delta,Q)} \frac{\cos\phi}{r^2}\, d[\omega\eta(w)] - \int_{w(\delta,Q)} \frac{\cos\phi'}{r^2}\, d[\omega\eta(w)] \right| \\ & + \left| \int_{w(\delta,Q)} \frac{\cos\phi'}{r^2}\, d[\omega\eta(w)] \right| \\ & + \left| A\gamma - 2\pi A \right| \\ & + \left| \int_{W(\delta,Q)} \frac{\cos\phi}{r^2}\, d\nu(w_P) - \int_{W(\delta,Q)} \frac{\cos(QP, n_P)}{QP^2}\, d\nu(w_P) \right| \\ & + \left| \int_{\omega(\delta,Q)} \frac{\cos\phi}{r^2} A\, d\omega \pm A\gamma \right| \end{aligned}$$

where with the notation $\pm$ the $+$ sign is taken if M is on $+n_Q$, the $-$ sign if M is on $-n_Q$. By taking δ small enough each of the first four expressions in absolute value signs, of the right hand member, may be made $\leq \epsilon$, by means of (6.10), (6.7), (6.9), (6.11), independently of the position of M; with δ so fixed, M may be taken on $\pm n_Q$ near enough to Q so that each of the last two expressions may be made $\leq \epsilon_1$, by means of (6.12), (6.13). Hence, given ϵ and ϵ_1 arbitrarily small, we can choose M near enough to Q so that

$$R \leq 4\epsilon + 2\epsilon_1.$$

But this establishes the equation (6.1) of the theorem.

By means of similar inequalities, taking account also of (6.6), we establish (6.2). This completes the demonstration of the theorem.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

SUFFICIENT CONDITIONS IN THE PROBLEM OF LAGRANGE WITH VARIABLE END CONDITIONS.

By MARSTON MORSE.*

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1. *Introduction.* A general formulation of problems of this type has been given by Bolza † who obtained necessary conditions analogous to the Euler equations and the transversality conditions. Bliss ‡ lightened Bolza's hypotheses, and gave a new form to the problem. In its simpler aspects in the plane it was recognized by Hilbert § and others that the conditions analogous to the Jacobi conditions could be given in terms of characteristic roots of an auxiliary boundary value problem. A necessary condition of the latter sort has been recently obtained by Cope.¶ The author obtains a similar necessary condition in somewhat simpler form.

Sufficient conditions in this general problem have never been established. The present paper gives and establishes such conditions.

The author has also studied the complete problem of which the minimum problem is a special case, namely, the problem of finding an extremal which gives to a fundamental quadratic form a prescribed type number. The solution is in terms of characteristic roots of a linear boundary value problem.

Finally the results obtained lead to new types of separation and oscillation theorems involving the relative distribution of characteristic roots of two different auxiliary boundary value problems. This is in contrast with earlier comparison theorems, more geometric in nature involving focal points and conjugate points. See Morse II.** Other theorems of this nature even more general in character will be published shortly by the author.

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* Presented to the Society, September 12 (1930).

† Bolza, *Mathematische Annalen*, Vol. 74 (1913), p. 430.

‡ Bliss, *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 305. Also see Bliss, *American Journal of Mathematics*, Vol. 52 (1930), p. 674.

§ In this connection see the following:

Lovitt, *Linear Integral Equations*, p. 207, for work of Hilbert; Richardson, *Mathematische Annalen*, Vol. 68 (1910), p. 279; Plancherel, *Bulletin des Sciences Mathématiques*, Vol. 47 (1923), p. 376; Bliss, *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 317.

¶ Cope, [illegible]

** Certain results in the following papers will be used.

Morse I, *Transactions of the American Mathematical Society*, Vol. 31 (1929), p. 379. This paper shows the connections with a theory in the large.

THE ACCESSORY PROBLEM.

2. *The transversality conditions.* In the space of the variables x and

$$(y) = (y_1, \cdots, y_n)$$

let there be given a curve

$$y_i = \bar{y}_i(x) \qquad a^1 \leqq x \leqq a^2 \qquad (i = 1, \cdots, n) \tag{2.1}$$

of class C'. Points neighboring the initial and final end points of g will be denoted respectively by

$$(x^s, y_1^s, \cdots, y_n^s) = (x^s, y^s) \qquad (s = 1, 2) \tag{2.2}$$

where $s = 2$ at the final end point and 1 at the initial end point.

We consider curves of class C' neighboring g. Such curves will be called *differentially admissible* if they satisfy m differential equations of the form

$$\phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m < n). \tag{2.3}$$

We suppose g is differentially admissible, and that along g the functional matrix of the functions (2.3) with respect to the variables y_i' is of rank m.

A curve neighboring g will be said to be *terminally admissible* if its end points are given for some value of (α) by the functions

$$x^s = x^s(\alpha_1, \cdots, \alpha_r) \quad y_i^s = y_i^s(\alpha_1, \cdots, \alpha_r) \qquad 0 < r \leqq 2n + 2 \tag{2.4}$$

Morse II. *Mathematische Annalen*, Band 103 (1930), p. 52. Here are separation theorems.

Morse III. "Sufficient Conditions in the Problem of Lagrange with Fixed End points," *Annals of Mathematics*, Vol. 32 (1931).

In this paper sufficient conditions are derived, it is believed for the first time, under the following hypotheses regarding normalcy. If the extremal g be defined on the closed interval (ab) of the x axis it shall be normal relative to the Euler conditions on every subinterval of (ab). Previous methods of proof break down because now a set of extremals through a point p on g's extension just before g *need not form a field* near g. Previously it was assumed that g was normal on every sub-interval of an interval including (ab) in its interior. For a definition of normalcy see the following paper.

Morse and Myers, *Proceedings of the American Academy of Arts and Sciences*, Vol. 66 (1931), p. 235. Here the Euler and transversality conditions are derived in forms necessary for the present paper.

Other references follow:

Bolza, *Vorlesungen über Variationsrechung*, hereafter referred to as Bolza; Hadamard, *Leçons sur le calcul des variations*, Vol. 1, Paris, 1910; Carathéodory, "Die Methode der geodätischen Äquidistanten und das Problem von Lagrange," *Acta Mathematica*, Vol. 47 (1926), p. 199; J. Radon, "Über die Oszillationstheoreme der konjugierten Punkte beim Problem von Lagrange," *Sitzungsberichte der mathnaturwissenschaftlichen Abteilung der Bayrischen Akademie der Wissenschaften zu München* (1927), p. 243.

where these functions of (α) are defined for (α) near (0) and reduce to the end points of g for $(\alpha) = (0)$. We assume that the functional matrix of the $2n + 2$ functions in (2.4) is of rank r for $(\alpha) = (0)$.

A curve that is both differentially and terminally admissible will be called admissible.

We seek first the conditions under which g affords a minimum for the expression

$$J = \int_{x^1}^{x^2} f(x, y, y')\,dx + G(x^s, y^s) \tag{2.5}$$

among admissible curves of class C'.

The functions f and ϕ_β are to be of class C''' while the functions G and the end point functions in (2.4) need be of no more than class C''.

It is known * that if g affords a minimum for J there exists a constant λ_0 and m functions $\lambda_\beta(x)$ not all identically zero if $\lambda_0 = 0$, such that g satisfies the equations

$$(d/dx)F_{y_i'} - F_{y_i} = 0 \qquad (i = 1, \cdots, n) \tag{2.6}$$

where †

$$F = \lambda_0 f + \lambda_\beta \phi_\beta \qquad (\beta = 1, \cdots, m)$$

while the following transversality relations hold

$$\lambda_0 dG + [(F - \bar{y}_i' F_{y_i'})\,dx^s + F_{y_i'}\,dy_i^s]_1^2 = 0. \tag{2.7}$$

Here dx^s and dy_i^s are the differentials of the functions (2.4) evaluated for $(\alpha) = (0)$, while dG is to be evaluated for (x^1, y^1) and (x^2, y^2) at the respective ends of g. Here and elsewhere $[\ ]_1^2$ shall mean the difference between the value of the bracket evaluated for $s = 2$ with (x, y, y') at the final end point of g, and the corresponding evaluation at the initial end point of g.

It will be convenient to set

$$G[x^s(\alpha), y^s(\alpha)] \equiv g(\alpha). \tag{2.8}$$

If we regard (2.7) as an identity in the independent differentials $d\alpha_h$ we obtain the following relations

$$\lambda_0 g_h + [(F - \bar{y}_i' F_{y_i'})\,x_h^s + F_{y_i'}\,y_{ih}^s]_1^2 = 0 \tag{2.9}$$
$$(h = 1, \cdots, r;\ i = 1, \cdots, n)$$

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* See Morse and Myers, § 4, *loc. cit.*

† The summation convention of tensor analysis is to be used throughout.

where the subscript h indicates differentiation with respect to α_h.

3. *The second variation.* It is convenient to set

$$(3.1) \qquad P_{ij}(x) \equiv F_{y_i y_j} \qquad Q_{ij} \equiv F_{y_i y_j'} \qquad R_{ij} \equiv F_{y_i' y_j'}$$

where the partial derivatives of F are to be evaluated along g. As is con ventional we then set

$$(3.2) \qquad 2\omega(\eta, \eta') = P_{ij}\eta_i\eta_j + 2Q_{ij}\eta_i\eta_j' + R_{ij}\eta_i'\eta_j'.$$

A set (α) in (2.4) determines a set of end points. Let us take a set of points (α) of the form

$$(3.3) \qquad \alpha_h = \alpha_h(e) \qquad \alpha_h(0) = 0 \qquad (h = 1, \cdots, r)$$

where the functions $\alpha_h(e)$ are of class C'' for (e) near (0). Suppose that we have a family of admissible curves

$$(3.4) \qquad y_i = y_i(x, e) \qquad (i = 1, \cdots, n)$$

taking on the end points determined by (α) in (3.3). That is, we suppose that we have, subject to (3.3), the following identities in e

$$(3.5) \qquad y_i[x^s(\alpha), e] \equiv y_i^s(\alpha). \qquad (s = 1, 2;\ i = 1, \cdots, n).$$

We suppose that for $e = 0$, (3.4) gives the extremal g.

We shall need * the results of differentiating these identities with respect to e. Upon setting $\alpha_h' = u_h$ a first differentiation gives, $(h = 1, \cdots, r)$,

$$(3.6) \qquad y_{ie} = y_{ih}^s u_h - y_{ix} x_h^s u_h \qquad \alpha_h' = u_h.$$

A second differentiation and evaluation for $e = 0$ yields the result, (s not summed), $(h, k = 1, \cdots, r)$:

$$(3.7) \qquad y_{iee} + 2y_{iex}x_h^s u_h = (y^s_{ihk} - \bar{y}_i'' x_h^s x_k^s - \bar{y}_i' x_{hk}^s) u_h u_k + (y_{ih}^s - \bar{y}_i' x_h^s) u_h$$

We now suppose J evaluated along the curves (3.4) between the end points determined by (3.3). We have †

$$(3.8) \qquad \lambda_0 dJ/de \equiv \lambda_0 g_h u_h + (F x_h^s) \ \ u_h + \int_{x^1}^{x^2} (F_{y_i} y_{ie} + F_{y_i'} y_{ixe})\, dx.$$

* What we need for the general theory is to know that the second variation takes the form (3.9). For the general theory the reader need not follow through the algebra except to that end.

† The symbol $(\ )_1^2$ which is used in this place alone, means evaluation at the two ends of the curve $y_i = y_i(x, e)$.

We shall differentiate (3.8) with respect to e and set $e=0$. From differentiation of the integral in (3.8) and appropriate integration by parts we obtain terms of the form

$$A = [(F_{y_i}y_{ie} + F_{y_i'}y_{iex})x_h^s]_1^2 u_h + [y_{iee}F_{y_i'}]_1^2$$

together with the integral

$$B = 2\int_{a^1}^{a^2} \omega(\eta,\eta')\,dx \qquad\qquad \eta_i = y_{ie}(x,0).$$

Differentiation of the remaining terms in the right member of (3.8) gives terms of the form

$$\begin{aligned}C = \lambda_0 g_{hk}u_h u_k &+ [(F_x + \bar{y}_i'F_{y_i} + \bar{y}_i''F_{y_i'})x_h^s x_k^s + Fx_{hk}^s]_1^2 u_h u_k \\ &+ [(F_{y_i}y_{ie} + F_{y_i'}y_{ixe})x_h^s]_1^2 u_h + (\lambda_0 g_h + [Fx_h^s]_1^2)u_h'.\end{aligned}$$

We wish to reduce the sum $A + C$ to a quadratic form in (u). We first replace y_{ie} in A and C by the right member of (3.6). We thereby obtain additional quadratic terms of the form

$$D = 2[F_{y_i}(x_h^s y_{ik}^s - \bar{y}_i' x_h^s x_k^s)]_1^2 u_h u_k.$$

The terms remaining in $A + C$ which are not quadratic in (u) have the form

$$[F_{y_i'}(y_{iee} + 2y_{iex}x_h^s u_h)]_1^2 + \lambda_0 g_h u_h' + [Fx_h^s]_1^2 u_h'$$

which terms with the aid of (3.7) and the transversality relations reduce to the form

$$E = [F_{y_i'}(y^s_{ihk} - \bar{y}_i'' x_h^s x_k^s - \bar{y}_i' x_{hk}^s)]_1^2 u_h u_k.$$

Thus we have for $e=0$, setting $\alpha_h' = u_h$,

$$\lambda_0 d^2J/de^2 = b_{hk}u_h u_k + 2\int_{a^1}^{a^2}\omega(\eta,\eta')\,dx \qquad\qquad b_{hk} = b_{kh} \tag{3.9}$$

where $2b_{hk}$ is the sum of the coefficients of $u_h u_k$ and $u_k u_h$ in C, D, and E. We thereby find that

$$\begin{aligned}b_{hk} = \lambda_0 g_{hk} &+ [(F_x - \bar{y}_i'F_{y_i})x_h^s x_k^s \\ &+ (F - \bar{y}_i'F_{y_i'})x_{hk}^s + F_{y_i}(x_h^s y_{ik}^s + x_k^s y_{ih}^s) + F_{y_i'}y^s_{ihk}]_1^2.\end{aligned} \tag{3.10}$$

For $e=0$ let us set

$$(\partial/\partial e)x^s(\alpha(e)) = \gamma^s$$

and indicate evaluation of $\eta_i(x)$ at the respective ends of g by the superscript s. We shall prove the following lemma.

LEMMA. *The quadratic form*

$$b_{hk}u_h u_k \qquad (h, k = 1, \cdots, r)$$

is identically equal to a quadratic form in a suitable subset of r of the $2n+2$ variables γ^s, η_i^s.

Let us set (s not summed)

$$y_{ih}^s(0) - x_h^s(0)\bar{y}_i'(a^s) = c_{ih}^s \qquad (i = 1, \cdots, n;\ s = 1, 2). \tag{3.11}$$

From (3.6) we obtain then a fundamental relation

$$\eta_i^s = c_{ih}^s u_h. \tag{3.12}$$

A parallel relation is obtained from (2.4), namely

$$\gamma^s = x_h^s u_h \qquad (\alpha) = (0). \tag{3.13}$$

We need to consider the two matrices

$$\left\| \begin{matrix} c_{ih}^s \\ x_h^s \end{matrix} \right\| = \boldsymbol{p} \qquad \left\| \begin{matrix} y_{ih}^s \\ x_h^s \end{matrix} \right\| = \boldsymbol{q} \tag{3.14}$$

of which the first is the matrix of the coefficients of the variables (u) in the system (3.12) and (3.13), and the second is the functional matrix of the system (2.4), evaluated for $(\alpha) = (0)$.

By hypothesis $\boldsymbol{q}$ is of rank r. It follows that $\boldsymbol{p}$ is of rank r. For $\boldsymbol{p}$ can be readily obtained from $\boldsymbol{q}$ by adding suitable multiples of its last two rows to the preceding rows.

Hence the variables u_h may be expressed as a linear combination of r of the variables γ^s, η_i^s and the lemma follows directly.

4. *The accessory boundary problem.* From this point on we suppose that the extremal g is *normal* relative to the Euler equations and transversality conditions. See Morse and Myers, § 5. The constant λ_0 is not then zero. It can be taken as unity and this choice we suppose made.

A set of functions $\eta_i(x)$ of class C' which satisfy the differential conditions

$$\Phi_\beta(\eta) \equiv \phi_{\beta y_i}\eta_i + \phi_{\beta y'_i}\eta_i' = 0 \qquad (\beta = 1, \cdots, m;\ i = 1, \cdots, n) \tag{4.1}$$

and which with a set of constants (u) satisfy the terminal conditions

$$\eta_i^s = c_{ih}^s u_h \tag{4.2}$$

will be called *admissible*.

*The form of the second variation suggests** *the accessory problem of finding functions* $\eta_i(x)$ *and constants* (u) *which give a minimum to*

$$I(\eta, u, \sigma) = b_{hk}u_hu_k + 2\int_{x^1}^{x^2}[\omega(\eta, \eta') - \sigma\eta_i\eta_i]\,dx$$

for a given σ, *relative to all admissible sets* (η) *and constants* (u).

Suppose we have a solution of the accessory minimum problem for a given σ, in the form of n functions $\eta_i(x)$ of class C', and a set of constants (u). It follows from the fact that g is normal as stated that the solution (η) is normal in the accessory minimum problem relative to the new Euler and transversality conditions. For we note that for $\lambda_0 = 0$ the Euler and transversality conditions for the two problems are the same. In particular, upon noting that $x^s = a^s$ in the accessory problem, we find that the transversality conditions (2. 9) for both problems, if $\lambda_0 = 0$, may be given the form

$$[\lambda_\beta\phi_{\beta y_i'}c_{ih}^s]_1^2 = 0 \qquad (h = 1, \cdots, r).$$

There must then exist m multipliers $\mu_\beta(x)$ of class C' which with the functions $\eta_i(x)$ satisfy the differential equations

$$(4.3) \quad (d/dx)\Omega_{\eta_i'} - \Omega_{\eta_i} = 0, \quad \Phi_\beta = 0 \quad (i = 1, \cdots, n;\ \beta = 1, \cdots, m)$$

where

$$(4.4) \qquad \Omega(\eta, \eta', \mu, \sigma) = \omega(\eta, \eta') + \mu_\beta\Phi_\beta - \sigma\eta_i\eta_i.$$

The functions (η), (μ) and the constants (u) and σ must also satisfy the transversality conditions,

$$(4.5) \qquad b_{hk}u_k + [\Omega_{\eta_i'}c_{ih}^s]_1^2 = 0 \qquad (h, k = 1, \cdots, r)$$

and the given boundary conditions

$$(4.6) \qquad \eta_i^s = c_{ih}^su_h.$$

The differential equations (4. 3) *and the boundary conditions* (4. 5) *and* (4. 6) *define what will be called the accessory boundary problem.*

By a *solution* of the accessory boundary problem is meant a set of functions $\eta_i(x)$ of class C' which with multipliers $\mu_\beta(x)$ of class C' and constants (u) and σ satisfy the conditions of the problem. We have the theorem

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* The form of the integrand is suggested by the idea of dominating the sign of the second variation by new terms. It also occurs if one seeks to minimize the second variation subject to a condition that the integral of $\eta_i\eta_i$ over the given interval be unity. See Cope, *loc. cit.*

THEOREM 1. *A solution (η), (u) of our problem of minimizing $I(\eta, u, \sigma)$ for a fixed σ must possess multipliers (μ) which with (η), (u), and σ, give a solution of the accessory boundary problem.*

By a *characteristic solution* * of the accessory boundary problem will be meant a solution for which $(\eta) \not\equiv (0)$. The corresponding value of σ will be called a *characteristic root.*

5. *The necessary condition on the characteristic roots.* We shall prove the following lemma.

LEMMA. *If (η) is a characteristic solution with constants (u) and σ, $I(\eta, u, \sigma) = 0$.*

Since (η) satisfies $\Phi_\beta = 0$, we have

$$I(\eta, u, \sigma) = b_{hk} u_h u_k + 2 \int_{a^1}^{a^2} \Omega(\eta, \eta', \mu, \sigma)\, dx. \tag{5.1}$$

If we make use of the homogeneity of Ω and integrate by parts in the usual way, we find as a consequence of (4.3) that

$$I(\eta, u, \sigma) = b_{hk} u_h u_k + [\Omega_{\eta_i'} \eta_i{}^s]_1^2. \tag{5.2}$$

If finally we make use of (4.5) we see that $I(\eta, u, \sigma) = 0$ as was to be proved.

We now come to the following theorem.

THEOREM 2. *If a normal extremal g furnish a minimum for the given problem it is necessary that there exist no characteristic solution of the accessory boundary problem for which $\sigma < 0$.*

Suppose σ_1 were a negative characteristic root and (η) the corresponding characteristic solution with its constants (u). We have

$$I(\eta, u, 0) = I(\eta, u, \sigma_1) + 2 \int_{a^1}^{a^2} \sigma_1 \eta_i \eta_i \, dx. \tag{5.3}$$

By the preceding lemma $I(\eta, u, \sigma_1) = 0$ and since $(\eta) \not\equiv (0)$ we have from (5.3) that

$$I(\eta, u, 0) < 0.$$

We now seek an admissible family of curves of which $I(\eta, u, 0)$ is the second variation.

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* We shall presently assume that every segment of g is normal relative to the Euler conditions. Under this hypothesis the only solutions $(\eta, \mu) \not\equiv (0, 0)$ will be the solutions for which $(\eta) \not\equiv (0)$.

For the given set (u) let γ^s be defined by (3.13). The sets γ^s, (η), (u) then satisfy (3.12), (3.13) and (4.1). In the normal case * it is known that there will then exist an admissible family of curves of the form $y_i = y_i(x, e)$ and functions $\alpha_h(e)$, such as (3.3) and (3.4), such that γ^s, (η) and (u) are the corresponding variations for this family. But this is impossible if g furnish a minimum since $I(\eta, u, 0) < 0$.

Thus the theorem is proved.

6. *The non-tangency hypothesis.* In ordinary problems involving transversality of a manifold to a given extremal it is generally customary to assume that the manifold is not tangent † to the given extremal, or to insure this by other assumptions. There is here a corresponding assumption apparently hitherto unnoticed. The problem could be treated without this hypothesis, and a summary of results in such a case will be published separately. But sufficient conditions in the case where the hypothesis is made are much simpler than in the case where it is not made. Moreover simple examples in the plane will show the undesirable complexity and unimportance of the special case.

In both cases one is led to an accessory problem involving a parameter σ, but in the more general case the parameter need be introduced only into the integral and not into the end conditions.

In the space of the $2n + 2$ variables (x^s, y_i^s) consider the 2-dimensional manifold defined by the equations, (see 2.1),

$$y_i^s = \bar{y}_i(x^s) \qquad (s = 1, 2;\ i = 1, \cdots, n) \tag{6.1}$$

for x^s neighboring a^s. This manifold is essentially the arbitrary combination of a point of g near the final end of g with a point of g near the initial end of g. We now regard the equations (2.4) as defining another manifold

$$y_i^s = y_i^s(\alpha) \qquad x^s = x^s(\alpha) \tag{6.2}$$

in the same $(2n + 2)$-dimensional space. We term (6.1) the *extremal manifold* and (6.2) the *terminal manifold.*

Our non-tangency condition is simply that the extremal manifold and the terminal manifold possess no common tangent line at the point $(\alpha) = (0)$ *on the terminal manifold.*

* See Morse and Myers, § 5, Theorem 5.

† In the parametric form this assumption with the assumption that the Weierstrass function $F_1 \neq 0$, enables one to prove the existence of a field of extremals cutting the given manifold transversally.

We state the following lemma.

LEMMA A. *A necessary and sufficient condition for the non-tangency condition to hold is that the matrix* $\| c_{ih}{}^s \|$ *of* (3.11) *be of rank* r.

If one writes down as columns the direction numbers of the tangents to the parametric curves of the two manifolds concerned, one verifies the lemma readily. In particular a set of such direction numbers for the terminal manifold are given by the r columns of the matrix

$$\left\| \begin{matrix} x_h{}^s \\ y_{ih}{}^s \end{matrix} \right\| \qquad \begin{matrix} (i=1,\cdots,n). \\ (h=1,\cdots,r) \end{matrix}$$

This matrix is of rank r by hypothesis (§ 2). The corresponding matrix for the extremal manifold consists of two columns

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ \bar{y}_i' & 0 \\ 0 & \bar{y}_i' \end{matrix} \right\| \qquad (i=1,\cdots,n).$$

The non-tangency condition means that there is no linear relation between the columns of the two matrices which actually involves both matrices. That there is such a relation if and only if $\| c_{ih}{}^s \|$ is of rank less than r will be left to the reader to prove.

We shall point out some of the advantages of assuming the non-tangency condition which we shall assume from now on.

The variations (η) and constants (u) appearing in the second variation are related as follows:

$$\eta_i{}^s = c_{ih}{}^s u_h \tag{6.3}$$

as we have seen. In the case of non-tangency we can solve (6.3) for (u) in terms of a suitable subset of r of the variations $\eta_i{}^s$. Instead of the lemma of § 3 we then have the following.

LEMMA B. *In the case of non-tangency the second variation may be written in the form*

$$\lambda_0 d^2J/de^2 = q(\eta) + 2\int_{a^1}^{a^2} \omega(\eta, \eta')\, dx$$

where $q(\eta)$ *is a quadratic form in* r *of the variations* $\eta_i{}^s$.

7. *A canonical form for the accessory boundary problem.* The accessory

differential equations can be thrown into the Hamiltonian form, and in the case of non-tangency the boundary conditions considerably condensed.

We set

$$\zeta_i = \Omega_{\eta_i'}(\eta, \eta', \mu, \sigma) \qquad \Phi_\beta(\eta, \eta') = 0. \tag{7.1}$$

The relations (7. 1) can be solved for η_j' and μ_β in terms of ζ_i and η_i, provided we make the usual assumption that along g

$$\begin{vmatrix} R_{11} & \cdots & R_{1n} & \phi_{1y_1'} & \cdots & \phi_{my_1'} \\ \vdots & & \vdots & \vdots & & \vdots \\ R_{n1} & \cdots & R_{nn} & \phi_{1y_n'} & \cdots & \phi_{my_n'} \\ \phi_{1y_1'} & \cdots & \phi_{1y_n'} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_{my_1'} & \cdots & \phi_{my_n'} & 0 & \cdots & 0 \end{vmatrix} \neq 0. \tag{7.2}$$

With η_i' and μ_β taken as such functions of ζ_i and η_i, we set

$$H(x, \eta, \zeta, \sigma) = \zeta_i \eta_i' - \Omega(\eta, \eta', \mu, \sigma). \tag{7.3}$$

As is well known and easily verified the accessory differential equations take the form *

$$d\eta_i/dx = H_{\zeta_i}; \qquad d\zeta_i/dx = -H_{\eta_i}. \tag{7.4}$$

The boundary conditions (4. 5) and (4. 6) take the respective forms:

$$\zeta_i^1 c_{ih}^1 - \zeta_i^2 c_{ih}^2 = b_{hk} u_k \qquad (h, k = 1, \cdots, r) \tag{7.5}$$

$$\eta_i^s = c_{ih}^s u_h \qquad (s = 1, 2;\ i = 1, \cdots, n). \tag{7.6}$$

In the case of non-tangency these boundary conditions are equivalent to exactly $2n$ linearly independent conditions $L_p(\eta, \zeta) = 0$ on the variations η_i^s, ζ_i^s $(p = 1, \cdots, 2n)$.

For in the case of non-tangency the matrix $c = \| c_{ih}^s \|$ is of rank r, so that (7. 6) gives $2n - r$ independent conditions on the variables η_i^s. We can replace the variables (u) in (7. 5) by linear combinations of the variables η_i^s obtained from (7. 6). The resulting relations will be independent of each other because the matrix of the variables ζ_i^s in (7. 5) is obtained from c by interchanging rows and columns and then changing the sign of the last n columns. We thus have $2n$ independent relations as desired.

* See Bolza, p. 593, *loc. cit.*

The accessory boundary value problem thus takes the compact form

$$(7.7) \qquad d\eta_i/dx = H_{\zeta_i}; \qquad d\zeta_i/dx = -H_{\eta_i} \qquad (i = 1, \cdots, n)$$

$$(7.8) \qquad L_p(\eta, \zeta) = 0 \qquad (p = 1, \cdots, 2n).$$

SUFFICIENT CONDITIONS.

8. *The theorem.* By the *Clebsch sufficient condition* we shall hereafter mean the condition that

$$(8.1) \qquad R_{ij} w_i w_j > 0 \qquad (i, j = 1, \cdots, n)$$

for any set $(w) \neq (0)$ satisfying

$$(8.2) \qquad \phi_{\beta y_i'} w_i = 0 \qquad (\beta = 1, \cdots, m)$$

where the partial derivatives involved are evaluated along g.

By the *Weierstrass sufficient condition* we shall mean the condition

$$(8.3) \qquad E(x, y, \bar{y}', y', \lambda) > 0$$

for distinct admissible sets $(x, y, \bar{y}')$ and (x, y, y') for which $(x, y, \bar{y}', \lambda)$ is near the corresponding set on g.

The extremal g will be said to be *identically normal* if every subinterval of it is normal relative to the Euler conditions.

In § 12 we shall prove the following theorem.

THEOREM 3. *In order that an identically normal extremal g afford a proper, strong, relative minimum, it is sufficient that the Weierstrass and Clebsch sufficient conditions and the non-tangency condition hold, and that all characteristic roots be positive.*

9. *The fundamental quadratic form.* We are now assuming that g is identically normal and the Clebsch sufficient condition holds. On such an extremal (Morse III, § 4) there exists a positive lower bound of the distances between a point and its nearest conjugate point. With this understood let

$$(9.1) \qquad x = a_0, \cdots, x = a_{p+1} \qquad a_0 = a^1 \qquad a_{p+1} = a^2$$

be a set of successive n-planes cutting the x-axis in the order of their subscripts, and placed so near together that no point and its first following conjugate point lie on a closed segment of g cut out by two successive n-planes.

On the intermediate n-planes

$$(9.2) \qquad x = a_1, \cdots, x = a_p$$

respectively let

$$(9.3) \qquad P_1, \cdots, P_p$$

be a set of points neighboring g. Let (v) be a set of $pn + r = \delta$ variables of which the first r equal $(\alpha_1, \cdots, \alpha_r)$. The next n shall be the respective differences between the y coördinates of P_1 and of the point on g at $x = a_1$, the next n the differences between the y coördinates of P_2 and of the point on g at $x = a_2$, the next n a similar set for P_3 and so on to P_p.

A set (α) in (2.4) determines two end points

$$(9.4) \qquad (x^1, y^1) = P_0 \qquad (x^2, y^2) = P_{p+1}.$$

The complete set (v) determines the points

$$(9.5) \qquad P_0, \cdots, P_{p+1}.$$

If the points (9.5) be sufficiently near g they can be successively joined by extremal segments. Denote the resulting broken extremal by E. We shall say that (v) *determines* the above broken extremal E. The expression J taken along E will be denoted by $J(v)$. With the aid of the Euler equations one sees that the first partial derivatives of $J(v)$ with respect to $v_{r+1}, \cdots, v_\delta$ are all zero for $(v) = (0)$, and with the additional aid of the transversality conditions one sees that the remaining first partial derivatives are zero for $(v) = (0)$. Thus $J(v)$ has a critical point when $(v) = (0)$.

The terms of the second order of $J(v)$ now come to the fore. They will be obtained by means of the following identity in the variables $(z) = (z_1, \cdots, z_\delta)$

$$(9.6) \qquad J^0{}_{v_p v_q} z_p z_q = (d^2/de^2) J(ez_1, \cdots, ez_\delta) \qquad (p, q = 1, \cdots, \delta;\ e = 0)$$

where e is to be set equal to zero after the differentiation, and the partial derivatives on the left are to be evaluated for $(v) = (0)$, as indicated by the superscript 0.

Consider then the family of broken extremals E through the points (9.5) determined by a set $(v) = (ez_1, \cdots, ez_\delta)$ for a fixed (z) and variable e. Represent this family in the form

$$y_i = y_i(x, e) \qquad \alpha_h = ez_h \qquad (h = 1, \cdots, r;\ i = 1, \cdots, n).$$

Although the functions $y_i(x, e)$ fail in general to be of class C' at the corners of the broken extremals, one verifies the fact that (3.8) and (3.9) still hold. The derivation of [illegible] must be varied slightly in that we have here

$$(9.7) \qquad \int_{a_1}^{a_2} (F_{y_i} y_{iee} + F_{y_i'} y_{ixee})\, dx = [y_{iee} F_{y_i'}]_1^2 + \sum_{t=1}^{p} [y_{iee} F_{y_i'}]_{a_t+}^{a_t-}$$

while the last sum would not appear if there were no corners.

But even in the case of corners the last sum vanishes. For since the broken extremal is *determined* by (ez) we see, for example, for $x = a_1$, that

$$
\begin{array}{c}
y_1(a_1, e) - y_1(a_1, 0) = ez_{r+1} \\
\cdots\cdots\cdots\cdots \\
y_n(a_1, e) - y_n(a_1, 0) = ez_{r+n}
\end{array}
\tag{9.8}
$$

so that $y_{iee}(a_1, e) = 0$. Similarly

$$y_{iee}(a_t, e) = 0 \qquad (t = 1, \cdots, p)$$

from which it appears that the last sum in (9.7) vanishes just as in the absence of corners.

Now u_h was defined as $\alpha_h'(0)$. For the present family $u_h = z_h$ so that we can infer from (9.6) that

$$J^0{}_{v_p v_q} z_p z_q = b_{hk} z_h z_k + 2 \int_{a^1}^{a^2} \omega(\eta, \eta')\, dx \qquad \begin{array}{r}(p, q = 1, \cdots, \delta) \\ (h, k = 1, \cdots, r).\end{array} \tag{9.9}$$

A curve $\eta_i = \eta_i(x)$ in the (x, η) space will be called a *secondary extremal* provided it forms the part (η) of a set (η, μ) of functions of x of class C' which satisfy (4.3). From its origin in (9.9) we see that (η) in (9.9) defines a broken secondary extremal E'. To determine its ends and corners we note the following.

Since $u_h = z_h$, (3.12) becomes

$$\eta_i{}^s = c_{ih}{}^s z_h \qquad (i = 1, \cdots, n;\ s = 1, 2;\ h = 1, \cdots, r) \tag{9.10}$$

so that $(z_1, \cdots, z_r)$ determines the end points of E'. From (9.8) we see that

$$[\eta_1(a_1), \cdots, \eta_n(a_1)] = [z_{r+1}, \cdots, z_{r+n}].$$

Similarly we see that the set $\eta_i(a_2)$ equals the next set of n z's, and so on to the set $\eta_i(a_p)$ which equals the last n z's. Thus the set (z) determines both the end points and the corner points of the broken secondary extremal E'. The set (z) thus uniquely *determines* the broken secondary extremal E'.

We summarize in the following theorem.

THEOREM 4. *If $J(v)$ represents the value of J taken along the broken extremal determined by (v) its terms of second order have the form (9.9) where (η) is taken along the broken secondary extremal determined by (z).*

10. *Further introduction of σ.* The previous introduction of the para-

meter σ into the second variation will automatically result if we replace the integrand f by a one-parameter family of integrands

$$f - \sigma[y_i - \bar{y}_i(x)][y_i - \bar{y}_i(x)] \tag{10.1}$$

where $y_i = \bar{y}_i(x)$ as before represents the extremal g. For each value of σ, g will still be an extremal. To construct the broken extremal of the preceding section for this new problem we need the following lemma.

LEMMA. *A decrease of σ never causes a decrease of the distance on the x-axis between two successive conjugate points.*

First recall that we are assuming the Clebsch sufficient condition holds along g, and note that the Clebsch condition is independent of σ. Using Taylor's formula one sees that the Weierstrass sufficient condition is satisfied for the problem of minimizing $I(\eta, u, \sigma)$ subject to (4.1) along any secondary extremal. Moreover, g is assumed identically normal. It follows that any segment d of the x-axis free from successive conjugate points for $\sigma = \sigma_0$ will afford a proper minimum to $I(\eta, 0, \sigma_0)$ in the fixed end point problem. See Morse III.

From the way σ enters into the second variation it follows next that after σ is decreased from σ_0, d will still afford a proper minimum to $I(\eta, 0, \sigma)$.

If, however, there were for the decreased σ a pair of conjugate points on d, it would be possible to make $I(\eta, 0, \sigma)$ zero by taking it along the secondary extremal joining the two conjugate points, and along the x-axis for the rest of d. This is contrary to the nature of a proper minimum. Thus there can be no pairs of conjugate points on d.

The lemma follows readily.

By virtue of this lemma the broken extremal determined by (v) in § 9 for $\sigma = 0$ will also be similarly determined by (v) for each negative σ.

The value of J taken along the broken extremal determined by (v) will now be denoted by $J(v, \sigma)$, $\sigma \leqq 0$. We set

$$Q(z, \sigma) = J_{v_p v_q}(0, \sigma) z_p z_q \qquad (p, q = 1, \cdots, \delta). \tag{10.2}$$

By virtue of Theorem 4 extended to any negative σ we have

$$Q(z, \sigma) = [illegible] \, 2 \int_{a^1}^{a^2} \Omega([illegible] \, \sigma) \, dx \qquad ([illegible] = 1, \cdots, [illegible]) \tag{10.3}$$

where (η) is taken along the broken secondary extremal determined by (z) for the given σ.

11. *Properties of the form $Q(z, \sigma)$.* THEOREM 5. *The quadratic form $Q(z, \sigma)$ is singular if and only if σ is a characteristic root in the accessory boundary value problem.*

The conditions that the form $Q(z, \sigma)$ be singular are that the linear equations

$$Q_{z_p} = 0 \qquad (p = 1, \cdots, \delta) \tag{11.1}$$

have at least one solution $(z) \neq (0)$.

If such a solution (z) be given we shall first show that the broken secondary extremal E determined by (z), with the set (u) equal to the first r of the z's, gives a characteristic solution.

Let us first examine the geometric meaning of the conditions (11.1) for $p = r + 1, \cdots, r + n$. From (11.1) and (10.3) we see that

$$Q_{z_{r+i}} = 2[\Omega_{\eta_i'}]_{a_1^+}^{a_1^-} = 0 \qquad (i = 1, \cdots, n). \tag{11.2}$$

Equations (11.2) taken with (7.2) show that E has no corner at $x = a_1$. More generally we see that the conditions (11.1) with $p > r$ imply the absence of corners at each of the points of E at which

$$x = a_1, \cdots, x = a_p,$$

that is, the complete absence of corners on E.

There remain the conditions (11.1) for which $p \leqq r$. From (10.3) and (11.1) we see that

$$Q_{z_h} = 2b_{hk}z_k + 2[\Omega_{\eta_i'}\,\partial\eta_i^s/\partial z_h]_1^2 = 0 \qquad (h, k = 1, \cdots, r). \tag{11.3}$$

With the aid of the relations (9.10), namely,

$$\eta_i^s = c_{ih}^s z_h \qquad (h = 1, \cdots, r) \tag{11.4}$$

we see that (11.3) takes the form

$$[\Omega_{\eta_i'}\,c_{ih}^s]_2^1 = b_{hk}z_k. \tag{11.5}$$

But (11.4) and (11.5) regarded as boundary conditions on (η) have precisely the form of the boundary conditions (4.5) and (4.6) of the accessory boundary problem.

Finally we see that on E, $(\eta) \not\equiv (0)$ since $(z) \neq (0)$. If then Q is singular we have a characteristic solution (η) of the accessory boundary problem.

Conversely, let there be given for some σ a characteristic solution (η) with its constants (u). Let (z) be a set which determines the secondary extremal (η). In (z) the first r variables will be the set (u).

Conditions (11. 5) and (11. 4) are satisfied since (η) is a characteristic solution. Conditions (11. 3) then follow. All conditions such as (11. 2) are satisfied because of the absence of corners on the secondary extremal (η). Hence all conditions (11. 1) are satisfied. Moreover, $(z) \neq (0)$ since $(\eta) \neq (0)$.

Thus $Q(z, \sigma)$ is singular when σ is a characteristic root.

The theorem is thereby proved.

The number of linearly independent sets (η) which furnish characteristic solutions corresponding to a given value of σ will be called the index of σ.

For a given $\sigma < 0$ it is clear that linearly independent secondary extremals will determine and be determined by linearly independent sets (z). Since the nullity of the form Q is the number of linearly independent solutions (z) of the equations (11. 1) we have the following theorem.

THEOREM 6. *The nullity of the form $Q(z, \sigma)$ equals the index of the root σ.*

Let $M(\eta, \nu)$ be any form quadratic in η_i, ν_i with coefficients continuous in x, and with no terms quadratic in (ν) alone. We shall prove the following lemma involving Ω and $M(\eta, \nu)$.

LEMMA A. *For σ sufficiently large and negative the form*

$$N = 2\Omega(\eta, \nu, \mu, \sigma) + M(\eta, \nu) \tag{11. 6}$$

is positive definite in its variables (η, ν, μ), subject to the conditions

$$\phi_{\beta y_i}\eta_i + \phi_{\beta y_i'}\nu_i = 0 \qquad (i = 1, \cdots, n;\ \beta = 1, \cdots, m). \tag{11. 7}$$

Subject to (11. 7) the form N may be written as follows:

$$N = R_{ij}\nu_i\nu_j + 2Q_{ij}\eta_i\nu_j + P_{ij}\eta_i\eta_j + M(\eta, \nu) - 2\sigma\eta_i\eta_i. \tag{11. 8}$$

If we set

$$-\sigma = 1/\rho^2 \qquad \eta_i = \rho\omega_i \qquad \rho \neq 0 \tag{11. 9}$$

we have in place of (11. 8)

$$N = R_{ij}\nu_i\nu_j + \rho^2 q_{ij}\omega_i\omega_j + \rho a_{ij}\omega_i\nu_j + 2\omega_i\omega_i \tag{11. 10}$$

[illegible] conditions (11. 7) become

$$\rho\phi_{\beta y_i}\omega_i + \phi_{\beta y_i'}\nu_i = 0. \tag{11. 11}$$

For $\rho = 0$ the form (11.10), taken subject to (11.11), becomes the form

$$N = R_{ij}\nu_i\nu_j + 2\omega_i\omega_i \qquad \phi_{\beta y_i'}\nu_i = 0,$$

and is positive definite by virtue of the Clebsch condition. It follows that for ρ sufficiently small the form (11.10) is positive definite * subject to (11.11), and hence for σ sufficiently large and negative, (11.8) is positive definite subject to (11.7).

Thus the lemma is proved.

We come to a fundamental theorem.

THEOREM 7. *If an extremal g is identically normal, while the Clebsch sufficient condition and the non-tangency condition hold, then the form $Q(z, \sigma)$ will be positive definite for sufficiently large negative values of σ.*

According to Lemma B of § 6, in the case of non-tangency we have

$$Q(z, \sigma) = q(\eta) + 2\int_{a^1}^{a^2} \Omega(\eta, \eta', \mu, \sigma)\, dx \tag{11.12}$$

where $q(\eta)$ is a form quadratic in the variables $\eta_i{}^s$.

Now any such form as q will satisfy a relation

$$q(\eta) \geqq -c[\eta_i{}^2\eta_i{}^2 + \eta_i{}^1\eta_i{}^1] \qquad (i = 1, \cdots, n) \tag{11.13}$$

provided simply that c be a positive constant sufficiently large.

Let $h(x)$ be any function of x of class C' on the closed interval (a^1, a^2), taking on the values 1 and -1 respectively at $x = a^2$ and $x = a^1$. Then (11.13) may also be written in the form

$$q(\eta) \geqq -c\int_{a^1}^{a^2} (d/dx)\,(\eta_i\eta_i h(x))]\, dx \tag{11.14}$$

where (η) represents any set of functions $\eta_i(x)$ of class D' taking on the end values $\eta_i{}^s$.

From (11.12) and (11.14) we see that

$$Q(z, \sigma) \geqq \int_{a^1}^{a^2} [2\Omega(\eta, \eta', \mu, \sigma) - c(d/dx)\,(\eta_i\eta_i h(x))\, dx \tag{11.15}$$

* This statement can be proven as follows. By virtue of (11.11), for a fixed x, m of the variables (ν) can be eliminated and N reduced to a form L in the remaining variables with coefficients continuous in ρ. The form L is now subject to no auxiliary conditions. For $\rho = 0$ L is positive definite, and according to the ordinary theory of quadratic forms will still be positive definite for ρ sufficiently small.

By virtue of Lemma A of this section the integrand in (11.15) will be positive definite in (η, η', μ), subject to (4.1), provided only that σ be sufficiently large and negative. For such a σ, for $(z) \neq (0)$ and hence $(\eta) \neq (0)$, we have then

$$Q(z, \sigma) > 0.$$

Thus the theorem is proved.

12. *Sufficient condition for a minimum. We shall now prove Theorem* 3.

By virtue of Theorem 5, $Q(z, \sigma)$ is non-singular for $\sigma \leqq 0$, since all characteristic roots are positive. According to Theorem 7, $Q(z, \sigma)$ is positive definite for σ sufficiently large and negative. If we increase σ from large negative values to 0, Q will still remain positive definite since it remains non-singular. Hence under the hypothesis of Theorem 3

$$Q(z, 0) > 0 \qquad (z) \neq (0).$$

Since the extremal g is identically normal and the Clebsch and Weierstrass conditions hold, a neighborhood N of g exists so small that if the end points and corners of a broken extremal E determined by (z) lie in N, each of E's component segments will afford a minimum to J in N in the fixed end point problem with the given differential conditions.

Let g' now be an admissible curve lying in N. The end points of g' and its intersections with the n-planes

(12.1) $$x = a_1, \cdots, x = a_p$$

of § 9 will determine the set (v) of § 9. For our choice of the neighborhood N we have

(12.2) $$J_{g'} \geqq J(v).$$

On the other hand $Q(v, 0)$ gives the terms of second order in $J(v)$, so that for (v) sufficiently near $(v) = (0)$

(12.3) $$J(v) \geqq J(0)$$

where the equality holds only for $(v) = (0)$.

Thus if g' lies in a sufficiently small neighborhood of g

(12.4) $$J_{g'} \geqq J(0).$$

Now (12.4) becomes an equality only if both (12.2) and (12.3) become equalities. But (12.3) becomes an equality only if $(v) = (0)$, and (12.2)

then becomes an equality only if g' is identical with g. Hence the equality in (12.4) holds only if g' is identical with g.

Thus the minimum is proper and the theorem is proved.

If we recall that the Clebsch sufficient condition entails the Weierstrass sufficient condition in its weak form, that is for sets (x, y, y') neighboring those on g, we have the following corollary.

COROLLARY 1. *For a weak, relative minimum it is sufficient that g be identically normal, that the Clebsch* and non-tangency conditions hold, and that all characteristic roots be positive.*

It is now easy to obtain certain theorems about the sign of the second variation

$$I(\eta, u, 0) = b_{hk}u_h u_k + 2\int_{a^1}^{a^2} \omega(\eta, \eta')\,dx \qquad (h, k = 1, \cdots, r) \tag{12.5}$$

taken subject to the conditions

$$\Phi_\beta = 0 \qquad \eta_i{}^s = c_{ih}{}^s u_h \qquad (\beta = 1, \cdots, m;\ i = 1, \cdots, n). \tag{12.6}$$

In the first place we see from (10.3) that along a broken secondary extremal determined by (z) for $\sigma = 0$

$$I(\eta, u, 0) = Q(z, 0). \tag{12.7}$$

In the second place we recall that the Clebsch sufficient condition for g in the original problem entails the Weierstrass sufficient condition

$$E(x, \eta, \bar{\eta}', \eta', \mu) > 0 \qquad (\eta') \neq (\bar{\eta}')$$

set up for the second variation, for x on our interval, for (μ) unrestricted, and for differentially admissible sets (x, η, η') and $(x, \eta, \bar{\eta}')$.

Now let $I(\eta, u, 0)$ be taken along any curve g of class C' which satisfies (12.6). The resulting value of I will certainly be as great as the value of I taken along the broken secondary extremal with the same ends as g and the same intersections with the n-planes (12.1), as a use of the Weierstrass condition shows. But $Q(z, 0)$ is positive definite if all characteristic roots are positive. Hence the following theorem.

THEOREM 7. *For the second variation to be positive for every admissible $(\eta) \neq (0)$ it is sufficient that g be identically normal, that the Clebsch and non-tangency conditions hold, and that all the characteristic roots be positive.*

* We refer only to Clebsch and Weierstrass *sufficient* conditions from this point on.

If $\sigma = 0$ is a characteristic root of index q, but there are no negative characteristic roots, we see that the form $Q(z, 0)$ is still positive, except on the q-plane of points (z) satisfying (11.1), on which it is zero. Use of the Weierstrass condition leads to the following theorem.

THEOREM 8. *If g be identically normal, if the Clebsch and non-tangency conditions hold, and if $\sigma = 0$ is the smallest characteristic root, then the second variation will be positive for all admissible sets $(\eta) \not\equiv (0)$, except for those characteristic solutions for which $\sigma = 0$, and for these solutions the second variation will be zero.*

The following corollary is of interest.

COROLLARY. *In order that g afford a proper, strong, relative minimum, it is sufficient that it be identically normal, that the Clebsch, Weierstrass, and non-tangency conditions hold, and that the second variation be positive for all admissible $(\eta) \not\equiv (0)$.*

For if the second variation be positive for all admissible $(\eta) \not\equiv (0)$ there can be no negative root σ by virtue of the proof of Theorem 2, and no root $\sigma = 0$ by virtue of the preceding theorem. Theorem 3 then yields the corollary.

13. *The case $G \equiv 0$. The fixed end point problem.* It will be of interest to determine the nature of the accessory boundary value conditions (7.5) and (7.6) in certain important problems. One can find b_{hk} from (3.10).

In the fixed end point problem one finds that

$$b_{hk} = 0 \qquad c_{ih}{}^{s} = 0,$$

and that accordingly the accessory boundary value conditions reduce simply to $\eta_i{}^s = 0$.

The end points variable on two manifolds. We suppose the end points variable on two n-dimensional manifolds not tangent to g. It will be illuminating if we suppose the extremal carried into the x-axis and the tangent planes of the two manifolds into the planes $x = a^s$ respectively. Such a transformation can be readily set up and would be legitimate if the problem came to us in parametric form.

We take the y-coördinates on the respective manifolds as our terminal

parameters (α). We then have $x_h^s(0)=0$. The variations η_i^s are independent, and the accessory boundary conditions are readily seen to have the form

$$\zeta_i^1 = -[F]^1 x_{ij}^1 \eta_j^1$$
$$\zeta_i^2 = -[F]^2 x_{ij}^2 \eta_j^2$$

thus exhibiting the dependence of the accessory boundary conditions upon the curvature of the end manifolds.

The periodic case. Here we suppose that the integrand f and the functions ϕ_β have a period ω in x, and that $a^2 - a^1 = \omega$. We suppose further that an extremal g of period ω is given.

We compare g with neighboring curves of class C' whose end points are congruent, that is, whose y-coördinates at $x = a^2$ and $x = a^1$ are the same. We can take these common y-coördinates as the terminal parameters (α). Thus the terminal conditions may be taken as

$$y_i^s = \alpha_i \qquad x^s = a^s \qquad (i = 1, \cdots, n;\ s = 1, 2).$$

From (3.10) one sees that $b_{hk} = 0$. The accessory boundary conditions become

$$\zeta_i^1 = \zeta_i^2 \qquad \eta_i^1 = \eta_i^2$$

and thus require a characteristic solution to be periodic.

We have thus the following theorems.

THEOREM 9. *In order that a normal periodic extremal afford a minimum to J relative to neighboring differentially admissible curves joining congruent points, it is necessary that the accessory differential equations have no periodic solutions for $\sigma < 0$.*

THEOREM 10. *In order that a periodic extremal afford a proper, strong, minimum to J relative to neighboring differentially admissible curves joining congruent points, it is sufficient that it be identically normal, that it satisfy the Clebsch and Weierstrass sufficient conditions, and that the accessory differential equations possess no periodic solutions for $\sigma \leqq 0$.*

III. THE GENERAL PROBLEM.

14. *The problem defined.* Apart from conditions, such for example as the Clebsch condition, which in the most important problems are generally

fulfilled, the problem of minimizing J is essentially the problem of finding an extremal for which the type * number and nullity of $Q(z, 0)$ are zero.

This is a special case of the more general problem of finding an extremal for which the type number of $Q(z, 0)$ is a prescribed positive integer, and the nullity zero.

This problem will have more point if the type number and nullity of $Q(z, 0)$ are really determined by the extremal and boundary conditions alone, and not also by the number and position of the n-planes (9.1) we have used in constructing the broken extremal and defining $J(v, \sigma)$; and this is the case as we shall show. This is subject to the natural limitation that the n-planes (9.1) be sufficiently near together.

The study of this general problem exhibits more adequately the interrelation between the calculus of variations and the accessory boundary problem.

Up to this point the functions $J(v, \sigma)$ and the quadratic form $Q(z, \sigma)$ of § 10 have been defined only for $\sigma \leqq 0$. We can, however, define them also for $\sigma > 0$.

For each $\sigma > 0$ it will, however, first be necessary to choose the n-planes (9.1) nearer together than any two successive conjugate points. If the choice of these n-planes be made, say for $\sigma = \sigma_0$, the same choice will suffice, according to the lemma of § 10, for $\sigma < \sigma_0$. Thus the number δ of variables in the set (z) depends upon the choice of σ, but one choice can be made for all values of $\sigma \leqq \sigma_0$.

It will remove ambiguity if we now denote the form Q by $Q(z, \sigma, \delta)$, and term δ the *dimension* of the form Q.

In Part III we shall assume that the extremal g is identically normal, and that the Clebsch sufficient condition and the non-tangency condition hold.

15. *The theorem about* $Q(z, \sigma, \delta)$. A point $(z) \neq (0)$ at which all the partial derivatives of $Q(z, \sigma, \delta)$ with respect to (z) vanish will be called a *critical set* with characteristic σ and dimension δ. The value of σ will be a characteristic root as we have seen.

We now come to the following lemmas.

LEMMA 1. *The form Q has the property that*

* The type number of a quadratic form is the number of negative coefficients appearing in the form after it has been transformed by a real linear non-singular transformation into a sum of squared terms only.

$$Q(z, \sigma', \delta) < Q(z, \sigma'', \delta) \tag{15.1}$$

provided $(z) \neq (0)$ *and* $\sigma' > \sigma''$.

Let (η) represent the broken secondary extremal E determined by (z) when $\sigma = \sigma''$. From (5.1) and (10.3) we have

$$I(\eta, u, \sigma') = Q(z, \sigma'', \delta) + 2(\sigma'' - \sigma') \int_{a^1}^{a^2} \eta_i \eta_i dx \tag{15.2}$$

where (u) gives the first r variables in the set (z). From (15.2) we see that

$$I(\eta, u, \sigma') < Q(z, \sigma'', \delta). \qquad (z) \neq (0). \tag{15.3}$$

But from the minimizing properties of the component arcs of E

$$Q(z, \sigma', \delta) \leqq I(\eta, u, \sigma'). \tag{15.4}$$

The lemma follows from the last two inequalities.

By the *sum of a number of sets* (z) will be meant the set (z) obtained by adding sets (z) as if they were vectors.

LEMMA 2. *The form* $Q(z, \sigma, \delta)$ *is negative if evaluated for a sum* $(z) \neq (0)$ *of a finite number of critical sets with characteristic roots less than* σ.

Without loss of generality we can suppose the critical sets in the sum have distinct characteristic roots.

Let (z) be the sum. Let σ' be the largest of the characteristic roots and (z') the corresponding critical set. Let (z'') be the sum of the remaining critical sets so that $(z) = (z') + (z'')$.

From the preceding lemma we have

$$Q(z, \sigma, \delta) < Q(z, \sigma', \delta) \qquad \sigma' < \sigma \tag{15.5}$$

and this inequality proves the lemma if there is but one critical set in the sum, since the right hand form is then zero.

Now as a matter of algebra of quadratic forms

$$Q(z, \sigma', \delta) = Q(z', \sigma', \delta) + z_p'' Q_{z_p}(z', \sigma', \delta) + Q(z'', \sigma', \delta) \qquad (p = 1, \cdots, \delta). \tag{15.6}$$

But since (z') is a critical set for $\sigma = \sigma'$ this equality reduces to

$$Q(z, \sigma', \delta) = Q(z'', \sigma', \delta). \tag{15.7}$$

If we now adopt the method of mathematical induction and assume the

lemma true for a sum involving one less critical set than the original sum, the right hand form is as a consequence negative. The lemma then follows from (15. 5).

LEMMA 3. *The members of any finite ensemble of critical sets* (z) *with distinct characteristics* σ *are linearly independent.*

Suppose there were such a linear dependence. Let (z) be the linear combination which is zero. We can regard (z) as a sum of critical sets with distinct characteristics. Let (z') and (z'') now be defined as in the preceding lemma. Equations (15. 6) and (15. 7) hold as before. But the left hand member of (15. 7) is zero since $(z)=(0)$, and the right hand member is negative by virtue of the preceding lemma.

From this contradiction we infer the truth of the lemma.

For a fixed dimension δ there cannot be more than δ sets (z) which are independent, since there are δ variables in the sets (z). From this fact and the preceding lemma we deduce the following.

The number of characteristic roots less than σ_0 *is at most the minimum dimension number* δ *permissible for* σ_0.

From this lemma we also obtain the following:

LEMMA* 4. *The members of any finite set of characteristic solutions* (η) *with distinct roots* σ *are linearly independent.*

We come now to the fundamental theorem.

THEOREM 11. *The type number of the form* $Q(z, \sigma_0, \delta_0)$ *equals the number* h *of characteristic roots less than* σ_0, *counting each with a multiplicity equal to its index.*

We shall keep $\delta=\delta_0$ throughout the proof.

If σ be sufficiently large and negative we have seen that Q is positive definite. If σ be now increased, the form Q will remain non-singular except when σ passes through a characteristic root σ_1. According to Theorem 6 of § 11, the index q_1 of such a root equals the nullity of the form when $\sigma=\sigma_1$. As σ increases through σ_1, it follows from the theory of quadratic forms that the type number of $Q(z, \sigma, \delta_0)$ changes by at most q_1. Thus the type number of $Q(z, \sigma_0, \delta_0)$ is at most h, the sum of these indices.

* This lemma could also be proved directly from the differential equations and boundary conditions.

Corresponding to each characteristic root $\sigma < \sigma_0$ of index q, there are q linearly independent critical sets (z). According to Lemma 2 these sets will make $Q(z, \sigma_0, \delta_0)$ negative, as will any linear combination of them not (0), arising from different characteristic roots $\sigma < \sigma_0$.

But according to Lemma 3, the members of any finite ensemble of critical sets with distinct characteristics will be independent. Thus there are h critical sets with $\sigma < \sigma_0$ which are independent. These h critical sets regarded as points (z) taken with the point $(z) = (0)$ determine an h-plane in the space (z). On this h-plane $Q(z, \sigma_0, \delta_0)$ is negative definite.

It follows * that the type number of Q is at least h. But we have seen that is at most h. Thus the type number is exactly h and the theorem is proved.

16. *Comparison theorems.* We seek to connect characteristic roots with conjugate points.

A point $x = b$ will be said to be a conjugate point of $x = a$ of index q for a given value of σ, if there are just q linearly independent secondary extremals $(\eta) \not\equiv (0)$ which vanish at $x = a$ and $x = b$.

We come to the following theorem.

THEOREM 12. *In the fixed end point problem the form $Q(z, \sigma, \delta)$ is singular if and only if for the given σ, $x = a^2$ is conjugate to $x = a^1$. Moreover the nullity of $Q(z, \sigma, \delta)$ equals the index of $x = a^2$ as a conjugate point of $x = a^1$.*

The type number of $Q(z, \sigma, \delta)$ equals the number of conjugate points of $x = a^1$ *preceding $x = a^2$ for the given σ, counting conjugate points according to their indices.*

The first paragraph of the theorem is a consequence of Theorems 5 and 6 applied to the fixed end point problem, inasmuch as the accessory boundary conditions arising from the fixed end point problem are simply $\eta_i{}^s = 0$. The second paragraph of the theorem is proved by a repetition of the proof of Theorem 2 of Morse I, p. 392, except for obvious changes such as replacing the word "order" by "index."

When the boundary conditions reduce to $\eta_i{}^s = 0$ the accessory boundary problem will be called the *boundary problem with null end points.*

The latter half of Theorem 12 taken with Theorem 11 gives the following Theorem.

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* Morse I, p. 390, *loc. cit.*

THEOREM 13. *The number of conjugate points of $x = a^1$ which for $\sigma = \sigma_0$ precede $x = a^2$, equals the number of characteristic roots less than σ_0 in the accessory boundary problem with null end points.*

We shall now prove the following:

There exist arbitrarily many conjugate points of $x = a^1$ preceding $x = a^2$ for σ sufficiently large and positive.

Let (a, b) be any closed interval interior to the given interval. I say that for σ sufficiently large and positive there must be a conjugate point of $x = a^1$ on (a, b).

If this were not so the integral

$$\int_a^b \Omega(\eta, \eta', \mu, \sigma)\, dx \tag{16.1}$$

would be positive for all differentially admissible sets (η) not identically zero on (a, b), of class D', and null at a and b. Moreover there exists at least one set (η) in this class. For example one could take a finite succession of short arcs each of which is a secondary extremal when $\sigma = 0$.

Holding any such (η) fast let σ become positively infinite. The term

$$-2 \int_a^b \sigma \eta_i \eta_i \, dx$$

included in (16.1) will cause (16.1) to become negatively infinite. This is contrary to a previous assertion.

From this contradiction we infer that for σ sufficiently large and positive there will be at least one conjugate point on (a, b).

Hence the statement in italics is true.

We have compared characteristic roots with conjugate points. We can also compare characteristic roots in one boundary problem with characteristic roots in another such problem, as follows.

THEOREM 14. *The number of characteristic roots less than σ_0 in an accessory boundary problem involving r end parameters (u),* (see § 4), *lies between k and $k + r$ inclusive, where k is the number of characteristic roots less than σ_0 in the boundary problem with null end points.*

This theorem will be proved with the aid of the following lemma on quadratic forms.

LEMMA. *Let $Q_2(z)$ be a quadratic form obtained by setting the first r variables (z) in a quadratic form $Q_1(z)$ equal to zero. If the type number of $Q_2(z)$ is k, the type number of $Q_1(z)$ lies between k and $k+r$ inclusive.*

Let h be the type number of Q_1. It follows from the theory of quadratic forms that there is an h-plane, say π_h, through the origin of the space (z) on which Q_1 is negative definite. If the first r of the variables (z) be set equal to zero, these r conditions together with the linear conditions defining π_h will define a plane π of dimensionality at least $h-r$. On π however, Q_2 will be negative definite. Hence $k \geqq h-r$. See Morse I, Lemma I, p. 390.

The variables (z) which actually appear in Q_2 are those coördinates (z) which are arbitrary in the sub-space $(z_1, \cdots, z_r) = (0)$. Since Q_2 is of type k, there exists in this sub-space a k-plane π_k through the origin on which Q_2 is negative definite. Now in this sub-space $Q_1 \equiv Q_2$. Hence Q_1 is negative definite on π_k. Hence $h \geqq k$.

Thus the lemma is proved.

We come now to the proof of the theorem.

Let $Q_1(z)$ be the form $Q(z, \sigma, \delta)$ with $\sigma = \sigma_0$, set up for the given accessory boundary problem. Let $Q_2(z)$ be the form obtained from $Q_1(z)$ by setting the first r variables in $Q_1(z)$ equal to zero. Using the same intermediate n-planes (9.2) as were used in setting up $Q_1(z)$, let Q be now set up with $\sigma = \sigma_0$, for the fixed end point problem, and in the resulting form let r be added to the subscript of each variable. There will result the previous form $Q_2(z)$.

Now the type number of $Q_2(z)$ is the number k of characteristic roots less than σ_0 in the boundary problem with null end points. The theorem follows from the lemma and Theorem II.

We have the following corollary.

COROLLARY. *The number of characteristic roots less than σ_0 in any accessory boundary problem involving r end parameters, differs from the corresponding number for any other such problem involving s end parameters by at most the larger of the two numbers r and s.*

We shall prove the following theorem.

THEOREM 15. *The number of characteristic roots on any finite interval of the σ-axis, in any accessory boundary problem involving r end parameters, differs from the corresponding number for the boundary problem with null end points by at most r.*

In fact let h_1 and h_2 be respectively the numbers of characteristic roots less than σ_1 and σ_2 in a given boundary problem. $(\sigma_1 < \sigma_2)$. If k_1 and k_2 denote the corresponding numbers for the boundary problem with null end points we see from Theorem 14 that

$$(16.2) \qquad \begin{aligned} h_1 &= k_1 + m_1 & \quad 0 &\leqq m_1 \leqq r \\ h_2 &= k_2 + m_2 & \quad 0 &\leqq m_2 \leqq r. \end{aligned}$$

Now $h_2 - h_1$ is the number, say m, of characteristic roots of the given problem on the interval

$$(16.3) \qquad \sigma_1 \leqq \sigma < \sigma_2.$$

For this number m we have from (16.2) that

$$m = (k_2 - k_1) + (m_2 - m_1) \quad | m_1 - m_2 | \leqq r$$

which proves the theorem for the interval (16.3).

Now corresponding to any finite interval whatsoever there exists a closely approximating interval which is of the form (16.3), and which contains the same characteristic roots. Thus the theorem is true in general.

We note the following corollary.

COROLLARY. *The number of characteristic roots on any finite interval of the σ-axis for any accessory boundary problem differs from the corresponding number for any other such problem by at most the sum of the numbers of parameters in the end point conditions of the two problems.*

Except for this corollary the limits of the inequalities given in these theorems can be realized by examples, so that these limits are reduced as much as possible.

It is not so with this corollary. From this corollary it would appear that the number of characteristic roots on a finite interval for one boundary problem might differ from that for another by as much as $2n + 2n = 4n$, whereas the actual limit can be shown to be $2n$, and depending upon the end conditions may be less.

This question is one of a series of questions in the theory of these boundary problems which can be effectively treated by the methods of this paper. Among other things it involves a notion of *the boundary problem common to two problems.*

From the point of view of the theory of boundary value problems the degree of generality of the preceding results is evidenced by the following fact. In the definition of the accessory boundary problem of § 4, one can

prescribe the form $\omega(\eta, \eta')$, the differential conditions $\Phi_\beta = 0$, and the constants $c_{ih}{}^s$ and b_{hk}, subject only to the restriction that $b_{hk} = b_{kh}$. To avoid unnecessary complexity we shall also require that $\| c_{ih}{}^s \|$ be of maximum rank r. The problem of minimizing

$$\tfrac{1}{2} b_{hk} u_h u_k + \int_{a^1}^{a^2} \omega(\eta, \eta')\, dx$$

subject to the conditions

$$\phi_\beta = 0 \qquad \eta_i{}^s = c_{ih}{}^s u_h \qquad\qquad (i = 1, \cdots, n;\ h = 1, \cdots, r)$$

will admit the x axis between a^1 and a^2 as an extremal with multipliers $\lambda_0 = 1$, $\lambda_\beta \equiv 0$, and for this extremal the accessory boundary problem will have the prescribed form.

This fact also shows the appropriateness of the form into which the accessory boundary conditions were thrown.

This theory will be further developed from the point of view of the geometry of boundary value problems.

ON THE PROBLEM OF LAGRANGE.

By LAWRENCE M. GRAVES.

In recent years there has been a tendency to derive the functional equations which characterize the solutions of problems of the calculus of variations under the least restrictive hypotheses on those solutions. For the simplest problem in the plane in non-parametric form, Whittemore * derived an equation for the solutions in 1901, under the hypothesis that the minimizing function $y(x)$ has a derivative $y'(x)$ which is bounded, and also continuous except on a set of content zero. Tonelli derived this equation † under the weaker hypothesis that the minimizing function $y(x)$ has bounded difference quotients. Similar results for the problem in parametric form were found by Hahn and by Tonelli.‡ Tonelli obtains the Weierstrass condition also, § and for the non-parametric problem under the still weaker hypothesis that $y(x)$ is absolutely continuous.

In the present paper, we obtain a multiplier rule to characterize the minimizing functions $y_i(x)$ in the problem of Lagrange, supposing only that those functions have bounded difference quotients. Under the same hypothesis the analogue of the Weierstrass condition is derived in § 4. Some properties of normal intervals for admissible functions are derived in § 3. The corollary of the theorem in this section enables us to prove the analogue of the Weierstrass condition under hypotheses on the normality of the minimizing functions which are less restrictive than those usually made.¶ § 5 contains an additional remark on the relation between normality and the Weierstrass condition, as well as a remark on an extension of the condition of Mayer.

The methods of proof and notations are largely those used by Bliss,‖ hough some differences are necessary. The ordinary theorems on differential and other functional equations are inadequate for obtaining the results of this paper. However, the equations involved are a special case of those treated

* "Lagrange's Equation in the Calculus of Variations, and the Extension of a Theorem of Erdmann", *Annals of Mathematics*, Ser. 2, Vol. 2 (1901), pp. 130-6.

† *Fondamenti di calcolo delle variazioni*, Vol. II, pp. 318, 557.

‡ Hahn, "Ueber die Herleitung der Differentialgleichungen der Variationsrechnung", *Mathematische Annalen*, Vol. 63 (1907), pp. 253-72; Tonelli, *loc. cit.*, pp. 89, 486.

§ Tonelli, *loc. cit.*, pp. 83, 317, 511, 557.

¶ This possibility was first called to my attention by Dr. M. C. Boyce.

‖ "The Problem of Lagrange in the Calculus of Variations," *American Journal of Mathematics*, Vol. 52 (1930), pp. 673-744.

in my paper on *Implicit Functions and Differential Equations in General Analysis.*[*] I have obtained a direct treatment of these equations without such a general background, but it is more complicated. In studying these equations it is convenient to center attention on the derivatives of the functions y_i. Hence we shall denote these derivatives by z_i.

1. *Formulation of the problem.* We shall consider an integral † to be minimized,

$$I[z] = \int_{x_1}^{x_2} f(x, y, z)\, dx,$$

where

$$y_i(x) = y_{i1} + \int_{x_1}^{x} z_i(x)\, dx, \qquad (i = 1, \cdots, n). \tag{1}$$

This relation (1) between $y(x)$ and $z(x)$ is assumed to hold thruout the paper. For simplicity we shall assume that the integrand function f is defined for (x, y) in an $n+1$-dimensional region $\mathfrak{R}$ and for all values of z, and that f is bounded on every bounded domain. We suppose also that for each (y, z) the function f is measurable in x on every measurable subset of its range of definition, and that f is of class $\mathfrak{C}'$ in (y, z) uniformly on every bounded domain of (x, y, z) points. The last statement means that the partial derivatives f_{y_i} and f_{z_i} are bounded and continuous in (y, z) uniformly on every bounded domain of (x, y, z) points. We consider also m functions ϕ_a, $(m < n)$, having the same properties as f. The functions $f(x, y(x), z(x))$, $\phi_a(x, y(x), z(x))$ are measurable whenever the $z_i(x)$ are bounded and measurable and have $(x, y(x))$ interior to $\mathfrak{R}$.‡

We shall call two measurable functions z_1 and z_2 *equivalent* if they differ only on a set of measure zero, and use the sign $z_1 \sim z_2$ in place of $z_1 = z_2$.

Admissible functions $z(x)$ are bounded and measurable, have $(x, y(x))$ interior to the region $\mathfrak{R}$, where $y(x)$ is defined by (1), and satisfy

$$\phi_a(x, y(x), z(x)) \sim 0, \qquad (a = 1, \cdots, m). \tag{2}$$

They also satisfy the condition

H) there exists a positive number μ such that for almost all x the matrix

* *Transactions of the American Mathematical Society,* Vol. 29 (1927), pp. 514-552. See especially Theorem XIII, p. 542.

† All integrals are to be understood in the Lebesgue sense.

‡ See Caratheodory, *Vorlesungen ueber reelle Funktionen,* p. 665; Graves, "Some Theorems Concerning Measurable Functions", *Bulletin of the American Mathematical Society,* Vol. 32 (1926), pp. 529-533.

$\phi_{\alpha z_i}(x, y(x), z(x))$ has a minor determinant whose absolute value is not less than μ.

2. THE MULTIPLIER RULE. *If $z_0(x)$ minimizes $I[z]$ in the class of admissible functions z making $y(x_2) = y_2$, then there exist constants $(l_0, b_1, \cdots, b_n)$ and bounded measurable functions $\lambda_1, \cdots, \lambda_m$, with $(l_0, \lambda_1, \cdots, \lambda_m)$ not all equivalent to zero, such that*

$$F_{z_i} \sim \int_{x_1}^{x} F_{y_i}\, dx + b_i, \qquad (i = 1, \cdots, n), \tag{3}$$

where $F(x, y, z, \lambda, l_0) \equiv l_0 f(x, y, z) + \lambda_\alpha \phi_\alpha(x, y, z)$.

From the hypothesis H) it is plain that $n - m$ additional functions $\phi_r(x, y, z)$ can be adjoined, having the same properties as the functions ϕ_α, so that the determinant

$$| \phi_{iz_j}(x, y_0(x), z_0(x)) |$$

has absolute value not less than μ on $(x_1 x_2)$. Then we have

$$\phi_i(x, y_0(x), z_0(x)) - w_{i0}(x) = 0, \qquad (i = 1, \cdots, n),$$

where $w_{\alpha 0}(x) \sim 0$, $(\alpha = 1, \cdots, m)$. If we apply theorem XIII of my paper * to the equations

$$\phi_i(x, y_1 + \int_{x_1}^{x} z\, dx, z) = w_i(x), \tag{4}$$

we find that for w near w_0 they have a unique solution $Z[w, x]$ near $z_0(x)$, which is bounded and measurable in x, and of class $\mathfrak{C}'$ in w uniformly.† Here the norm $|| w ||$ of a set of bounded measurable functions $w_i(x)$ is taken as the upper bound for all i and x of $| w_i(x) |$. The functional

$$Y[w, x] \equiv y_1 + \int_{x_1}^{x} Z[w, x] dx \tag{5}$$

is continuous in x and of class $\mathfrak{C}'$ in w uniformly, and the differentials $\eta = dY[w, x; \omega]$, $\zeta = dZ[w, x; \omega]$ are the unique solutions of the equations of variation

$$\phi_{iy_j}\eta_j + \phi_{iz_j}\zeta_j = \omega_i, \quad \eta_i = \int_{x_1}^{x} \zeta_i dx.$$

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* *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 542.

† See Hildebrandt and Graves, "Implicit Functions and Their Differentials in General Analysis", *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 12[illegible] definition of the class $\mathfrak{C}'$ of functionals. We [illegible] brackets to enclose the arguments of functionals. In writing the arguments of the differentials of functionals, we shall revert to a usage customary in the calculus of variations by writing ω for dx, η for dy, ζ for dz.

It is readily verified that the integral $I[z]$ is of class $\mathfrak{C}'$ on the region of the space of bounded measurable functions $z(x)$ where it is defined. Hence the functional

$$J[w] \equiv I[Z[w]] = \int_{x_1}^{x_2} f(x, Y, Z)\,dx$$

is of class $\mathfrak{C}'$ near w_0, and

$$dJ[w;\omega] = \int_{x_1}^{x_2} (f_{y_j} dY_j + f_{z_j} dZ_j)\,dx. \tag{6}$$

If we multiply (6) by a constant l_0 and add

$$\int_{x_1}^{x_2} (\lambda_i \phi_{iy_j} dY_j + \lambda_i \phi_{iz_j} dZ_j - \lambda_i \omega_i)\,dx = 0, \tag{7}$$

where the λ_i are arbitrary bounded measurable functions, we obtain

$$l_0 dJ[w;\omega] = \int_{x_1}^{x_2} (F_{y_j} dY_j + F_{z_j} dZ_j - \lambda_i \omega_i)\,dx, \tag{8}$$

where $F(x, y, z, \lambda, l_0) = l_0 f(x, y, z) + \lambda_i \phi_i(x, y, z)$.

The equations

$$F_{z_i}(x, Y, Z, \lambda, l_0) = \int_{x_2}^{x} F_{y_i}(x, Y, Z, \lambda, l_0)\,dx - c_i \tag{9}$$

are linear in λ, l_0, c, for every w near w_0. Hence, by the same theorem as before * they have a unique solution $\lambda_i = \Lambda_i[w, l_0, c, x]$ bounded and measurable in x. Using (9) to integrate by parts in (8), we find

$$l_0 dJ[w;\omega] + c_i dY_i[w, x_2; \omega] = -\int_{x_1}^{x_2} \Lambda_i[w, l_0, c, x]\omega_i(x)\,dx \tag{10}$$

for every w, l_0, c, ω.

If we now make use of the hypothesis that z_0 minimizes $I[z]$ in the class of admissible functions making $y(x_2) = y_2$, we find that w_0 must minimize $J[w]$ in the class of bounded measurable functions whose first m components are equivalent to zero and which make

$$Y[w, x_2] = y_2. \tag{11}$$

Hence the matrix

$$\left\| \begin{matrix} dJ[w_0;\omega] \\ dY_i[w_0, x_2;\omega] \end{matrix} \right\|$$

has rank less than $n + 1$ when ω ranges over the bounded measurable functions whose first m components are zero, since otherwise, the equations (11)

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* Graves, *Implicit Functions and Differential Equations*, Theorem XIII, p. 542.

with $$J[w] = J[w_0] + u$$

would have solutions for every u near u_0, by the ordinary implicit function theorem. From this we see that there must exist $n+1$ constants l_0 and c_i, not all zero, such that $l_0 dJ[w_0;\omega] + c_i dY_i[w_0, x_2;\omega] = 0$ for the class of functions ω just mentioned. On comparing with equation (10) we find $\Lambda_r[w_0, l_0, c, x] \sim 0$, $(r = m+1, \cdots, n)$. The functions l_0, Λ_α cannot all be equivalent to zero, since then the constants c_i would also be zero, by (9). This proves the multiplier rule.

COROLLARY 1. *If the functions f and ϕ_α are continuous in x, then the functions z_0 and λ may be redefined at the points of a set of measure zero so as to satisfy equations* (1) *and*

$$(2') \qquad \phi_\alpha(x, y(x), z(x)) = 0 \qquad (\alpha = 1, \cdots, m),$$

$$(3') \qquad F_{z_i} = \int_{x_1}^{x} F_{y_i} dx + b_i \qquad (i = 1, \cdots, n),$$

everywhere on $(x_1 x_2)$.

The process of redefinition is in terms of limiting values of the functions z_0 and λ. Obviously this process will not introduce any new discontinuities of z_0 or λ.

COROLLARY 2. *If the functions f, ϕ_α, f_{z_i}, and $\phi_{\alpha z_i}$ are of class $\mathfrak{C}'$ in all their arguments, then the set of points where the determinant*

$$R(x) \equiv \begin{vmatrix} F_{z_i z_j} & \phi_{\alpha z_i} \\ \phi_{\beta z_j} & 0 \end{vmatrix}$$

is not zero and the minimizing functions z_0 are continuous constitutes an open set O. On O the functions z_0 are of class $\mathfrak{C}'$ and the functions λ are continuous, and equations (3') *may be differentiated with respect to x.*

This corollary is the extension of the Hilbert theorem on the differentiability of minimizing functions and is proved as usual by means of the implicit function theorem.*

3. *Normal intervals for admissible functions.* We shall say that an interval $(x_1 x_2)$ is *normal* † for an admissible function $z(x)$ in case the matrix

$$\| dY_i[w, x_2; \omega] \|$$

has rank n when ω ranges over the bounded measurable functions whose first

* See Bliss, *loc. cit.*, p. 684.

† See Bliss, *loc. cit.*, p. 687.

m components are zero. Here w is defined by equations (4), and dY is the differential of the functional (5).

THEOREM. *If* (x_1x_2) *is a normal interval for an admissible function* $z(x)$, *and if* z *has multipliers* l_0, λ_a *with which it satisfies* (3), *then* $l_0 \neq 0$. *If we require* $l_0 = 1$, *the multipliers* λ_a *are unique apart from sets of measure zero. Conversely, if* (x_1x_2) *is not a normal interval for* z, *then* z *has multipliers* $l_0 = 0$, λ_a *with which it satisfies* (3).

For from (3) we obtain (9) with $\lambda_r \sim 0$ $(r = m + 1, \cdots, n)$, and then (10). Hence l_0 cannot be zero if the interval is normal. If there were two sets of multipliers with $l_0 = 1$, their difference would be a set of multipliers with $l_0 = 0$, which has just been shown to be impossible. For the converse, we have that there exist constants c_i not all zero such that $c_i dY_i[w, x_2; \omega] = 0$ for every bounded and measurable ω whose first m components are zero. In equations (7) put $\lambda_i(x) = \Lambda_i[w, 0, c, x]$, where Λ is the solution of (9), and we find

$$\int_{x_1}^{x_2} \Lambda_i \omega_i dx = 0,$$

and hence $\Lambda_r \sim 0$, $(r = m + 1, \cdots, n)$.

COROLLARY. *If* (x_1x_2) *is a normal interval for an admissible function* z, *then every interval containing* (x_1x_2) *is also normal.*

For, suppose an interval (x_1x_3) contains (x_1x_2) and is not normal. Then the multipliers λ_a of the last part of the theorem for the interval (x_1x_3) are all equivalent to zero on the interval (x_1x_2). From this we see that the constants c_i of equations (9) must all be zero, and then from the uniqueness of the solution of these equations we find $\lambda_a \sim 0$ on the larger interval (x_1x_3).

4. *The analogue of the Weierstrass condition.* In this section we assume that the functions f and ϕ_a are continuous in x. We suppose also that z_0 minimizes $I[z]$ in the class of admissible functions z giving y the end-values y_1 and y_2, and that z_0 has multipliers $l_0 = 1$, $\lambda_a(x)$, with which it satisfies equations (2′) and (3′) of page 551. *Then if the sub-interval* (x_1x_3) *of* (x_1x_2) *is normal for* z_0, *the functions* z_0 *and* λ *can be modified at the points of a set of measure zero without disturbing the validity of equations* (2′) *and* (3′), *so that the Weierstrass function*

$$E(x, y_0(x), z_0(x), \lambda(x), \bar{z}) \geqq 0$$

everywhere on the interval (x_3x_2), *for every set of numbers* $\bar{z}$ *such that* $\phi_a(x, y_0(x), \bar{z}) = 0$ *while the matrix* $\phi_{az_i}(x, y_0(x), \bar{z})$ *has rank* m.

To prove this, consider the point set S where each $y_{i0}(x)$ has a derivative equal to $z_{i0}(x)$ and

$$\int_{x_1}^{x} f(x, y_0(x), z_0(x))\,dx$$

has a derivative equal to $f(x, y_0(x), z_0(x))$, and let x_4 be a point of S between x_3 and x_2, $y_{i4} = y_{i0}(x_4)$, $z_{i4} = z_{i0}(x_4)$, $\lambda_{a4} = \lambda_a(x_4)$. If $\bar{z}$ satisfies the conditions specified in the theorem, the equations

$$\phi_a(x, \ddot{y}, \ddot{z}) = 0, \quad \ddot{y}_i = y_{i4} + \int_{x_4}^{x} \ddot{z}_i\,dx,$$

have a continuous solution $\ddot{z}(x)$ near x_4, with $\ddot{z}(x_4) = \bar{z}$. Since the interval $(x_1 x_4)$ is normal by the corollary in § 3, the matrix

$$\| \; dY_i[w_0, x_4; \omega] \; \|$$

has rank n. Let $\omega_i^{(1)}, \cdots, \omega_i^{(n)}$, give it this rank. Then the equations $Y_i[w_0 + a_k\omega^{(k)}, x_5] = \ddot{y}_i(x_5)$ have a unique solution $a_k = a_k(x_5)$ for x_5 near x_4 and a_k near zero, and this solution is continuous. By a direct consideration of the difference quotients the derivatives $a_k'(x_4)$ can readily be shown to exist and to satisfy the equations $dY_i[w_0, x_4; a_k'\omega^{(k)}] = \bar{z}_i - z_{i4}$, since $\ddot{z}(x)$ is continuous and x_4 is a point of the set S.

Now set

$$\begin{aligned} \mathfrak{Z}(x, x_5) &= Z[w_0 + a_k(x_5)\omega^{(k)}, x], \quad \mathfrak{Y}(x, x_5) = Y[w_0 + a_k(x_5)\omega^{(k)}, x], \\ z(x, x_5) &= \mathfrak{Z}(x, x_5) \quad \text{on } x_1 \leqq x \leqq x_5, \\ &= \ddot{z}(x) \quad \text{on } x_5 < x \leqq x_4, \\ &= z_0(x) \quad \text{on } x_4 < x \leqq x_2. \end{aligned}$$

Then the function $z(x, x_5)$ is admissible for $x_5 \leqq x_4$, and satisfies the end conditions

$$\int_{x_1}^{x_2} z_i(x, x_5)\,dx = y_{i2} - y_{i1}.$$

Hence $K(x_5) \equiv I[z(x, x_5)] \geqq I[z_0(x)] = K(x_4)$. By a direct consideration of the difference quotient and application of the theorem of mean value, we find that K has a left-hand derivative at x_4, given by

$$K'(x_4) = -f(x_4, y_4, \bar{z}) + f(x_4, y_4, z_4) + \int_{x_1}^{x_4} (f_{y_i} dY_i + f_{z_i} dZ_i)\,dx, \tag{12}$$

where the arguments of f_{y_i} and f_{z_i} are $x, y_0(x), z_0(x)$, and those of dY_i and dZ_i are $w_0, x, a_k'\omega^{(k)}$. Now dY and dZ satisfy

$$\phi_{a y_i} dY_i + \phi_{a z_i} dZ_i = 0, \tag{13}$$

where the arguments are respectively the same as in (12). If we multiply

(13) by the multipliers λ_α belonging to z_0, integrate from x_1 to x_4, add to equation (12), and use equations (3′), we find

$$K'(x_4) = -f(x_4, y_4, \bar{z}) + f(x_4, y_4, z_4) + (\bar{z}_i - z_{i4})F_{z_i}(x_4, y_4, z_4, \lambda_4).$$

Now since

$$\phi_\alpha(x_4, y_4, z_4) = 0, \quad \phi_\alpha(x_4, y_4, \bar{z}) = 0,$$

the derivative

$$K'(x_4) = -E(x_4, y_4, z_4, \lambda_4, \bar{z}).$$

To show $E \geqq 0$ for limiting values of the functions z_0, λ, consider an infinite sequence $\{x_q\}$ of points of the set S, such that $\lim x_q = x^*$, $\lim z_0(x_q) = z^*$, $\lim \lambda(x_q) = \lambda^*$. If $\bar{z}$ satisfies the conditions of the theorem at the point x^*, then there is a sequence $\{\bar{z}_q\}$ such that $\bar{z}_q$ satisfies the conditions of the theorem at the point x_q, and $\lim \bar{z}_q = \bar{z}$. Hence $E(x_q, y_0(x_q), z_0(x_q), \lambda(x_q), \bar{z}_q) \geqq 0$, and since E is continuous, $E(x^*, y_0(x^*), z^*, \lambda^*, \bar{z}) \geqq 0$.

5. *Remarks.* An example in which the preceding proof leads to the condition of Weierstrass when the proofs usually given do not, may be obtained by considering the isoperimetric problem. Let $z_0(x)$ have a single discontinuity at x_3, and minimize the integral I in the class of curves giving a second integral

$$G[z] = \int_{x_1}^{x_2} g(x, y, z)\, dx$$

a prescribed value. In case $z_0(x)$ is a mimimizing function for G on every interval not containing x_3, then every such interval is abnormal. But if $g_z(x, y_0(x), z_0(x))$ has a non-removable discontinuity at x_3, every interval containing x_3 is normal, and the Weierstrass condition holds on (x_1x_2). Cases of this sort would be those considered by Caratheodory in the second part of his dissertation,* in which the extremals for the two integrals I and G are the same.

I have also considered the second variation and obtained an extension of the necessary condition of Mayer. However, this extension is not satisfying, and the case when the derivatives have only a finite number of ordinary discontinuities † shows that a more penetrating study must be made before a consistent set of sufficient conditions can be given for the case when there are infinitely many discontinuities.

THE UNIVERSITY OF CHICAGO.

* *Ueber die diskontinuierlichen Lösungen in der Variationsrechnung*, Dissertation, Göttingen, 1904.

† See Graves, "Discontinuous Solutions in Space Problems of the Calculus of Variations", *American Journal of Mathematics*, Vol. 52 (1930), pp. 1-28.

PERSPECTIVE ELLIPTIC CURVES.

By Elizabeth Morgan Cooper.

1. *Introduction.* Two curves in a plane are said to be perspective if the points of one and the tangents of the other can be put into one-to-one correspondence in such a way that each point of the one lies on the corresponding tangent of the other. The property is not confined to curves in the plane, (for a discussion of many possible cases of perspective space curves and surfaces, see Segre [13, 14]), but this paper will deal with perspective plane curves only.

A curve perspective to a curve C is a birational transform of C, for if an envelope E is perspective to C a line τ of E, incident with the point t of C, will have, in general, a unique point of contact t' with E. Therefore the points of contact of E will be in one-to-one correspondence with the points of C, and their locus C', the dual form of E, will be a birational transform of C. Hence two perspective curves have, necessarily, the same genus. If, on the other hand, C' is a birational transform of C, the joins of corresponding points of C and C' envelope a curve E perspective to both C and C'. Two birational transforms, C' and C'', of C will, however, lead to the same envelope perspective to C if it happens that corresponding points of C, C' and C'' are collinear.*

Some, at least, of the curves perspective to C are obtained as transforms of C by quadratic null systems, i. e., quadratic correlations in which corresponding point and line are incident.

A conic generated by two perspective points, a cubic generated by a line and conic, a quartic by two conics, or, if it is "singular," by a line and cubic (Schroters [11]) are simple examples of a curve generated by two of its perspective curves. The fact that a curve can be so generated leads to a classification of rational curves by means of their perspective curves. Cf. Haase,[5] Meyer,[9] Stahl [15]). Take, for example, the rational sextic. A rational curve has ∞^{2m-n+1} perspective m-ics (Brill,[1] Coble [4]), hence the general rational sextic has ∞^1 perspective cubics and is the locus of points of intersection of corresponding lines of any two of these cubics. A condition on a sextic

* [illegible] of orders m, n and p respectively, each perspective to the other two, if it happens $m+n+p+1$ times that corresponding points are collinear, such points are always collinear.

gives it a perspective conic. It can then be generated by that conic and by one of its ∞^3 perspective quartics, while a sextic with a five-fold point can be generated by that point and a perspective quintic. There are, then, three types of rational sextics, the classification depending on the curve of lowest class perspective to the sextic. From this point of view perspective curves are important, and it is desirable to know the distribution of curves of given class (or order) perspective to a curve of given order (or class) and given genus. In this paper the number and arrangement of curves perspective to a curve of genus 1 is found. (p. 562).

It has been proved by Meyer [7, 8] that if two rational class curves, of lowest class,

$$\rho\xi_i = (\alpha_i\tau)^{m_1} \quad \text{and} \quad \rho\xi_i = (\beta_i\tau)^{m_2}, \qquad (i = 0, 1, 2),$$

are perspective to a curve C given by

$$\mu x_i = (a_i t)^n, \qquad (i = 0, 1, 2),$$

i. e., if

$$(a_0 t)^n (\alpha_0\tau)^{m_1} + (a_1 t)^n (\alpha_1\tau)^{m_1} + (a_2 t)^n (\alpha_2\tau)^{m_1} \equiv 0 \quad \text{when} \quad \tau = t$$

and

$$(a_0 t)^n (\beta_0\tau)^{m_2} + (a_1 t)^n (\beta_1\tau)^{m_2} + (a_2 t)^n (\beta_2\tau)^{m_2} \equiv 0 \quad \text{when} \quad \tau = t$$

then all curves of class m $\left(m > \dfrac{m_1}{m_2}\right)$ perspective to C are represented by

$$\rho\xi_i = (\gamma_i\tau)^{m-m_1}(\alpha_i\tau)^{m_1} + (\delta_i\tau)^{m-m_2}(\beta_i\tau)^{m_2}, \quad (i = 0, 1, 2),$$

where $(\gamma_i\tau)^{m-m_1}$ and $(\delta_i\tau)^{m-m_2}$ are arbitrary binary forms of the order indicated. For elliptic curves it is possible to write down perspective curves of a given order m in terms of two perspective curves of lower orders m_1 and m_2 in a similar manner, if and only if $m > \dfrac{m_1+1}{m_2+1}$. This restriction is needed, of course, because there is no first order elliptic function to take the place of the linear binary form.

A great deal of work on rational perspective curves has already been done, particularly by Stahl [15, 16]. Coble [4] has put in simpler and more interesting form the work of Stahl by showing its connection with apolarity, a property which does not, unfortunately, carry over to the elliptic case. Coble [4] discusses also the case of a curve which is doubly perspective to a given curve, i. e., a curve each of whose lines cuts the given curve twice in the point corresponding to the line. Such a curve is tangent to the given

curve at every point, and is, in fact, the line form of the curve itself. The condition that a curve have one or more cusps is merely the condition that it have a doubly perspective curve of sufficiently low class and this condition for rational curves Coble [11] writes down in very simple form.

It is proved by Haase [5] and Coble [4] that a rational n-ic and one of its perspective m-ics have $m+n-2$ contacts. Coble [4] finds that they lie in an involution $I_{2m-n+1,2n-3-m}$. The analogous situation for elliptic curves is discussed hereafter. (p. 563).

Some special applications of perspective rational curves are given by Coble,[2,3] St. Jolles,[6] Schumacher,[12] Study.[17] With the exception of the work of Segre,[13,14] who gives no specific results which are applicable in the plane, the articles we have located deal with the binary case only.

The elliptic case is more difficult to handle than the rational case because elliptic functions do not lend themselves as conveniently to the working out of the perspective theory. Considerable difficulty arises from the fact that the parametric expression of the coördinates of a curve C_n, given by

$$\mu x_i = R_i[p(u),\ p'(u)], \qquad (i = 0, 1, 2),$$

where R_i is a rational function, or by

$$\mu x_i = a_{i0} + a_{i2}p(u) + a_{i3}p'(u) + \cdots + a_{in}p^{n-2}(u), \qquad (i = 0, 1, 2),$$

suffers a serious alteration when the parameter change

$$u' = \pm u + k,$$

corresponding to the general transformation of the curve into itself, is introduced.

When, in the binary case, we ask that $\rho\xi_i = (\alpha_i\tau)^m$ be perspective to $\mu x_i = (a_i t)^n$, assumed given, the determination of the coefficients of $(\alpha_i\tau)^m$ *includes* the proper parameter choice, for $\rho\xi_i = (\alpha_i\tau)^m$ merely becomes $\rho'\xi_i = (\alpha_i'\tau')^m$ when the transformation

$$\tau = (a\tau' + b)/(c\tau' + d)$$

is made. In the elliptic case, on the other hand, though an appropriate choice of constants in $\rho\xi_i = R_i'[p(v),\ p'(v)]$ will make the curve it represents perspective to $\mu x_i = R_i[p(u),\ p'(u)]$ in the sense that the line v of the one and the point u of the other are incident, we can not, by merely changing *coefficients* and keeping the same *form* of the coördinate expressions, use $u' = \pm u + k$ to replace u as parameter. Indeed when we substitute $u' - k$ for u and use the addition formulae to express μx_i in terms of $p(u')$ and

$p'(u')$, we get expressions which are rational but no longer integral. By multiplying through by a common denominator we obtain, for each coördinate, a rational integral function of $p(u')$ and $p'(u')$, but these functions have higher orders than n, and we can not isolate the extraneous factors which have been introduced and which complicate matters greatly. The use of sigma functions does not seem to be profitable as an alternative, as it gives a still more troublesome form for the incidence relation. The most useful way of writing the coördinates seems to be in terms of the p-function and its derivatives.

It turns out that the elliptic parameter k appears as an independent parameter in the family of m-ics perspective to a given n-ic and leads to a distribution of these m-ics in a manner significantly different from the arrangement for rational curves.

2. *The m-ics Perspective to a Given n-ic.* The general elliptic class m-ic E_m, given by

$$(1) \quad \rho\xi_i = \alpha_{i0} + \alpha_{i2}p(v) + \alpha_{i3}p'(v) + \cdots + \alpha_{im}p^{m-2}(v), \qquad (i = 0, 1, 2),$$

will be perspective to a given order n-ic C_n, given by

$$(2) \quad \begin{aligned} \mu x_i = a_{i0} + a_{i2}p(-u+k) + a_{i3}p'(-u+k) \\ + \cdots + a_{in}p^{n-2}(-u+k), \qquad (i = 0, 1, 2), \end{aligned}$$

if and only if the incidence condition

$$(3) \qquad (x\xi)_{v=u} = 0$$

vanishes identically. If, instead of $(-u+k)$, we take $(u+k)$ for canonical parameter on C_n, we get the same curves E_m with canonical parameter $-v$ instead of v.

The incidence relation (3) is an elliptic function with an m-th order pole at $u = 0$ and an n-th order pole at $u = k$. The vanishing of the constant terms and of the coefficients of the principal parts of both expansions is a necessary and sufficient condition of perspectivity and gives $m + n + 2$ equations which are linear and homogeneous in α_{ij} and elliptic in k. Of these equations only $m + n - 1$ are needed to put on (3) the condition that it be a constant. We can, for example, require the vanishing of the m coefficients of $1/u^j$ $(j = 1, 2, 3, \cdots, m)$ in the expansion of (3) about $u = 0$ and of the $n - 1$ coefficients of $1/(u-k)^i$ $(i = 2, 3, \cdots, n)$ in the expansion about $u = k$. One further condition will require that that constant

be zero, hence it is clear that not more than $m+n$ of the $m+n+2$ equations are independent, and we shall show that the rank of the matrix formed from the equations is always $m+n$.

To investigate the rank of this matrix, which we shall call M_{mn}, we use not the general n-ic but the special case

$$\mu x_0 = 1, \quad \mu x_1 = p^{h-2}(-u+k), \quad \mu x_2 = p^{n-2}(-u+k),$$

since for special values of a_{ij} the rank will be less than or equal to the rank in the general case. Since the rank, we find, can never be greater than $2m+h$, we take $h=m$ when $m \leqq n$ and when n is being increased, otherwise, for simplicity, we choose $h=2$. In forming the matrix we take the equations in this order:

$$\begin{array}{ll}
A_0 = 0, & \\
(-1)^j A_j/(j-1)! = 0, & (j = 2, 3, 4, \cdots, m), \\
B_j/(j-1)! = 0, & (j = h, h-1, h-2, \cdots, 3, 2), \\
B_1 = 0, & \\
B_j/(j-1)! = 0, & (j = n, n-1, n-2, \cdots, h+2, h+1),
\end{array}$$

where A_j, $(j=0, 1, 2, \cdots, m)$, is the coefficient of $1/u^j$ and B_j, $(j=0, 1, 2, \cdots, n)$, is the coefficient of $1/(u-k)^j$ in the expansions of $(x\xi)_{v=u}$ about $u=0$ and $u=k$ respectively.

The matrix $M_{m,n+1}$ is formed from the matrix M_{mn} by the addition of one row and by changing, in numerical coefficient and by differentiation with respect to k, certain of the elements of the original matrix. The matrices M_{44} and M_{36} will serve to indicate the general situation in the two cases when $h=2$ and $h=m$ respectively. The heavy figures, besides indicating a new row, show wherein the corresponding elements of M_{43} and M_{35} have been altered in a numerical coefficient or in the order of a derivative.

$$\left\| \begin{array}{cccccccccccc}
1 & 0 & 0 & 2!c_2 & p & p''/2 & p'''/3 & (2!c_2+p^{IV}/4) & p'' & p^{IV}/2 & p^{V}/3 & (2!c_2+p^{VI}/4) \\
0 & 1 & 0 & 0 & 0 & p & \binom{2}{1}p' & \binom{3}{1}p'' & 0 & p'' & \binom{2}{1}p''' & \binom{3}{1}p^{IV} \\
0 & 0 & 1 & 0 & 0 & 0 & p & \binom{3}{2}p & 0 & 0 & p'' & \binom{3}{2}p''' \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & p & 0 & 0 & 0 & p'' \\
0 & 0 & 0 & 0 & 1 & p & p' & p'' & 0 & \binom{3}{2}p'' & \binom{3}{2}p''' & \binom{3}{2}p^{IV} \\
0 & 0 & 0 & 0 & 0 & p' & p'' & p''' & 0 & \binom{3}{3}p''' & \binom{3}{3}p^{IV} & \binom{3}{3}p^{V} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \binom{3}{\text{[illegible]}}p & \binom{3}{\text{[illegible]}}p' & \binom{3}{\text{[illegible]}}p'' \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{3}{1}p' & \binom{3}{1}p'' & \binom{3}{1}p'''
\end{array} \right\|,$$

$$M_{36} = \left\| \begin{matrix} 1 & 0 & 0 & p' & p'''/2 & p^{\mathrm{IV}}/3 & p^{\mathrm{IV}} & p^{\mathrm{VI}}/2 & p^{\mathrm{VII}}/3 \\ 0 & 1 & 0 & 0 & p' & \binom{2}{1}p'' & 0 & p^{\mathrm{IV}} & \binom{2}{1}p^{\mathrm{V}} \\ 0 & 0 & 1 & 0 & 0 & p' & 0 & 0 & p^{\mathrm{IV}} \\ 0 & 0 & 0 & 1 & \binom{2}{0}p & \binom{2}{0}p' & 0 & \binom{5}{3}p''' & \binom{5}{3}p^{\mathrm{IV}} \\ 0 & 0 & 0 & 0 & \binom{2}{1}p' & \binom{2}{1}p'' & 0 & \binom{5}{4}p^{\mathrm{IV}} & \binom{5}{4}p^{\mathrm{V}} \\ 0 & 0 & 0 & 0 & \binom{2}{2}p'' & \binom{2}{2}p''' & 0 & \binom{5}{5}p^{\mathrm{V}} & \binom{5}{5}p^{\mathrm{VI}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \binom{5}{0}p & \binom{5}{0}p' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{5}{1}p' & \binom{5}{1}p'' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{5}{2}p'' & \binom{5}{2}p''' \end{matrix} \right\| .$$

Here, as elsewhere in this paper, $p, p', \cdots p^n$ refer to $p(k), p'(k), \cdots p^n(k)$ when no other argument is indicated. The element standing in the $(m+1)$-th row and $(2m+2)$-th column is $\binom{n-1}{n-h} p^{n-h}$. The constant c_2 comes from the expansion

$$p(k) = 1/k^2 + c_2 k^2 + c_3 k^4 + \cdots .$$

The lower right hand block of such a matrix has the form

$$(4) \qquad \left\| \begin{matrix} p' & p'' & p''' & \cdot\;\cdot\;\cdot & p^{m-1} \\ p'' & p''' & p^{\mathrm{IV}} & \cdot\;\cdot\;\cdot & p^{m} \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ p^{n-h-1} & p^{n-h} & p^{n-h+1} & \cdot\;\cdot\;\cdot & p^{n-h+m-3} \end{matrix} \right\|$$

after the i-th row has been divided by $\binom{n-1}{i}$, and the block composed of the elements just below the $(m+1)$-th row and to the right of the $(m+1)$-th column has the form

$$(5) \qquad \left\| \begin{matrix} p' & p'' & p''' & \cdot\;\cdot\;\cdot & p^{m-1} \\ p'' & p''' & p^{\mathrm{IV}} & \cdot\;\cdot\;\cdot & p^{m} \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ p^{h-1} & p^{h} & p^{h+1} & \cdot\;\cdot\;\cdot & p^{h+m-3} \end{matrix} \right\|$$

when the i-th row has been divided by $\binom{h-1}{i}$. To investigate the rank of such a matrix we substitute for each element the first term of its expansion about $k=0$ and get

$$\left\| \begin{matrix} \dfrac{-2\,!}{u^3} & \dfrac{3\,!}{u^4} & \dfrac{-4\,!}{u^5} & \cdots & \dfrac{(-1)^{m-1}m\,!}{u^{m+1}} \\ \dfrac{3\,!}{u^4} & \dfrac{-4\,!}{u^5} & \dfrac{5\,!}{u^6} & \cdots & \dfrac{(-1)^{m}(m+1)\,!}{u^{m+2}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{(-1)^{s-1}s\,!}{u^{s+1}} & \dfrac{(-1)^{s}(s+1)\,!}{u^{s+2}} & \dfrac{(-1)^{s+1}(s+2)\,!}{u^{s+3}} & \cdots & \dfrac{(-1)^{m+s-2}(m+s-2)\,!}{u^{m+s+1}} \end{matrix} \right\|,$$

where $s = n - h$ in the matrix (4) and $s = h$ in (5). A simple manipulation gives

$$\left\| \begin{matrix} 1/u^3 & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\ 0 & 1/u^5 & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\ 0 & 0 & 1/u^7 & \cdots & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1/u^{s+\epsilon} & \cdot & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1/u^{s+2+\epsilon} & \cdots \end{matrix} \right\|.$$

where $\epsilon = 1$, s even, $\epsilon = 0$, s odd.

Therefore the matrices (4) and (5) can be put into such form as to have zero elements below and to the left of the diagonal line, and diagonal elements which are non-vanishing in general. This means that the last row of the matrix $M_{m,n+1}$ (i. e., the row added to M_{mn}) can be so manipulated as to have zero elements to the left of the diagonal line, and, as diagonal element, a function which is non-vanishing for an arbitrary choice of k. Hence the rank of $M_{m,n+1}$ is one more than the rank of M_{mn}.

If m, instead of n, is increased by 1, the new matrix has one new row, the m-th, and three new columns, the m-th, $2m$-th, and $3m$-th. We write down the matrix M_{65}. Deleting the elements in heavy type gives the matrix M_{55}.

$$M_{65} = \left\| \begin{matrix} 1 & 0 & 0 & 2\,!\,c_2 & 0 & \mathbf{4!\,c_3} & p & p''/2 & p'''/3 & (2\,!\,c_2+p^{\mathrm{IV}}/4) & p^{\mathrm{V}}/5 & \mathbf{4!\,c_3+p''/6} \\ 0 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & p & \binom{2}{1}p' & \binom{3}{1}p'' & \binom{4}{1}p''' & \binom{5}{1}\mathbf{p^{IV}} \\ 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & p & \binom{3}{2}p' & \binom{4}{2}p'' & \binom{5}{2}\mathbf{p'''} \\ 0 & 0 & 0 & 1 & 0 & \mathbf{0} & 0 & 0 & 0 & p & \binom{4}{3}p' & \binom{5}{3}\mathbf{p''} \\ 0 & 0 & 0 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 & p & \binom{5}{4}\mathbf{p'} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{p} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 1 & p & p' & p'' & p''' & \mathbf{p^{IV}} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & p' & p'' & p''' & p^{\mathrm{IV}} & \mathbf{p^{V}} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{matrix} \right\|$$

$$
\left.\begin{array}{cccccc}
p''' & p^{\mathrm{V}}/2 & p^{\mathrm{VI}}/3 & (2!\,c_2+p^{\mathrm{VII}}/4) & p^{\mathrm{VIII}}/5 & 4!\,c_3+p^{\mathrm{IX}}/6 \\
0 & p''' & \binom{2}{1}p^{\mathrm{IV}} & \binom{3}{1}p^{\mathrm{V}} & \binom{4}{1}p^{\mathrm{VI}} & \binom{5}{1}p^{\mathrm{VII}} \\
0 & 0 & p''' & \binom{3}{2}p^{\mathrm{IV}} & \binom{4}{2}p^{\mathrm{V}} & \binom{5}{2}p^{\mathrm{VI}} \\
0 & 0 & 0 & p''' & \binom{4}{3}p^{\mathrm{IV}} & \binom{5}{3}p^{\mathrm{V}} \\
0 & 0 & 0 & 0 & p''' & \binom{5}{4}p^{\mathrm{IV}} \\
0 & 0 & 0 & 0 & 0 & p''' \\
0 & \binom{4}{3}p''' & \binom{4}{3}p^{\mathrm{IV}} & \binom{4}{3}p^{\mathrm{V}} & \binom{4}{3}p^{\mathrm{VI}} & \binom{4}{3}p^{\mathrm{VII}} \\
0 & \binom{4}{4}p^{\mathrm{IV}} & \binom{4}{4}p^{\mathrm{V}} & \binom{4}{4}p^{\mathrm{VI}} & \binom{4}{4}p^{\mathrm{VII}} & \binom{4}{4}p^{\mathrm{VIII}} \\
1 & \binom{4}{0}p & \binom{4}{0}p' & \binom{4}{0}p'' & \binom{4}{0}p''' & \binom{4}{0}p^{\mathrm{IV}} \\
0 & \binom{4}{1}p' & \binom{4}{1}p'' & \binom{4}{1}p''' & \binom{4}{1}p^{\mathrm{IV}} & \binom{4}{1}p^{\mathrm{V}} \\
0 & \binom{4}{2}p'' & \binom{4}{2}p''' & \binom{4}{2}p^{\mathrm{IV}} & \binom{4}{2}p^{\mathrm{V}} & \binom{4}{2}p^{\mathrm{VI}}
\end{array}\right\|
$$

The new last column always falls entirely to the right of the diagonal line and so can not affect the rank; the new $2m$-th column is equally ineffective, while the new m-th row and m-th column supply a 1 as diagonal element and nothing but zero elements to the left of it and below it. Hence the rank of $M_{m+1,n}$ is one more than the rank of M_{mn}. It has been verified (p. 566) that the rank of M_{33} is $3+3=6$. Therefore, for arbitrary k, the rank of M_{mn} is $m+n$. We have, then, $m+n$ linear relations connecting the $3m$ homogeneous constants α_{ij} and, for an arbitrary value of k, ∞^{2m-n-1} m-ics perspective to C_n. That k is an independent parameter can be shown as follows.

Suppose that k and the $2m-n-1$ essential parameters of the family of m-ics corresponding to k are all functions of δ_i $(i=1, 2, 3, \cdots, 2m-n-1)$ and these only. Then $k=k(\delta_i)=0$ would imply a relation among the δ_i's and we should have, not ∞^{2m-n-1} m-ics corresponding to $k=0$, but only ∞^{2m-n-2}. We have, therefore, proved the following theorem:

THEOREM I. *There are* ∞^{2m-n} m-ics *perspective to a given n-ic. Corresponding to a given transformation of the curve into itself there are* ∞^{2m-n-1} *perspective m-ics which lie in a linear system.*

The division of the set of perspective m-ics into families corresponding to a given value of k, corresponding, that is, to a given birational transformation of the curve into itself is a peculiarity of the elliptic case. It would not occur in the case of hyperelliptic curves which are transformed into themselves by a finite number of birational transformations * and it does not occur in the rational case.

* *Pascal Repertorium*, Chapter XV, § 6.

3. *The contacts of two Perspective Curves.*

THEOREM II. *An elliptic n-ic and an m-ic perspective to it have, in general, $m+n$ contacts.*

The contacts of C_n, (2), and E_m (1), will be the zeros of a function obtained by differentiating the incidence relation $(x\xi)$ with respect to u before putting $v=u$. The coefficient of $1/u^m$, the first term of the principal part of the expansion of this function about $u=0$, is a function of k which, in general, does not vanish. The coefficient of $1/(u-k)^{n+1}$, the first term of the principal part of the expansion about $u=k$, is B_n where B_n is one of the expressions (cf. p. 559) whose vanishing is a condition of perspectivity, and the coefficient of $1/(u-k)^n$ is non-vanishing in general. Hence the function has $m+n$ poles, and $m+n$ zeros, and the curves have $m+n$ contacts, the sum of whose parameters is nk. For a fixed k these contacts lie in an involution, $I_{2m-n-1,2n-m+1}$ since $2m-n-1$ of them will determine E_m and the rest of the $m+n$ contacts.

If $n/2 < m < 2n$ and if $2m-n$ contacts are given, these, substituted in $[d(x\xi)/du]_{v=u}=0$ will give $2m-n$ homogeneous, linear equations in the $2m-n$ parameters of the family of perspective m-ics. The coefficients are elliptic functions of k, of order M_j+n+1, where M_j is the highest order in k occurring in A_{0j}, A_{1j}, and A_{2j}, where E_m is written

$$\rho\xi_i = \sum_{j=1}^{2m-2} a_i A_{ij}, \qquad (i=0,1,2),$$

and where a_j, $(j=1,2,3,\cdots,2m-n)$, are the homogeneous parameters. When $m=n=3$, (cf. p. 570), $M_j=3$ for each of the three values of j, but its value in general has not been determined. If M is the maximum value of the set M_j, the elimination of a_j gives an elliptic function of order

$$\mu \leqq (2m-n)(M+n+1).$$

Then μ is the number of sets of $2n-m$ more contacts, i. e., the number of perspective m-ics determined by the $2m-n$ given contacts.

4. *The Family of Cubics Perspective to a Given Cubic.* If in a $(1,1)$ correspondence between two cubics with three self-corresponding common points, corresponding points are joined, the envelope is a cubic perspective to both. Such a correspondence exists between a cubic and its transform by a collineation with three fixed points on the cubic, and also between a cubic and its transform by a quadratic transformation with the fundamental points and three of the fixed points on the cubic. There are, then, many perspective

cubics associated with a given cubic, and we ask what conditions on the coefficients of the general elliptic cubic envelope E_3 given by

$$\rho\xi_i = \alpha_{i0} + \alpha_{i1}p(v) + \alpha_{i2}p'(v) \qquad (i = 0, 1, 2)$$

will make it perspective to the elliptic cubic C_3, which may be written

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= p(-u+k) \\ \mu x_2 &= p'(-u+k). \end{aligned}$$

The point $u = s$ of C_3 and the line $v = s$ of E_3 are incident for all values of s if and only if, when $v = u$,

$$(x\xi) \equiv x_0\xi_0 + x_1\xi_1 + x_2\xi_2 \equiv 0.$$

This is a sixth order elliptic function with 3rd order poles at $u = 0$ and at $u = k$.

Expanded about $u = 0$, the coördinates of C_3 are

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= p(k) - p'(k)u + p''(k)u^2/2! - \cdots, \\ \mu x_2 &= p'(k) - p''(k)u + p'''(k)u^2/2! - \cdots, \end{aligned}$$

and, for E_3, the expansion about $v = 0$ is

$$\begin{aligned} \rho\xi_i = \alpha_{i0} &+ \alpha_{i1}(1/v^2 + c_2v^2 + c_3v^4 + \cdots) \\ &+ \alpha_{i2}(-2!/v^3 + 2c_2v + 4c_3v^3 + \cdots), \quad (i = 0, 1, 2), \end{aligned}$$

so that the expansion for $(x\xi)_{v=u}$ about $u = 0$ will be

$$\begin{aligned} &(1/u^3)\{-2[\alpha_{02} + \alpha_{12}p(k) + \alpha_{22}p'(k)]\} \\ &+ (1/u^2)\{\alpha_{01} + \alpha_{11}p(k) + \alpha_{21}p'(k) + 2[\alpha_{12}p(k) + \alpha_{22}p'(k)]\} \\ &+ (1/u)\{-[\alpha_{11}p'(k) + \alpha_{21}p''(k) + \alpha_{12}p''(k) + \alpha_{22}p'''(k)]\} \\ &+ \{\alpha_{00} + \alpha_{10}p(k) + \alpha_{20}p'(k) \\ &+ (1/2)[\alpha_{11}p''(k) + \alpha_{21}p'''(k)] + (1/3)[\alpha_{11}p'''(k) + \alpha_{22}p^{\mathrm{IV}}(k)]\} \\ &\qquad + \text{a power series in } u, \end{aligned}$$

or, for brevity,

$$H_3/u^3 + H_2/u^2 + H_1/u + H_0 + \text{a power series in } u.$$

Expanded about $v = k$, $u = k$

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= 1/(u-k)^2 + c_2(u-k)^2 + c_3(u-k)^4 + \cdots \\ \mu x_2 &= 2!/(u-k)^3 - 2c_2(u-k) - 4c_3(u-k)^3 - \cdots \end{aligned}$$

and

$$\rho\xi_i = \alpha_{i0} + \alpha_{i1}[p(k) + p'(k)(u-k) + p''(k)(u-k)^2/2! + \cdots]$$
$$+ \alpha_{i2}[p'(k) + p''(k)(u-k) + p'''(k)(u-k)^2/2! + \cdots],$$
$$(i = 0, 1, 2),$$

so that the expansion of $(x\xi)_{v=u}$ is

$$[1/(u-k)^3]\{2[\alpha_{20} + \alpha_{21}p(k) + \alpha_{22}p'(k)]\}$$
$$+ [1/(u-k)^2]\{\alpha_{10} + \alpha_{11}p(k) + \alpha_{12}p'(k) + 2[\alpha_{21}p'(k) + \alpha_{22}p''(k)]\}$$
$$+ [1/(u-k)]\{\alpha_{11}p'(k) + \alpha_{12}p''(k) + \alpha_{21}p''(k) + a_{22}p'''(k)]\}$$
$$+ \{\alpha_{00} + \alpha_{01}p(k) + \alpha_{02}p'(k)$$
$$+ (1/2)[\alpha_{11}p''(k) + \alpha_{12}p'''(k)] + (1/3)[\alpha_{21}p'''(k) + \alpha_{22}p^{\mathrm{IV}}(k)]\}$$
$$+ \text{ a power series in } (u-k).$$

or

$K_3/(u-k)^3 + K_2/(u-k)^2 + K_1/(u-k) + K_0 +$ a power series in $(u-k)$.

Equating to zero the coefficients H_j and K_j $(j = 1, 2, 3)$ of the principal parts of these expansions gives the condition that $(x\xi)_{v=u}$ be a constant, and, by requiring this constant to be zero, we shall have the condition that E_3 be perspective to C_3.

The eight equations

$$H_j = 0, \qquad K_j = 0, \qquad\qquad (j = 0, 1, 2, 3),$$

which are linear and homogeneous in α_{ij} are not independent, (cf. p. 559). The sum of the residues is zero. Therefore

$$H_1 + K_1 \equiv 0$$

and we find that the relation

$$(H_0 - K_0) + p(k)(H_2 - K_2) - [p'(k)/2!](H_3 + K_3) \equiv 0$$

holds.

The six equations

(6) $H_0 = 0, \quad H_2 = 0, \quad -H_3/2 = 0, \quad K_2 = 0, \quad K_1 = 0, \quad K_3/2 = 0$

have the matrix

$$\left\Vert \begin{array}{ccccccccc} 1 & 0 & 0 & p & p''/2 & p'''/3 & p' & p'''/2 & p^{\mathrm{IV}}/3 \\ 0 & 1 & 0 & 0 & p & 2p' & 0 & p' & 2p'' \\ 0 & 0 & 1 & 0 & 0 & p & 0 & 0 & p' \\ 0 & 0 & 0 & 1 & p & p & 0 & 2p' & 2p'' \\ 0 & 0 & 0 & 0 & p' & p'' & 0 & p'' & p''' \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p & p' \end{array} \right\Vert$$

Since $p'(k)$ and $p''(k)$, in the fifth row of the matrix, have no common zeros, the matrix has rank six, so that the equations (6) give six independent homogeneous linear conditions on the nine coefficients α_{ij} of E_3. Therefore, for every value of k, there are ∞^2 class cubics perspective to C_3. We have already shown (p. 562) that k is an independent parameter, so that there are ∞^3 cubics E_3 perspective to C_3.

Solving the equations (6) we find that these perspective cubics are

$$(7) \qquad \rho\xi_i = aA_i(k, v) + bB_i(k, v) + cC_i(k, v), \qquad (i = 0, 1, 2),$$

where

$$a = \alpha_{12} - \alpha_{21}, \qquad b = \alpha_{02} - \alpha_{20}, \qquad c = \alpha_{01} - \alpha_{10},$$

and where

$$\begin{aligned}
A_0(k, v) &= 3g_3p(k) + g_2^2/4 + [g_2p(k) + 3g_3]\,p(v) \\
A_1(k, v) &= g_2p(k) + 3g_3 + g_2p(v) + p'(k)p'(v) \\
A_2(k, v) &= p'(k)p(v) + p'(v) \\
B_0(k, v) &= g_2p(k) + 3g_3 + g_2p(v) - p'(k)p'(v) \\
B_1(k, v) &= g_2 - 12p(k)p(v) \\
B_2(k, v) &= -p'(k) + p'(v) \\
C_0(k, v) &= -p'(k)p(v) - p(k)p'(v) \\
C_1(k, v) &= p'(k) + p'(v) \\
C_2(k, v) &= p(k) - p(v).
\end{aligned}$$

If $a = b = 0$, $c = 1$, E_3 becomes $\rho\xi_i = C_i(k, v)$, i. e., the point

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & p(-k) & p'(-k) \\ 1 & p(v) & p'(v) \end{vmatrix} = 0,$$

which is the point $u = 2k$ of C_3 doubly covered.

These cubics may also be written

$$(7') \qquad \rho\xi_i = l_i(a, b, c, v) + p(k)m_i(a, b, c, v) + p'(k)n_i(a, b, c, v), \qquad (i = 0, 1, 2),$$

where

$$\begin{aligned}
l_0 &= ag_2^2/4 + 3bg_3 + [3ag_3 + bg_2]\,p(v) \\
l_1 &= 3ag_3 + bg_2 + ag_2p(v) + cp'(v) \\
l_2 &= -cp(v) + bp'(v) \\
m_0 &= 3ag_3 + bg_2 + ag_2p(v) - cp'(v) \\
m_1 &= ag_2 - 12bp(v) \\
m_2 &= c - ap'(v)
\end{aligned}$$

$$\begin{array}{llll} n_0 = & & cp(v) & + bp'(v) \\ n_1 = -c & & & - ap'(v) \\ n_2 = & b & - ap(v). & \end{array}$$

If, in (7′), we expand in powers of k, multiply throughout by $-k^3/3$ and then put $k = 0$ we get

$$\rho\xi_i = n_i(a, b, c, v), \qquad (i = 0, 1, 2),$$

i. e., the point

$$(8) \qquad \begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & p(-v) & p'(-v) \\ a & b & c \end{vmatrix} = 0,$$

or, as a, b and c vary, the ∞^2 points of the plane triply covered.

These triply covered points may also be obtained as follows. Putting $k = 0$ in C_3 and writing the coördinates.

$$\mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the incidence relation becomes

$$\alpha_{00} + (\alpha_{10} + \alpha_{01})p(u) + (\alpha_{20} + \alpha_{02})p'(u) \\ + \alpha_{11}p^2(u) + (\alpha_{21} + \alpha_{12})p(u)p'(u) + \alpha_{22}p'^2(u),$$

which vanishes identically when and only when

$$\alpha_{ij} = -\alpha_{ji}$$

and

$$\alpha_{ij} = 0 \quad \text{when} \quad i = j,$$

or when E_3 is

$$\begin{array}{llll} \rho\xi_0 = & \alpha_{01}p(v) + \alpha_{02}p'(v) & = n_0(\alpha_{12}, \alpha_{02}, \alpha_{01}, v) \\ \rho\xi_1 = -\alpha_{01} & - \alpha_{12}p'(v) & = n_1(\alpha_{12}, \alpha_{02}, \alpha_{01}, v) \\ \rho\xi_2 = -\alpha_{02} + \alpha_{12}p(v) & & = n_2(\alpha_{12}, \alpha_{02}, \alpha_{01}, v), \end{array}$$

i. e., when E_3 is one of the points (8).

If two cubics, given respectively by

$$(9) \qquad \rho\xi_i = aA_i(k, -v + k) + bB_i(k, -v + k) + cC_j(k, -v + k)$$

$$(10) \qquad \rho\xi_i = a'A_i(k', -v + k') + b'B_i(k', -v + k') + c'C_i(k', -v + k'), \\ (i = 0, 1, 2),$$

are each perspective to C_3:

$$(11) \qquad \mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the intersection of the line $v = s$ of the one and the line $v = s$ of the other

will give the point $u=s$ of C_3. But this fact is not easy to verify algebraically.

Rewriting (9) and (10) respectively as

$$(9') \qquad \rho\xi_i = a\alpha_i(k,v) + b\beta_i(k,v) + c\gamma_i(k,v) = a\alpha_i + b\beta_i + c\gamma_i$$

and

$$(10') \qquad \rho\xi_i = a'\alpha_i(k',v) + b'\beta_i(k',v) + c'\gamma_i(k',v) = a'\alpha_i' + b'\beta_i' + c'\gamma_i',$$

(where $\alpha_i(k,v) \equiv A_i(k,-v+k)$ and is obtained from it by means of the addition formulae for $p(-v+k)$ and $p'(-v+k)$, and where β_i and γ_i are new forms for B_i and C_i similarly obtained) the generated curve will be

$$\begin{aligned}
\mu x_i = aa' &\begin{vmatrix} \alpha_j & \alpha_j' \\ \alpha_k & \alpha_k' \end{vmatrix} + bb' \begin{vmatrix} \beta_j & \beta_j' \\ \beta_k & \beta_k' \end{vmatrix} + cc' \begin{vmatrix} \gamma_j & \gamma_j' \\ \gamma_k & \gamma_k' \end{vmatrix} \\
(12) \qquad + bc' &\begin{vmatrix} \beta_j & \gamma_j' \\ \beta_k & \gamma_k' \end{vmatrix} + ca' \begin{vmatrix} \gamma_j & \alpha_j' \\ \gamma_k & \alpha_k' \end{vmatrix} + ab' \begin{vmatrix} \alpha_j & \beta_j' \\ \alpha_k & \beta_k' \end{vmatrix} \\
- b'c &\begin{vmatrix} \beta_j' & \gamma_j \\ \beta_k' & \gamma_k \end{vmatrix} - c'a \begin{vmatrix} \gamma_j' & \alpha_j \\ \gamma_k' & \alpha_k \end{vmatrix} - a'b \begin{vmatrix} \alpha_j' & \beta_j \\ \alpha_k' & \beta_k \end{vmatrix},
\end{aligned}$$

where $i, j, k = 0, 1, 2$, no two of the three being equal. We shall have proved that this is the curve C_3 if we can show that each of the nine curves obtained by putting equal to zero any four of the six constants a, b, c, a', b', c' is identical with C_3. The work involved in each case is, however, exceedingly troublesome and has been carried through only in a few of the simpler cases when $k=0$ or a proper half period.

When $k=k'$, some of the determinants in (12) vanish and others duplicate each other. Moreover the work can be simplified by writing (9) and (10) as

$$(9'') \qquad \rho\xi_i = aA_i(k,v) + bB_i(k,v) + cC_i(k,v)$$

$$(10'') \qquad \rho\xi_i = a'A_i(k',v) + b'B_i(k',v) + c'C_i(k',v),$$

and writing C_3

$$\mu x_0 = 1, \qquad \mu x_1 = p(-u+k), \qquad \mu x_2 = p'(-u+k).$$

The generated curve will be

$$(13) \qquad x_0 : x_1 : x_2 =$$

$$\begin{vmatrix} M_{A_0} & M_{B_0} & M_{C_0} \\ a & b & c \\ a' & b' & c' \end{vmatrix} : \begin{vmatrix} M_{A_1} & M_{B_1} & M_{C_1} \\ a & b & c \\ a' & b' & c' \end{vmatrix} : \begin{vmatrix} M_{A_2} & M_{B_2} & M_{C_2} \\ a & b & c \\ a' & b' & c' \end{vmatrix},$$

where M_{A_0} is the minor of $A_0(k, v)$ in the matrix

$$(14) \qquad \left\| \begin{matrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{matrix} \right\| .$$

If

$$(15) \qquad M_{A_0} : M_{A_1} : M_{A_2} = M_{B_0} : M_{B_1} : M_{B_2} \\ = M_{C_0} : M_{C_1} : M_{C_2} = 1 : p(-u+k) : p'(-u+k),$$

then

$$x_0 : x_1 : x_2 = 1 : p(-u+k) : p'(-u+k).$$

The verification of (15) can be done by forming proportions of the types

$$(16) \qquad M_{A_1}/M_{A_0} = p(-v+k)/1$$

$$(17) \qquad M_{A_2}/M_{A_1} = p'(-v+k)/p(-v+k)$$

and showing, by means of the identity

$$p'^2(k) = 4p^3(k) - g_2 p(k) - g_3$$

that the coefficients of $[p(v)]^j$, $(j = 0, 1, 2, 3, 4, 5)$, and of $p'(v)[p(v)]^i$, $(i = 0, 1, 2)$, vanish, together with the constant terms. This has been done for the proportion (16).

When $k = 0$, the matrix (14) will be

$$\left\| \begin{matrix} 0 & -p'(v) & -p(v) \\ p'(v) & 0 & 1 \\ p(v) & -1 & 0 \end{matrix} \right\|$$

so that for (15) we have

$$1 : p(v) : -p'(v) = p(v) : p^2(v) : -p(v)p'(v) \\ = p'(v) : p(v)p'(v) : -p'^2(v) = 1 : p(-v+0) : p'(-v+0).$$

For $k =$ a proper half period we have carried through the verification of (15) completely with the help of the identities

$$e_1 + e_2 + e_3 = 0, \quad g_2 = 4(e_1e_2 + e_2e_3 + e_3e_1), \quad g_3 = 4e_1e_2e_3$$

and of

$$4e_1^3 - g_2e_1 - g_3 = p'^2(\omega_1/2) = 0,$$
$$12e_1^3 - g_2e_1 = 2g_2e_1 + 3g_3 = 4e_1(2e_1^2 + e_2e_3) \\ = 4e_1(e_1 - e_2)(e_1 - e_3) = 2e_1p''(\omega_1/2),$$
$$p(-v + \omega_1/2) = e_1 + (e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]$$
$$p'(-v + \omega_1/2) = p'(v)(e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]^2$$

and similar expressions involving $\omega_2/2$ and e_3 or $(\omega_1 + \omega_2)/2$ and e_2. We

give a sample of the calculation here because it seems to show a sort of raison d'etre for the complicated functions of g_2, g_3, and k that appear in (7).

When $k = \omega_1/2$,

$$\begin{aligned}
\mu x_0 = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} &= \begin{vmatrix} g_2 - 12e_1 p(v) & p'(v) \\ p'(v) & e_1 - p(v) \end{vmatrix} \\
&= -\{[p(v) - e_1][g_2 - 12e_1 p(v)] + p'^2(v)\} \\
&= -4[p(v) - e_1]\{e_1^2 - e_2 e_3 - 3e_1 p(v) \\
&\quad + [p(v) - e_2][p(v) - e_3]\} \\
&= -4[p(v) - e_1]^3,
\end{aligned}$$

$$\begin{aligned}
\mu x_1 = \begin{vmatrix} B_2 & C_2 \\ B_0 & C_0 \end{vmatrix} &= \begin{vmatrix} p'(v) & e_1 - p(v) \\ g_2 e_1 + 3g_3 + g_2 p(v) & -e_1 p'(v) \end{vmatrix} \\
&= e_1[4p^3(v) - g_2 p(v) - g_3] \\
&\quad - g_2 e_1^2 - 3g_3 e_1 + 3g_3 p(v) + g_2 p^2(v) \\
&= -4e_1 p^3(v) + g_2[p^2(v) + e_1 p(v) - e_1^2] \\
&\quad + g_3[3p(v) - 2e_1] \\
&= -4\{e_1 p^3(v) + (e_2 e_3 - e_1^2)[p^2(v) + e_1 p(v) - e_1^2] \\
&\quad - e_1 e_2 e_3[3p(v) - 2e_1]\} \\
&= -4\{e_1[p(v) - e_1]^3 + (2e_1^2 + e_2 e_3)[p(v) - e_1]^2\} \\
&= -4[p(v) - e_1]^3\{e_1 + (e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]\} \\
&= -4[p(v) - e_1]^3 p(-v + \omega_1/2),
\end{aligned}$$

$$\begin{aligned}
\mu x_2 = \begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} &= \begin{vmatrix} g_2 e_1 + 3g_3 + g_2 p(v) & -e_1 p'(v) \\ g_2 - 12e_1 p(v) & p'(v) \end{vmatrix} \\
&= p'(v)\{2g_2 e_1 + 3g_3 - (12e_1^2 - g_2)p(v)\} \\
&= p'(v)\{4e_1(2e_1^2 + e_2 e_3) - 4(e_1^2 + e_2 e_3)p(v)\} \\
&= -4[p(v) - e_1]^3 p'(v)(e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]^2 \\
&= -4[p(v) - e_1]^3 p'(-v + \omega_1/2).
\end{aligned}$$

5. *The Contacts of Perspective Cubics.* Two perspective cubics have 6 contacts, by Theorem II, p. 563. If k is known and C_3 given, two contacts determine the ratios $a : b : c$ in (7), and so fix E_3 and the other four contacts.

If three contacts are given,

$$u = u_i, \qquad (i = 1, 2, 3),$$

then $[d(x\xi)/du]_{v=u=u_i}$ gives the three equations

$$\begin{aligned}
&[ag_2 p(k) + 3g_3 + bg_2 - cp'(k)]p'(u_i - k) + [ag_2 - 12bp(k)]p(u_i)p'(u_i - k) \\
&\quad + [ap'(k) + c]p'(u_i)p'(u_i - k) + [bp'(k) + cp(k)]p''(u_i - k) \\
&\quad + [ap'(k) - c]p(u_i)p''(u_i - k) + [-ap(k) + b]p'(u_i)p''(u_i - k).
\end{aligned}$$

We have underlined the terms of highest order in k occurring in the coefficient of a of b or of c, showing that the coefficients of a and of b are 7th order elliptic functions in k, while that of c is of the 6th order. Elimination of a, b, c gives, therefore, an elliptic function of order 20 in k. This means that, when three contacts are given, twenty sets of three more contacts, or twenty perspective cubics, are determined.

6. *Perspective Cubics Associated with Collineation and with Quadratic Transformation.* We naturally wish to compare the set of cubics (7) with the sets obtained by joining the points of C_3 with the corresponding points of its transforms by a quadratic transformation with fundamental points and three fixed points on C_3 and by a collineation with fixed points on C_3. An expression for the cubics in the latter case has been obtained in terms of sigma functions and it is shown that they are ∞^3 in number. The method is as follows.

Putting on the collineation the condition that it leave fixed the lines joining the fixed points u_0, u_1 and u_2 of C_3, which is written

$$\mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the desired envelope is expressed in terms of D_i and certain minors of D_i, where D_i is the determinant obtained from

$$D = \begin{vmatrix} 1 & p(u_0) & p'(u_0) \\ 1 & p(u_1) & p'(u_1) \\ 1 & p(u_2) & p'(u_2) \end{vmatrix}$$

by substituting the row $1, p(u), p'(u)$ for the row $1, p(u_i)\ p'(u_i)$.

Expressions for D_i and for the minor

$$\begin{vmatrix} 1 & p(u) \\ 1 & p(u_i) \end{vmatrix}$$

in terms of sigma functions are well known. The other minors needed were obtained by putting first $u_j = \omega_1/2$ and then $u_j = \omega_2/2$ $(j \neq i)$ in D_i and solving simultaneously the two equations so obtained. In this way the perspective envelopes can be expressed, in a rather unpromising manner, in terms of sigma functions. They involve, besides u_0, u_1 and u_2 three homogeneous parameters, and, by setting two of the latter equal to zero it is easy to show that there are just three effective parameters.

In the case of quadratic transformation, by considering the condition on A_1, A_2, and A_3, the fundamental points, and on B_1, B_2, and B_3, the fixed points on C_3, that A_1B_1, A_2B_2, A_3B_3 be concurrent, a proof is obtained that

there are ∞^4 quadratic transformations of this type. By noting that the points of C_3 which lie on a conic through A_1, A_2 and A_3 will be collinear on the transform of C_3, we obtain k in terms of the parameters of A_1, A_2, and A_3. This makes it necessary to exclude from among the permissible values of k, those values for which A_1, A_2, and A_3 (or B_1, B_2, and B_3) are collinear. We find, in fact, that $k = 0$ or a proper third period cannot be associated with curves obtained in this manner.

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CONJUGATE NETS AND THE LINES OF CURVATURE.

By Ernest P. Lane.

1. *Introduction.* The purpose of this paper is twofold; first, to make some contributions to the projective differential geometry of a general conjugate net on an analytic surface in ordinary space; second, to connect this geometry with the metric differential geometry of the lines of curvature.

Two sections are concerned with purely projective geometry. An analytic basis for the projective differential theory of a parametric conjugate net on a surface in ordinary space is established in Section 2, where the parts of this theory that are essential for what follows are summarized. In Section 3, as one of the principal contributions of this paper, two quadrics called *conjugate osculating quadrics* are associated with each point of a given curve on the surface sustaining a given conjugate net. The reader who is familiar with the *asymptotic osculating quadrics,* defined by Bompiani and Klobouček, will observe that the definitions of the two kinds of quadrics are quite similar. An asymptotic osculating quadric at a point of a curve on a surface is determined by three consecutive asymptotic tangents of one family, whereas in defining a conjugate osculating quadric we shall use instead three consecutive tangents of the curves of one family of a given conjugate net. The equations of these quadrics will be derived and their properties briefly studied.

The last three sections are more or less metric in character. Section 4 is taken up with a summary of the elements of the metric differential geometry of the lines of curvature on a surface, preparatory to what follows. In Section 5, as a second contribution of this paper, a transformation is devised, *from* the local projective homogeneous coördinates based on a certain projectively covariant tetrahedron at a general point of a surface, which are used in the projective theory of a general conjugate net, *to* the local cartesian non-homogeneous coördinates based on a certain metrically covariant trihedron at a general point of a surface, which are used in the metric theory of the lines of curvature. This transformation makes it possible to study in Section 6 for the particular metrically defined conjugate net called the lines of curvature various configurations that are considered in the projective theory of a general conjugate net.

2. *Conjugate Nets.* In this section we establish an analytic basis for

the projective differential geometry of a parametric conjugate net on a surface in ordinary space. Some portions of this geometry, which will be needed later on, are summarized. The union curves of the axis congruence of a conjugate net appear. The equations of the ray-point cubic and ray conic of a pencil of conjugate nets are written, as well as the equation of the quadric of Lie. By *net* we shall mean *conjugate net* unless otherwise indicated.

The projective differential geometry of conjugate nets was studied by G. M. Green, who based his theory* on a system of differential equations of the form

$$\begin{aligned} y_{uu} &= a y_{vv} + b y_u + c y_v + d y, \\ y_{uv} &= \phantom{a y_{vv} +{}} b' y_u + c' y_v + d' y. \end{aligned} \tag{1}$$

The obvious lack of symmetry in these equations has led Slotnick to use† a different system. We shall use here a system which differs only notationally and in the choice of proportionality factor from that of Slotnick.

Let the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x on a surface S referred to a conjugate net N_x in ordinary space be given as analytic functions of two independent variables u, v. The axis at the point P_x of the net N_x is the line of intersection of the osculating planes of the parametric curves C_u, C_v at P_x. Let P_y be the point which is the harmonic conjugate of P_x with respect to the two foci of the axis regarded as generating a congruence (the axis congruence) when the point P_x varies over the surface S. Then x and y satisfy a system of equations of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + a x_u + b x_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0). \end{aligned} \tag{2}$$

We shall use this system as the basis of our projective theory.

From the equations

$$(x_{vv})_u = (x_{uv})_v, \qquad (x_{uu})_v = (x_{uv})_u$$

we obtain

$$y_u = fx - n x_u + s x_v + Ay, \qquad y_v = gx + t x_u + n x_v + By, \tag{3}$$

where we have placed

* Green, "Projective Differential Geometry of One-Parameter Families of Curves, and Conjugate Nets on a Curved Surface," First Memoir, *American Journal of Mathematics*, Vol. 37 (1915), p. 215; Second Memoir, Vol. 38 (1916), p. 287.

† Slotnick, "On the Projective Differential Geometry of Conjugate Nets," *American Journal of Mathematics*, Vol. 53 (1931), p. 143.

$$
(4)\quad
\begin{aligned}
fN &= c_v + ac + bq - c\delta - qu, & gL &= c_u + bc + ap - c\alpha - p_v,\\
-nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v,\\
sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p,\\
A &= b - (\log N)_u, & B &= a - (\log L)_v.
\end{aligned}
$$

From the equation $(y_u)_v = (y_v)_u$ we could obtain in light of the fact that the four points x, x_u, x_v, y are not coplanar, four integrability conditions, which we do not need to write here.

The ray-points, or Laplace transformed points, ρ, σ of the curves C_u, C_v respectively at the point P_x are defined by the formulas

$$
(5)\qquad \sigma = x_v - ax, \qquad \rho = x_u - bx.
$$

The Laplace-Darboux point invariants H, K, the tangential invariants $\boldsymbol{H}$, $\boldsymbol{K}$, the Weingarten invariants $W^{(u)}$, $W^{(v)}$, the invariants $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}$ of Green, and the invariant r of Eisenhart * are expressed in terms of the coefficients of systems (2) by the formulas

$$
(6)\quad
\begin{aligned}
H &= c + ab - a_u, & K &= c + ab - b_v,\\
\boldsymbol{H} &= sN, & \boldsymbol{K} &= tL,\\
W^{(u)} &= \boldsymbol{H} - K, & W^{(v)} &= \boldsymbol{K} - H,\\
8\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & 8\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u,\\
\mathfrak{D} &= -2nL, & r &= N/L.
\end{aligned}
$$

The invariant r is the reciprocal of Green's invariant a.

We shall have occasion to use the covariant tetrahedron x, ρ, σ, y as a local tetrahedron of reference with a unit point chosen so that a point

$$
x_1 x + x_2 \rho + x_3 \sigma + x_4 y
$$

has local coördinates proportional to $x_1, \cdots, x_4$. In this coördinate system the equations of the osculating planes of the curves C_u, C_v at the point P_x are respectively $x_3 = 0$ and $x_2 = 0$. The equation of the osculating plane of the curve C_λ of the family defined by the equation

$$
(7)\qquad dv - \lambda du = 0
$$

at the point P_x is found by making use of the fact that this plane is determined by the points x, x', x'', where

$$
(8)\quad x' = x_u + x_v\lambda, \quad x'' = x_{uu} + 2x_{uv}\lambda + x_{vv}\lambda^2 + x_v\lambda' \qquad (\lambda' = \lambda_u + \lambda\lambda_v).
$$

Th[illegible]

$$
(9)\quad (L + N\lambda^2)(x_3 - \lambda x_2) - \lambda\left[\tfrac{1}{4}(\mathfrak{C}' - \lambda\mathfrak{B}') + (\log \lambda r^{1/2})'\right] x_1 = 0,
$$

* Eisenhart, *Transformations of Surfaces*, Princeton University Press, [illegible]

wherein $\mathfrak{B}'$, $\mathfrak{C}'$ are two invariants appearing in equations (6). This plane coincides with the tangent plane, $x_4 = 0$, at each point of C_λ if, and only if, C_λ is an asymptotic curve; in this case λ satisfies the equation

$$L + N\lambda^2 = 0. \tag{10}$$

The union curves of the axis congruence of a net are those curves such that at each point of one of them its osculating plane contains the axis through the point. The curve C_λ is a union curve of the axis congruence of the net N_x in case the coefficient of x_4 in equation (9) vanishes. This condition can be written in the form

$$\lambda' = -[4\mathfrak{C}' + (\log r^{1/2})_u]\lambda + [4\mathfrak{B}' - (\log r^{1/2})_v]\lambda^2. \tag{11}$$

If λ is replaced by dv/du this equation becomes an equation of the second order for v as a function of u along a union curve. So we obtain the well-known result that there is a two-parameter family of union curves of a given axis congruence. The fact that equation (11) is an equation of Riccati for λ leads to the remark that the cross ratio of four particular solutions is constant, and hence to the following theorem, apparently unobserved before and capable of generalization to any two-parameter family of hypergeodesics containing the parametric curves on a surface:

Any four one-parameter families of union curves of the axis congruence of a conjugate net on a surface in ordinary space have the property that the cross ratio of the tangents of the four curves of the families at a point of the surface is the same at every point of the surface.

Replacing λ by dv/du in equation (10) we obtain *the differential equation of the asymptotic curves* on the surface S, namely,

$$Ldu^2 + Ndv^2 = 0. \tag{12}$$

In order that the differential equation

$$(dv - \lambda du)(dv - \mu du) = 0 \tag{13}$$

may represent a conjugate net the two directions defined by this equation must separate harmonically the two asymptotic directions satisfying equation (12). A condition necessary and sufficient therefor is the following, which we shall suppose satisfied whenever we employ the function μ hereinafter:

$$\mu = -1/\lambda r \qquad (r = N/L). \tag{14}$$

The two curves of such a conjugate net, at a point of a surface, may be designeted as C_λ, C_μ respectively.

The ray-points ρ_λ, σ_μ of the curves C_λ, C_μ at a point P_x can be shown * to be represented by the formulas

$$\begin{aligned}(15)\qquad \rho_\lambda &= (1+\lambda^2 r)(\rho+\mu\sigma) - [\lambda P - Q + (\log \lambda^2 r)']\, x,\\ \sigma_\mu &= (1+\lambda^2 r)(\rho+\lambda\sigma) + \lambda(P+\lambda r Q)x,\end{aligned}$$

wherein P, Q are defined by placing

$$(16)\qquad P = 4\mathfrak{B}' - (\log \lambda r^{1/2})_v, \quad Q = 4\mathfrak{C}' + (\log \lambda r^{1/2})_u.$$

The equations of *the ray-point cubic* of the pencil of conjugate nets determined by the conjugate net (13) at a point P_x are †

$$\begin{aligned}(17)\qquad x_4 = (x_2^2 + r x_3^2)(x_1 - \mathfrak{C}' x_2 - \mathfrak{B}' x_3)&\\ + \mathfrak{C}' x_2^3 - 3\mathfrak{B}' x_2^2 x_3 - 3r\mathfrak{C}' x_2 x_3^2 + r\mathfrak{B}' x_3^3 &= 0,\end{aligned}$$

and the equations of *the ray conic* of the pencil are

$$(18)\qquad x_4 = (\mathfrak{B}'^2 + r\mathfrak{C}'^2)(x_2^2 + r x_3^2) - r(x_1 - \mathfrak{C}' x_2 - \mathfrak{B}' x_3)^2 = 0.$$

The equation of any quadric of Darboux at a point P_x of a surface S referred to a conjugate net N_x is

$$(19)\qquad L x_2^2 + N x_3^2 + x_4(-2x_1 + 4\mathfrak{C}' x_2 + 4\mathfrak{B}' x_3 + k x_4) = 0,$$

where k is arbitrary. For *the quadric of Lie* the value of k is given ‡ by

$$(20)\qquad LNk = 2(L\mathfrak{B}'^2 + N\mathfrak{C}'^2) + L\mathfrak{B}'(\log \mathfrak{B}' r^{1/2})_v + N\mathfrak{C}'(\log \mathfrak{C}'/r^{1/2})_u.$$

The entire theory of surfaces in ordinary space can, of course, be developed on the basis of a parametric conjugate net, in spite of the fact that it may ordinarily be more convenient to take the asymptotic curves as parametric. For some further developments in this direction the reader may consult the last two references cited.

3. *The Conjugate Osculating Quadrics.* In order to formulate a definition let us consider a conjugate net N on a surface S in ordinary space, and on S any curve C not belonging to N. At a point P of the curve C, and at two neighboring points P_1, P_2 on C, let us construct the tangents of the curves of one family of the net N. These three tangents determine a quadric, and the limit of this quadric as the points P_1, P_2 approach P along C is a *conjugate osculating quadric* at the point P of the curve C on the net N. The other

* Lane, "Bundles and Pencils of Nets on a Surface," *Transactions of the American Mathematical Society*, Vol. 28 (1926), p. 161.

† Lane, *loc. cit.*, p. 163.

‡ Miss Hagen, Chicago Master's Thesis, 1926, p. 12.

conjugate osculating quadric at P of C on N is defined similarly by using tangents of the other family of the net N. The first problem is to find the equations of the quadrics just defined. Then their geometrical properties can be investigated analytically.

We now set out to find the equation of the conjugate osculating quadric Q_u determined by three consecutive u-tangents at a point P_x of a curve C_λ of the family represented by equation (7) on the parametric net N_x on an integral surface S of system (2). We should like eventually to have this equation referred to the covariant tetrahedron x, ρ, σ, y. From system (2) by differentiation and substitution one obtains

$$\begin{aligned}
x_{uuu} &= (p_u + \alpha p + fL)x + (\alpha_u + \alpha^2 + p - nL)x_u \\
&\quad + sLx_v + (L_u + \alpha L + AL)y, \\
x_{uuv} &= (c_u + bc + ap)x + (a_u + c + ab + a\alpha)x_u + (b_u + b^2)x_v + aLy, \\
x_{uvv} &= (c_v + ac + bq)x + (a_v + a^2)x_u + (b_v + c + ab + b\delta)x_v + bNy.
\end{aligned} \tag{21}$$

Similarly any derivative of x can be expressed as a linear combination of x, x_u, x_v, y. Any point X on the curve C_λ and near the point P_x can be defined by a power series in the increment Δu corresponding to displacement on C_λ from P_x to the point X, of the form

$$X = x + x'\Delta u + x''\Delta u^2/2 + \cdots.$$

Then we find

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 y,$$

where

$$\begin{aligned}
x_1 &= 1 + \cdots, \qquad x_2 = \Delta u + \cdots, \\
x_3 &= \lambda\Delta u + (\lambda' + 2b\lambda + \delta\lambda^2)\Delta u^2/2 + \cdots, \\
x_4 &= (L + N\lambda^2)\Delta u^2/2 + \cdots.
\end{aligned} \tag{22}$$

These series represent the local coördinates $x_1, \cdots, x_4$ of the point X, referred to the tetrahedron x, x_u, x_v, y with suitably chosen unit point, to terms of as high degree in Δu as will be needed in this paper. Similarly, expanding X_u, we have

$$\begin{aligned}
X_u &= x_u + (x_u)'\Delta u + (x_u)''\Delta u^2/2 + \cdots \\
&= x_1 x + x_2 x_u + x_3 x_v + x_4 y,
\end{aligned}$$

where

$$\begin{aligned}
x_1 &= (p + c\lambda)\Delta u + \cdots, \qquad x_2 = 1 + (\alpha + a\lambda)\Delta u + \cdots, \\
x_3 &= b\lambda\Delta u + [sL + 2(b_u + b^2)\lambda \\
&\quad + (b_v + c + ab + b\delta)\lambda^2 + b\lambda']\,\Delta u^2/2 + \cdots, \\
x_4 &= L\Delta u + (L_u + \alpha L + AL + 2aL\lambda + bN\lambda^2)\Delta u^2/2 + \cdots.
\end{aligned} \tag{23}$$

In order to calculate power series expansions for the local coördinates $x_1, \cdots, x_4$ of any point $hX + kX_u$ on the u-tangent XX_u at the point X, it is sufficient to multiply the series (22) by h and the series (23) by k and add corresponding series. Then writing the equation of the most general quadric surface and demanding that it be satisfied by the power series thus calculated, identically in h, k and in Δu as far as the terms of the second degree, we obtain *the equation of the quadric* Q_u *referred to the tetrahedron* x, x_u, x_v, y, namely,

$$(24) \qquad Dx_3^2 + Ex_3x_4 + Fx_4^2 - G[x_2x_3 - \lambda x_4(x_1 + bx_2)/L] = 0,$$

the ratios of the coefficients D, E, F, G being defined by

$$(25) \qquad \begin{aligned} 2\lambda LD &= G(L - N\lambda^2), \\ 2LE &= G[\alpha + \delta\lambda - b - A - (\log L) + (\log \lambda)'], \\ L^2F &= G\lambda[b_u + b^2 - b\alpha - p + (\lambda/2)(b_v - c - ab + sL/\lambda^2)]. \end{aligned}$$

For the purpose of writing the equation of the quadric Q_u referred to the tetrahedron x, ρ, σ, y a simple computation shows that it is sufficient to replace x_1 in equation (24) by $x_1 - bx_2 - ax_3$. Making this substitution and simplifying the coefficients by means of equations (4), (6), we arrive at the desired *equation of the conjugate osculating quadric* Q_u, *referred to the tetrahedron* x, ρ, σ, y, namely,

$$(26) \qquad \begin{aligned} &(L - N\lambda^2)x_3^2 - \lambda[4(\mathfrak{C}' + \lambda\mathfrak{B}') - (\log \lambda r^{1/2})']x_3x_4 \\ &\quad + \lambda(2n\lambda - \lambda^2K/L + \boldsymbol{H}/N)x_4^2 + 2\lambda(\lambda x_1x_4 - Lx_2x_3) = 0. \end{aligned}$$

The equation of the conjugate osculating quadric Q_v at the point P_x of the curve C_λ on the net N_x can be written by interchanging u and v and making the appropriate symmetrical interchanges of the other symbols. The result is

$$(27) \qquad \begin{aligned} &(L - N\lambda^2)x_2^2 + [4(\mathfrak{C}' + \lambda\mathfrak{B}') + (\log \lambda r^{1/2})']x_2x_4 \\ &\quad + (2n + H/\lambda N - \lambda\mathbf{K}/L)x_4^2 - 2(x_1x_4 - \lambda Nx_2x_3) = 0. \end{aligned}$$

Several interesting theorems can be easily established. For example, the tangent plane, $x_4 = 0$, intersects the quadric Q_u in the u-tangent, $x_4 = x_3 = 0$, of course, and also in the residual line

$$(28) \qquad x_4 = (L - N\lambda^2)x_3 - 2\lambda Lx_2 = 0.$$

Similarly, the tangent plane intersects Q_v in the v-tangent, $x_4 = x_2 = 0$, and in the residual line

$$(29) \qquad x_4 = (L - N\lambda^2)x_2 + 2\lambda Nx_3 = 0.$$

The line (28) coincides with the v-tangent,—and the line (29) with the u-tangent,—if, and only if, the curve C_λ is such that

$$L - N\lambda^2 = 0. \tag{30}$$

If this happens at every point of C_λ, then C_λ is a curve of *the associate conjugate net* of the parametric net N_x, that is, the conjugate net whose tangents at each point of the surface S separate the tangents of N_x harmonically. So we have proved the theorem:

Each of the two conjugate osculating quadrics at every point of a curve on a net intersects the tangent plane of the net in the tangents of the net if, and only if, the curve belongs to the associate conjugate net.

When $L - N\lambda^2 \neq 0$ the two lines (28), (29) coincide at every point of a curve C_λ if, and only if, $L + N\lambda^2 = 0$. But in this case the curve C_λ is an asymptotic curve. Moreover, in this case the two lines coincide with the tangent, $x_4 = x_3 - \lambda x_2 = 0$, of the curve C_λ. Hence we have the theorem:

The two residual lines in which the tangent plane intersects the two conjugate osculating quadrics at every point of a curve on a net coincide if, and only if, the curve is an asymptotic curve on the surface sustaining the net; then the lines coincide with the tangent of the curve.

When $(L - N\lambda^2)(L + N\lambda^2) \neq 0$, the cross ratio of the two tangents of the net and the two residual lines (28), (29), in one of the possible orders, is

$$(L + N\lambda^2)^2/(L - N\lambda^2)^2.$$

The intersections of the quadrics Q_u, Q_v with the other three faces of the covariant tetrahedron of reference are of some interest, but we shall not prolong these considerations to include the details beyond mentioning the following facts.

The osculating plane, $x_3 = 0$, at a point P_x of a curve C touches the corresponding quadric Q_u of a curve C_λ in the ray-point ρ of the curve C_v. One of the generators in the plane $x_3 = 0$ is the u-tangent, $x_3 = x_4 = 0$. The other coincides with the line $x_3 = x_1 = 0$ if, and only if,

$$2n\lambda - \lambda^2 K/L + \boldsymbol{H}/N = 0.$$

The equation of *the cone projecting from the point P_x the curve of intersection of the two quadrics Q_u, Q_v* is obtained by eliminating x_1 from equations (26), (27). The result is

$$(31)\qquad (L - N\lambda^2)(x_3 - \lambda x_2)^2 - \lambda[4(\mathfrak{C}' + \lambda\mathfrak{B}') - (\log \lambda r^{1/2})']x_3x_4$$
$$+ \lambda^2[4(\mathfrak{C}' + \lambda\mathfrak{B}') + (\log \lambda r^{1/2})']x_2x_4$$
$$+ \lambda[4n\lambda - \lambda^2(K + \mathbf{K})/L + (H + \mathbf{H})/N]x_4^2 = 0.$$

This cone is clearly tangent to the tangent plane, $x_4 = 0$, along the tangent line, $x_4 = x_3 - \lambda x_2 = 0$, of the curve C_λ. Its discriminant vanishes, so that it is a pair of planes and the intersection is two conics, in case

$$(32)\qquad (L - N\lambda^2)(\lambda^2 r)' = 0.$$

When the first factor vanishes one of the planes is the tangent plane, $x_4 = 0$, and the conic in it is composed of the tangents of the net N_x, as we have already seen. The equation of the other plane can easily be read from equation (31). When the second factor vanishes the product $\lambda^2 r$ is *constant along the curve* C_λ, and λ satisfies the equation

$$(33)\qquad \lambda' = -\lambda(\log r^{1/2})_u - \lambda^2(\log r^{1/2})_v.$$

Evidently this equation is satisfied if $\lambda^2 r =$ const. *over the surface.* Then the family (7) is the most general family such that at each point of the surface its tangent and either one of the associate conjugate tangents form with the tangents of the net N_x a cross ratio which is the same for every point of the surface. The equations of the two planes when the second factor of equation (32) vanishes are easily obtained by factoring the left member of equation (31), and it may be observed that these planes intersect in the tangent of the curve C_λ.

The relations of two conjugate osculating quadrics Q_u for the curves of two conjugate families are of some interest. The equation of the quadric Q_u for the curve C_μ, with μ satisfying equation (14), can be written at once by replacing λ by $-1/r\lambda$ in equation (26). The result, after some simplification, is

$$(34)\quad r\lambda(L - N\lambda^2)x_3^2 + \lambda[4(\mathfrak{B}' - r\lambda\mathfrak{C}') + (\log \lambda r^{1/2})_v - r\lambda(\log \lambda r^{1/2})_u]x_3x_4$$
$$-(2n\lambda + K/N - r\lambda^2\mathbf{H}/N)x_4^2 - 2\lambda(x_1x_4 + N\lambda x_2x_3) = 0.$$

Eliminating x_1 between equations (26), (34) we obtain the equation of *the cone projecting from the point P_x the curve of intersection of the quadrics Q_u for the curves of two conjugate families,* namely,

$$(35)\quad (1 + r\lambda^2)[(L - N\lambda^2)x_3^2 - 4\lambda\mathfrak{C}'x_3x_4 + \lambda W^{(u)}x_4^2/N - 2\lambda L x_2x_3]$$
$$+ \lambda[(1 - r\lambda^2)(\log \lambda r^{1/2})_u + 2\lambda(\log \lambda r^{1/2})_v]x_3x_4 = 0.$$

As is geometrically obvious, the cone (35) is indeterminate, so that the quadrics (26), (34) coincide, in case $1 + r\lambda^2 = 0$, that is, in case the curve

C_λ is an asymptotic curve. Otherwise, the cone (35) is a pair of planes, so that the quadrics intersect in two conics, in case $W^{(u)} = 0$, that is, in case the u-curves of the net N_x form a W congruence. Then one of the planes is the plane $x_3 = 0$, and the equation of the other is easily read from equation (35). If $W^{(u)} \neq 0$, the cone (35) is tangent to the osculating plane, $x_3 = 0$, of the u-curve, touching it along its tangent line, $x_3 = x_4 = 0$.

4. *The Local Trihedron.* In the metric differential geometry of a surface in ordinary space it is convenient for some purposes to take the lines of curvature for the parametric curves and to employ a local trihedron at a point of the surface, whose edges are the tangents of the lines of curvature and the normal at the point of the surface. This section is designed to introduce these conceptions and to collect some formulas which will be used later on in this paper.

Let us consider in ordinary metric space a surface whose parametric equations in cartesian coordinates are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v). \tag{36}$$

Let the lines of curvature be the parametric curves on this surface. Then its first and second fundamental forms, written in the customary notation, are

$$E du^2 + G dv^2 \qquad D du^2 + D'' dv^2. \tag{37}$$

The differential equations satisfied by the coördinates x, y, z of a variable point on the surface and by the direction cosines X, Y, Z of the normal of the surface at the point take the form

$$\begin{aligned} x_{uu} &= (\log E^{1/2})_u x_u - (E_v/2G) x_v + DX, \\ x_{uv} &= (\log E^{1/2})_v x_u + (\log G^{1/2})_u x_v, \\ x_{vv} &= -(G_u/2E) x_u + (\log G^{1/2})_v x_v + D''X, \\ X_u &= -x_u/R_1, \qquad X_v = -x_v/R_2, \end{aligned} \tag{38}$$

where *the principal radii of normal curvature* R_1, R_2 at a point of the surface are defined by the formulas

$$R_1 = E/D, \qquad R_2 = G/D''. \tag{39}$$

The corresponding *radii of geodesic curvature* ρ_1, ρ_2 of the u-curve and of the v-curve respectively at the point are given * by the formulas

$$1/\rho_1 = -(\log E)_v/2G^{1/2}, \qquad 1/\rho_2 = (\log G)_u/2E^{1/2}. \tag{40}$$

* Eisenhart, *Differential Geometry*, Ginn and Co., 1909, p. 134.

The three integrability conditions of equations (38) are *the equation of Gauss*

$$-2(EG)^{1/2}K_t = [G_u/(EG)^{1/2}]_u + [E_v/(EG)^{1/2}]_v, \tag{41}$$

wherein the total curvature K_t is defined by

$$K_t = 1/R_1R_2, \tag{42}$$

and *the two equations of Codazzi,*

$$\begin{aligned}(1/R_1)_v &= (1/R_2 - 1/R_1)(\log E^{1/2})_v,\\ (1/R_2)_u &= (1/R_1 - 1/R_2)(\log G^{1/2})_u.\end{aligned} \tag{43}$$

As a local trihedron of reference at a point (x, y, z) of the surface, we shall take the origin at this point, the ξ-axis along the u-tangent, the η-axis along the v-tangent, and the ζ-axis along the normal. If $\bar{x}, \bar{y}, \bar{z}$ are the general coördinates of a point having local coördinates ξ, η, ζ, the equations of transformation between the general and local cartesian systems can be written in the form

$$\begin{aligned}\bar{x} - x &= \xi x_u/E^{1/2} + \eta x_v/G^{1/2} + X\zeta,\\ \bar{y} - y &= \xi y_u/E^{1/2} + \eta y_v/G^{1/2} + Y\zeta,\\ \bar{z} - z &= \xi z_u/E^{1/2} + \eta z_v/G^{1/2} + Z\zeta.\end{aligned} \tag{44}$$

The equations of the inverse transformation are

$$\begin{aligned}\xi &= (\bar{x} - x, \quad x_v/G^{1/2}, \quad X),\\ \eta &= (\bar{x} - x, \quad X, \quad x_u/E^{1/2}),\\ \zeta &= (\bar{x} - x, \quad x_u/E^{1/2}, \quad x_v/G^{1/2}),\end{aligned} \tag{45}$$

parentheses indicating determinants of which only a typical row is written.

5. *Transformation of Coordinates.* The lines of curvature on a surface in ordinary space are a metrically defined conjugate net; precisely, they are the only orthogonal conjugate net on the surface. Besides their special metric properties they, of course, also possess all the general projective properties of an arbitrary conjugate net. Moreover, the configurations associated with a point of an arbitrary conjugate net in the projective differential theory certainly are defined for the lines of curvature and may have interesting metric properties. The investigation of this situation is facilitated by employing a transformation of coördinates, which it is the purpose of this section to

More in detail, the transformation with which we are concerned here connects the local homogeneous coordinates of the projective theory with the

local non-homogeneous cartesian coördinates of the metric theory. The projective homogeneous coordinates are based on the tetrahedron whose vertices are the points x, ρ, σ, y, in the notation of Section 3, with suitably chosen unit point. The cartesian coördinates are those described in Section 4.

In order to prepare for the calculation of the equations of the transformation we observe that elimination of X from the first three of equations (38) leads to a system of the form (1) whose coefficients are defined by the formulas

$$\begin{aligned} a &= 1/r = D/D'', \quad b = (\log E^{1/2})_u + E^{1/2}R_2/R_1\rho_2, \\ d &= d' = 0, \quad c = -(1/r)[(\log G^{1/2})_v - G^{1/2}R_1/R_2\rho_1], \\ b' &= -G^{1/2}/\rho_1, \qquad c' = E^{1/2}/\rho_2. \end{aligned} \tag{46}$$

The ray-points ρ, σ of the lines of curvature are therefore given by the formulas

$$\rho = x_u - c'x, \qquad \sigma = x_v - b'x, \tag{47}$$

while the point τ (hitherto denoted by y) which is the harmonic conjugate of the point x with respect to the foci of the axis of the lines of curvature is given * by two equal expressions,

$$\tau = (1/D)[x_{uu} - bx_u + Mx] = (1/D'')[x_{vv} + rcx_v + r(M + d)x] \tag{48}$$

in which M is defined by placing

$$-2M = d + ab'^2 + c'^2 + ab_v' + c_u' + b'c - bc'. \tag{49}$$

In the first expression for τ let us substitute the value of x_{uu} taken from the first of equations (38), and then let us use the values of the coefficients of system (1) that are defined by the formulas (46). After a somewhat lengthy reduction whose details need not be recorded here, we find

$$\tau = (1/D)[-(E^{1/2}R_2/R_1\rho_2)x_u + (E/G^{1/2}\rho_1)x_v + DX + Mx], \tag{50}$$

where M is given by

$$-2M = D[(R_2 - R_1 + R_2\rho_{1v}/G^{1/2})/\rho_1^2 + (R_1 - R_2 - R_1\rho_{2u}/E^{1/2})/\rho_2^2]. \tag{51}$$

To continue the process of expressing quantities in terms of R_1, R_2, ρ_1, ρ_2 and their derivatives, we may, if we like, use the formulas

$$E^{1/2} = \rho_2(1/R_2)_u/(1/R_1 - 1/R_2), \quad G^{1/2} = -\rho_1(1/R_1)_v/(1/R_2 - 1/R_1). \tag{52}$$

The actual calculation of the equations of the transformation proceeds

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* Green, Second Memoir, p. 292.

as follows. Let $x_1, \cdots, x_4$ be the local projective homogeneous coordinates of a point P referred to the tetrahedron x, ρ, σ, τ at a point (x, y, z) of the surface under consideration, with the unit point chosen so that the general projective coördinates of the point P are represented by the expression

$$x_1 x + x_2 \rho + x_3 \sigma + x_4 \tau.$$

Let us replace ρ, σ, τ in this expression by their values from equations (47), (50) and then in place of (x, X) substitute in turn $(1, 0)$, (x, X), (y, Y), (z, Z). Let us divide the first expression thus obtained into each of the remaining three expressions. The resulting ratios are the general cartesian coördinates $\bar{x}, \bar{y}, \bar{z}$ of the point P. Denoting by ξ, η, ζ the local cartesian coördinates of P, we arrive by way of the calculations just indicated at a system of equations of precisely the same form as (44), with ξ, η, ζ defined by the formulas

$$\begin{aligned} \xi &= (E^{1/2} x_2 - R_2 x_4/\rho_2)/T, \\ \eta &= (G^{1/2} x_3 + R_1 x_4/\rho_1)/T, \\ \zeta &= x_4 T, \end{aligned} \tag{53}$$

where the denominator T is defined by

$$T = x_1 - E^{1/2} x_2/\rho_2 + G^{1/2} x_3/\rho_1 + M x_4/D. \tag{54}$$

These are the equations of the transformation which we were seeking. Solving them for the ratios of $x_1, \cdots, x_4$ we get the equations of the inverse transformation,

$$\begin{aligned} x_1 &= 1 + \xi/\rho_2 - \eta/\rho_1 + (\zeta/2)[(R_1 + R_2 \\ &\qquad + R_2 \rho_{1v}/G^{1/2})/\rho_1^2 + (R_2 + R_1 - R_1 \rho_{2u}/E^{1/2})/\rho_2^2], \\ x_2 &= (\xi + R_2 \zeta/\rho_2)/E^{1/2}, \\ x_3 &= (\eta - R_1 \zeta/\rho_1)/G^{1/2}, \\ x_4 &= \zeta. \end{aligned} \tag{55}$$

6. *Metric Results.* The formulas and the transformation discussed in the preceding sections will now be applied in studying for the lines of curvature some of the configurations associated with a general conjugate net in the projective theory. The usual invariants that occur in the projective theory will be calculated in terms of metric quantities for the lines of curvature, and a few metric theorems will be stated.

Since the equations of the osculating planes of the lines of curvature

C_u, C_v at a point P of a surface are $x_3=0$ and $x_2=0$ respectively in the homogeneous coördinate system, it follows immediately from equations (55) that the equations of the same planes in the cartesian system are

$$\rho_1\eta - R_1\zeta = 0, \qquad \rho_2\xi + R_2\zeta = 0. \tag{56}$$

The axis of the lines of curvature C_u, C_v at the point P is the line of intersection of these two planes and therefore passes through the point

$$(-R_2/\rho_2, \quad R_1/\rho_1, \quad 1).$$

The coördinates of the point τ on the axis result from multiplying each of these coördinates by D/M. The direction cosines of the axis can easily be written. The projective homogeneous local coördinates of the ray-points ρ, σ of the lines of curvature at the point P are $(0, 1, 0, 0)$, $0, 0, 1, 0)$, and hence the cartesian local coördinates of the ray-points are $(-\rho_2, 0, 0)$, $(0, \rho_1, 0)$. The distance between these points and the direction cosines of the ray can easily be written. *The cosine of the angle θ between the ray and axis of the lines of curvature* may thus be expressed by the formula

$$\cos\theta = (R_1 - R_2)/(\rho_1^2 + \rho_2^2)^{1/2}(1 + R_1^2/\rho_1^2 + R_2^2/\rho_2^2)^{1/2}. \tag{57}$$

The ray and axis are ordinarily not orthogonal.

The invariants appearing in equations (6) are given for the lines of curvature by the following formulas:

$$\begin{aligned}
&H = G^{1/2}(1/\rho_1)_u, && K = -E^{1/2}(1/\rho_2)_v,\\
&\boldsymbol{H} = H + (\log R_1)_{uv}, && \boldsymbol{K} = K + (\log R_2)_{uv},\\
&W^{(u)} = \boldsymbol{H} - K, && W^{(v)} = \boldsymbol{K} - H,\\
&8\mathfrak{B}' = (\log R_1^3/R_2)_v, && 8\mathfrak{C}' = (\log R_2^3/R_1)_u,\\
&\mathfrak{D} = D[(R_2 - R_1 + R_2\rho_{1v}/G^{1/2})/\rho_1^2 + (R_2 - R_1 + R_1\rho_{2u}/E^{1/2})\rho_2^2], &&\\
&r = D''/D. &&
\end{aligned} \tag{58}$$

The projective theorem that *a curve C_u is a cone curve* (i. e. is the curve of contact of a cone circumscribing the surface) *in case $H=0$*, becomes the metric theorem that *the curve C_u is a cone curve in case it has constant geodesic curvature.* But the dual projective theorem that the curve C_u is a plane curve in case $\boldsymbol{H}=0$ does not seem to have so simple a metric formulation.

Since necessary and sufficient conditions that a net be quadratic are $\mathfrak{B}' = \mathfrak{C}' = 0$, it follows that *necessary and sufficient conditions that a surface be a quadric are*

$$R_1 = cU^3/V, \qquad R_2 = cU/V^3, \tag{54}$$

where c is an arbitrary constant and U, V are arbitrary functions of u alone and of v alone respectively. The u-curves and v-curves form W congruences in case $W^{(u)} = W^{(v)} = 0$, and then the net is called an R net; it follows that *the lines of curvature on a surface form an R net if, and only if,*

$$H - K + (\log R_1)_{uv} = 0, \qquad K - H + (\log R_2)_{uv} = 0. \tag{55}$$

In this case it follows that $(\log K_t)_{uv} = 0$.

Some calculation, which will be omitted, results in the formulas

$$\begin{aligned} &(\log E/G)_{uv} = -2(H-K), \\ &(\log D/D'')_{uv} = (\log R_2/R_1)_{uv} - 2(H-K). \end{aligned} \tag{56}$$

Thus we arrive at the well-known theorem that the lines of curvature are isothermally orthogonal in case $H = K$, and also at the theorem that *the lines of curvature are isothermally conjugate if, and only if,*

$$(\log R_2/R_1)_{uv} = 2(H-K). \tag{57}$$

It follows that *the lines of curvature are both isothermally orthogonal and isothermally conjugate if, and only if,*

$$H = K, \qquad (\log R_2/R_1)_{uv} = 0. \tag{58}$$

In this case the lines of curvature are a conjugate net of the type called a *net of Jonas,* since they are isothermally conjugate and have equal point invariants H, K.

The equations of *the ray-point cubic of the pencil of conjugate nets determined by the lines of curvature* are easily found, by carrying out the transformation (55) on equations (17), to be

$$\begin{aligned} \zeta = {} & (\xi^2/R_1 + \eta^2/R_2)[1 + (1/\rho_2 - \mathfrak{C}'/E^{1/2})\xi - (1/\rho_1 + \mathfrak{B}'/G^{1/2})\eta)] \\ & + \mathfrak{C}'\xi^3/R_1E^{1/2} - 3\mathfrak{B}'\xi^2\eta/R_1G^{1/2} - 3\mathfrak{C}'\xi\eta^2/R_2E^{1/2} + \mathfrak{B}'\eta^3/R_2G^{1/2} = 0. \end{aligned} \tag{59}$$

Similarly the equations of *the ray conic of the pencil determined by the lines of curvature* are found from equation (18) to be

$$\begin{aligned} \zeta = {} & (\mathfrak{B}'^2 + r\mathfrak{C}'^2)(\xi^2/R_1 + \eta^2/R_2) \\ & - D''[1 + (1/\rho_2 - \mathfrak{C}'/E^{1/2})\xi - (1/\rho_1 + \mathfrak{B}'/G^{1/2})\eta]^2 = 0. \end{aligned} \tag{60}$$

The equations (20) of the quadric of Darboux [illegible], [illegible] the [illegible] coordinate system,

$$\xi^2/R_1 + \eta^2/R_2 + 2\zeta(-1 + k_2\xi + k_3\eta + k\zeta) = 0, \tag{61}$$

where k is arbitrary and k_2, k_3 are given by

$$(62) \qquad 4E^{1/2}k_2 = (\log K_t)_u, \qquad 4G^{1/2} = (\log K_t)_v.$$

From these formulas we can easily prove the well-known theorem that the line of centers of the quadrics of Darboux coincides with the normal at every point of a surface if, and only if, the surface has constant total curvature. The value of k for the quadric of Lie is expressed * by the formula

$$(63) \qquad \begin{aligned} 4k = {} & 2(k_2{}^2R_1 + k_3{}^2R_2) + (1/R_1 + 1/R_2)(k_2{}^2R_1{}^2 \sin^2 A \\ & + k_3{}^2R_2{}^2 \cos^2 A + 1) + (k_2R_1 \sin A)_u/E^{1/2} \sin A \\ & + (k_3R_2 \cos A)_v/G^{1/2} \cos A, \end{aligned}$$

where A is an angle such that

$$(64) \qquad (R_1 - R_2)\sin^2 A = -R_2, \qquad (R_1 - R_2)\cos^2 A = R_1.$$

We conclude with the remark that the conjugate osculating quadrics (26), (27) of Section 2, when defined as on the lines of curvature, are quite analogous to the asymptotic osculating quadrics as to their covariantive properties, the projectively covariant asymptotic curves in the latter instance being replaced by the metrically covariant lines of curvature in the former.

UNIVERSITY OF CHICAGO,
CHICAGO, ILL.

* Demoulin, "Sur quelques propriétés des surfaces courbes," *Comptes Rendus*, Vol. 147 (1908), p. 566.

ON THE EXPANSION OF HARMONIC FUNCTIONS IN TERMS OF NORMAL-ORTHOGONAL HARMONIC POLYNOMIALS.*

By GAYLORD M. MERRIMAN.†

Consider a closed, rectifiable Jordan curve, C, lying in the (x, y)-plane, possessing continuous curvature, and of length σ. Let $f(x, y)$ be a function which is harmonic inside C and continuous in the closed region $\bar{C}$ consisting of C and its interior. The present paper is concerned with the development of $f(x, y)$ in terms of a set of harmonic polynomials $P_m(x, y)$, which are linear combinations of the functions $1, x, y, x^2 - y^2, xy, \cdots$. Let

$$(1) \qquad f(x, y) \sim \sum_{j=0}^{\infty} c_j P_j(x, y)$$

be such an expansion, the c_j being constants obtained later in the Fourier fashion:

$$c_j = (1/\sigma) \int_C f(x, y) P_j(x, y)\, d\sigma, \qquad (j = 0, 1, 2, \cdots).$$

It is natural, of course, to desire that the n-th partial sum of the development:

$$s_n(x, y) = \sum_{j=0}^{n} c_j P_j(x, y)$$

should give among all polynomials $S_n(x, y)$ of the same degree a "best" approximation to $f(x, y)$, that is, say, that the integral

$$(2) \qquad (1/\sigma) \int_C [f(x, y) - S_n(x, y)]^2 d\sigma$$

should be minimized by the substitution $S_n(x, y) = s_n(x, y)$. It is well-known, however, that such a result can be accomplished by normalizing and orthogonalizing the polynomials $P_m(x, y)$: here it will be done with respect to the perimeter of C ‡:

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† This paper was begun while the author was National Research Fellow in Mathematics, Harvard University, 1926-1928.

‡ St. Bergmann *Mathematische Annalen*, Vol. 86 (1922), pp. 238-271, has normalized and orthogonalized polynomials with respect to an *area*; he considers in general expansions which can be obtained from real parts of analytic functions.

$$(3)\quad (1/\sigma)\int_C P_m(x,y)P_n(x,y)\,d\sigma = \left\{\begin{matrix} 0, & m \neq n \\ 1, & m = n \end{matrix}\right\}, \quad (m, n = 0, 1, \cdots).$$

We are first interested in exhibiting such a set of polynomials.

Next to discuss is the convergence of the set of normalized and orthogonalized polynomials; in this connection we shall prove the following theorem:

THEOREM I. *Let C be a closed, rectifiable Jordan curve possessing continuous curvature, and let $f(x,y)$ be harmonic interior to C and continuous in the closed region $\bar{C}$ consisting of C and its interior. Then the series* (1) *of harmonic polynomials normalized and orthogonalized with respect to the contour C converges uniformly to $f(x,y)$ in any closed region wholly interior to C.*

Finally, we prove the expansion (1) to be "overconvergent" in the sense of J. L. Walsh,* in that it converges uniformly in a larger region than that originally considered, thus extending the region of definition of $f(x,y)$.† In this connection we shall establish

THEOREM II. *Let C be a closed, rectifiable Jordan curve in the (x,y)-plane, and possessing continuous curvature, and let $w = \phi(z)$, $z = x + iy$, map the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve in the z-plane corresponding to $w = R$, $R > 1$. If $f(x,y)$ is harmonic interior to C and continuous in the closed region $\bar{C}$ consisting of C and its interior, then the series* (1) *of normal-orthogonal polynomials converges to $f(x,y)$ throughout the interior of C_R, and uniformly in any closed region wholly interior to C_R.*

The method of normalization and orthogonalization for harmonic polynomials has been used previously in an incomplete way by S. Bernstein,

* Two papers of Walsh, important in what follows, will henceforth be referred to as Walsh (i) and (ii): Walsh (i): "On the Overconvergence of Sequences of Polynomials of Best Approximation," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 794-815; Walsh (ii): "On the Degree of Approximation to a Harmonic Function," *Bulletin of the American Mathematical Society*, Sept.-Oct. 1927. The present reference is, of course, to the first of these, p. 794.

† Another point of view is, of course, to consider $f(x,y)$ defined outside C by harmonic extension.

Picone, and Brillouin.* Szégö † has used normalized and orthogonalized polynomials to expand analytic functions of a complex variable; in the paper just referred to, however, he considered only analytic Jordan curves as boundaries, a restriction which can be removed to some extent by a more recent theorem of Walsh,‡ so that some of his convergence theorems hold for the general rectifiable Jordan curve. In comparison, it is of interest that the present treatment of a similar problem for harmonic functions is, by the methods of proof, limited to rectifiable Jordan curves possessing continuous curvature.

It will be noted that the results of the paper furnish a solution of the Dirichlet problem for the region bounded by C and the boundary values $f(x,y)$ on C; the theoretical and practical value of such solutions of that problem has been remarked at the end of Walsh (i).

I. *The Set of Polynomials.* Let $r(z^m)$ be the real part, and $p(z^m)$ the coefficient of i, in z^m, $z=x+iy$, $i=(-1)^{1/2}$. Let $a_{j,k}$ be defined as follows:

$$a_{0,0}=1,$$

$$a_{0,2k-1}=(1/\sigma)\int_C p(z^k)\,d\sigma, \qquad a_{0,2k}=(1/\sigma)\int_C r(z^k)\,d\sigma, \qquad (k=1,2,\cdots),$$

$$a_{2j-1,0}=(1/\sigma)\int_C p(z^j)\,d\sigma, \qquad a_{2j,0}=(1/\sigma)\int_C r(z^j)\,d\sigma, \qquad (j=1,2,\cdots),$$

$$a_{2j-1,2k-1}=(1/\sigma)\int_C p(z^j)p(z^k)\,d\sigma, \qquad a_{2j-1,2k}=(1/\sigma)\int_C p(z^j)r(z^k)\,d\sigma, \qquad (j,k\geqq 1),$$

$$a_{2j,2k-1}=(1/\sigma)\int_C r(z^j)p(z^k)\,d\sigma, \qquad a_{2j,2k}=(1/\sigma)\int_C r(z^j)r(z^k)\,d\sigma, \qquad (j,k\geqq 1).$$

These numbers are the coefficients of symmetric, positive-definite quadratic bilinear forms in n variables, $n=1,2,3,\cdots$, since, for example,

* S. Bernstein, *Comptes Rendus*, Vol. 148, pp. 1306-1308; Picone, *Rendiconti dei Lincei* (1922), pp. 357-359; Brillouin, *Annales de Physique*, Vol. 6 (1916), pp. 137-223.

† G. Szégö, "Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören," *Mathematische Zeitschrift*, Vol. 9 (1921), pp. 218-270.

‡ Walsh, "Über die Entwicklung einer analytischen Funktion nach Polynomen," *Mathematische Annalen*, Vol. 96, pp. 430-436.

$$\sum_{j,k=0}^{2q} a_{j,k}t_jt_k = (1/\sigma)\int_C [t_0 + t_1 p(z) + t_2 r(z) + \cdots + t_{2q} r(z^q)]^2 d\sigma.$$

Also, the determinants D_n of these forms are positive for all n.*

The polynomials $P_m(x, y)$ are defined in the following manner:

$$P_0(x, y) = 1,$$

$$P_{2m-1}(x,y) = \gamma_{2m-1} \begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{2m-1,0} \\ a_{0,1} & a_{1,1} & a_{2,1} & \cdots & a_{2m-1,1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-2} & a_{1,2m-2} & a_{2,2m-2} & \cdots & a_{2m-1,2m-2} \\ 1 & p(z) & r(z) & \cdots & p(z^m) \end{vmatrix},$$

$$P_{2m}(x,y) = \gamma_{2m} \begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{2m-1,0} & a_{2m,0} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & a_{2,2m-1} & \cdots & a_{2m-1,2m-1} & a_{2m,2m-1} \\ 1 & p(z) & r(z) & \cdots & p(z^m) & r(z^m) \end{vmatrix},$$

for $m = 1, 2, \cdots$, the γ's being constants to be determined later.

It is verified from the definitions that

(4a)
$$(1/\sigma)\int_C P_{2m}(x,y) r(z^k)\, d\sigma = \gamma_{2m} \begin{vmatrix} a_{0,0} & a_{1,0} & \cdots & a_{2m,0} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & \cdots & a_{2m,2m-1} \\ a_{0,2k} & a_{1,2k} & \cdots & a_{2m,2k} \end{vmatrix}$$

$$= \left\{ \begin{matrix} 0, & k < m \\ \gamma_{2m} D_{2m}, & k = m \end{matrix} \right\}, \qquad (m = 1, 2, \cdots).$$

Also,

(4b)
$$(1/\sigma)\int_C P_{2m}(x,y) p(z^k)\, d\sigma = \gamma_{2m} \begin{vmatrix} a_{0,0} & a_{1,0} & \cdots & a_{2m,0} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & \cdots & a_{2m,2m-1} \\ a_{0,2k-1} & a_{1,2k-1} & \cdots & a_{2m,2k-1} \end{vmatrix}$$

$$= 0, \quad k \leqq m, \quad m = 1, 2, \cdots.$$

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* This is a well-known property of these forms; cf. Osgood's *Advanced Calculus*, p. 179.

Again,

$$(4c)\quad (1/\sigma)\int_C P_{2m-1}(x,y)p(z^k)\,d\sigma = \left\{\begin{array}{cc} 0, & k<m \\ \gamma_{2m-1}D_{2m-1}, & k=m \end{array}\right\}\quad (m=1,2,\cdots),$$

and finally,

$$(4d)\qquad (1/\sigma)\int_C P_{2m-1}(x,y)r(z^k)\,d\sigma = 0,\qquad (k<m,\ m=1,2,\cdots).$$

Equations (4) insure the orthogonality of the polynomials as defined; the polynomials will now be proved normalized by a proper choice of the γ's.

Thus, if the polynomials are to be normalized,

$$1=(1/\sigma)\int_C P^2{}_{2m}(x,y)\,d\sigma$$
$$=\gamma_{2m}D_{2m-1}\cdot(1/\sigma)\int_C P_{2m}(x,y)r(z^m)\,d\sigma=\gamma^2{}_{2m}D_{2m-1}D_{2m},$$

whence

$$\gamma_{2m}=1/(D_{2m-1}D_{2m})^{1/2},$$

the positive square root being in order if we assume that the leading coefficient of $P_{2m}(x,y)$ is positive. Similarly,

$$\gamma_{2m-1}=1/(D_{2m-2}D_{2m-1})^{1/2}.$$

With these values of γ_{2m} and γ_{2m-1} inserted in their definitions the polynomials $P_m(x,y)$, $m=0, 1, 2,\cdots$, are both normalized and orthogonalized with respect to C.

II. *Convergence of the Set Interior to C.* The function $f(x,y)$, harmonic in the interior of C and continuous in $\bar{C}$, is now considered to be developed formally as in (1) and inquiry is made as to the convergence of the series. The clue is furnished by a discussion of the minimum value of the integral (2) with

$$S_n(x,y)=s_{2n}(x,y)=\sum_{j=0}^{2n} c_jP_j(x,y).$$

Here,

$$(2')\qquad M_n(\{c_j\})=(1/\sigma)\int_C [f(x,y)-s_{2n}(x,y)]^2d\sigma \geqq 0.$$

The minimum of $M_n(\{c_j\})$, call it m_n, is actually attained, and in obtaining it in the usual way it is found that

$$(5)\qquad c_j=(1/\sigma)\int_C f(x,y)P_j(x,y)\,d\sigma,\qquad (j=0,1,2,\cdots),$$

in the Fourier fashion. In these circumstances the minimum value is

$$m_n = (1/\sigma) \int_C f^2(x,y)\,d\sigma - \sum_{j=0}^{2n} c_j^2. \tag{6}$$

It is next to be shown that m_n has the limiting value zero as n becomes infinite. In the first place, since $f(x,y)$ is harmonic in the interior of C and continuous in the closed region $\bar{C}$, it follows that, for a suitably chosen polynomial $P(x,y)$, it can be uniformly approximated in C*:

$$| f(x,y) - P(x,y) | < \epsilon, \tag{7}$$

where ϵ has been previously assigned. Hence,

$$(1/\sigma) \int_C [f(x,y) - P(x,y)]^2 d\sigma < \epsilon^2. \tag{8}$$

But consider

$$\lim_{n\to\infty} (1/\sigma) \int_C [f(x,y) - s_{2n}(x,y)]^2 d\sigma.$$

It is a monotonically decreasing function, and positive or zero. If, however, its limit is positive, say greater than ϵ^2, then $P(x,y)$ would, according to (8), give a better approximation to $f(x,y)$ than does $s_{2n}(x,y)$; this is impossible, so that the limit is zero. Hence, making use of (5) and (6),

$$(1/\sigma) \int_C f^2(x,y)\,d\sigma = \sum_{j=0}^{\infty} c_j^2,$$

and we have established

Lemma 1. *If $f(x,y)$, expanded formally as in* (1) *with coefficients* (5), *is harmonic in the interior of C and continuous in the closed region $\bar{C}$ consisting of C and its interior, then*

$$\lim_{n\to\infty} (1/\sigma) \int_C [f(x,y) - s_{2n}(x,y)]^2 d\sigma = 0,$$

and

$$(1/\sigma) \int_C f^2(x,y)\,d\sigma = \sum_{j=0}^{\infty} c_j^2.$$

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* This result has been proved for a general Jordan curve by Walsh: *Crelle's Journal*, Vol. 159 (1928), pp. 197-209, Satz II. It is this result which permits the extension of Szégö's work already alluded to.

COROLLARY. *If $\bar{f}(x, y)$ satisfies the conditions imposed on $f(x, y)$ in Lemma 1, and if the coefficients of its formal expansion in terms of the $P_m(x, y)$ are $\bar{c}_j$, $j = 0, 1, 2, \cdots$, then the $\bar{c}_j$ are not all zero unless $f(x, y) \equiv 0$.*

LEMMA 2. *If $f_n(x, y)$ is harmonic in the region bounded by C and is continuous in $\bar{C}$, and if*

$$\lim_{n \to \infty} \int_C f_n^2(x, y)\, d\sigma = 0,$$

then

$$\lim_{n \to \infty} f_n(x, y) = 0$$

uniformly in any closed region wholly interior to C.

Green's integral gives, for any point (x, y) inside C,

$$f_n(x, y) = (1/2\pi) \int_C f_n(\xi, \eta) [\partial G(\xi, \eta)/\partial N]\, d\sigma,$$

where (ξ, η) is a general boundary point of C, N is the normal at (ξ, η), and $G(x, y)$ is the Green's function belonging to the region. Because of the condition of continuous curvature of C, $\partial G/\partial N$ is continuous on C and less in absolute value than some constant M. By use of Schwarz's inequality,

$$(9) \qquad | f_n(x, y) | \leqq [M'(\sigma)^{1/2}/2\pi] \left[\int_C f_n^2(\xi, \eta)\, d\sigma \right]^{1/2}.$$

The lemma follows at once from (9).*

Theorem I is an immediate consequence of Lemmas 1 and 2, if the substitution

$$f_n(x, y) = [f(x, y) - s_{2n}(x, y)]/\sigma^{1/2}$$

is made in Lemma 2.

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* It will be noticed that all the results up to that of Lemma 2 can be proved for an arbitrary, rectifiable Jordan curve; for such a curve, however, lack of information concerning Green's function and its normal derivative (if it has one) preclude the use of Green's integral as above. We have therefore assumed the curve C as one having continuous curvature, in which case the normal derivative of the Green's function is continuous; this condition could, it is interesting to note, be lightened if conditions on $f(x, y)$ were correspondingly strengthened: cf. O. D. Kellogg, "Potential Functions" and "Double Distributions and the Dirichlet Problem," *Transactions of the American Mathematical Society*, Vol. 9 (1908), pp. 39 and 51, with cross references to previous results, especially those of Painlevé and Neumann.

III. *Overconvergence of the Set.* In connection with Theorem VII, p. 808, of Walsh (i) there is specific mention made of the result contained in Theorem II of the present paper (not then published), as being a previously established special case of the aforesaid Theorem VII; however, the asymptotic formula for $P_m(x, y)$ outside C, from which Theorem II was then deduced, has since collapsed. Inasmuch as Theorem VII of Walsh (i) indicates the truth of Theorem II, we include its statement merely for the sake of completeness of the present paper as first conceived. Walsh does not display a proof of Theorem VII, stating it as an analogue of Theorem III of the same paper, dealing with polynomials of a best approximation to a function of a complex variable; suffice it to say here that the methods of proof of the last mentioned theorem, in conjunction with Theorem I of Walsh (ii), a mapping theorem of Carathéodory,* and the use of Green's integral, establish Theorem II.

HARVARD UNIVERSITY,
THE UNIVERSITY OF CINCINNATI.

* *Mathematische Annalen* Vol. 72 (1912), pp. 126-127.

ON SOME PROBLEMS OF TCHEBYCHEFF.

By J. GERONIMUS.

We show in this paper that some results of Tchebycheff can be generalized and obtained without the use of algebraic continuous fractions. We base our investigation on some well known properties of orthogonal polynomials the theory of which can be made independent of continuous fractions.

1. *On functions having the least deviation from zero.* We, first, consider, with Tchebycheff, the following problem:

The polynomial of the n-th degree

$$y(x) = \sum_{i=0}^{n} \sigma_i x^{n-i} \tag{1}$$

is monotonic in the interval $(-1, +1)$. *Find the least deviation from zero of this polynomial in the interval* $(-1, +1)$ *its first coefficient* σ_0 *being given.**

Without loss of generality we may suppose that $y(-1) = 0$. Then our polynomial may be written

$$y(x) = \int_{-1}^{x} \phi(x)\,dx, \tag{2}$$

where $\phi(x) \geq 0$ for $-1 \leq x \leq 1$. We suppose that our monotonic polynomial is increasing in the interval $(-1, +1)$. It is easy to show further that $\phi(x)$ *is of the form*

$$\phi(x) = (1-x)^{\alpha}(1+x)^{\beta}u^2(x), \tag{3}$$

where $u(x)$ *is a polynomial, and* $\alpha = \beta = 0$ *or* $\alpha = 1$, $\beta = 1$ *if* n *is odd, and* $\alpha = 0$, $\beta = 1$ *or* $\alpha = 1$, $\beta = 0$ *if* n *is even.*

Thus we are to minimize the integral

$$y(1) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}u^2(x)\,dx. \tag{4}$$

In view of the well known minimum properties of orthogonal Tchebycheff

* P. Tchebycheff, "Sur les fonctions qui diffèrent le moins possible de zéro," *Oeuvres*, t. II (1907), pp. 189-215.

polynomials, we make use of the normalized Jacobi polynomials $P_k(x)$, corresponding to the interval $(-1, +1)$ with the characteristic function

$$p(x)=(1-x)^\alpha(1+x)^\beta.$$

We find, putting $u(x)=a_mP_m(x)=a_m(d_mx^m+\cdots)$:

$$\min y(1)=a_m^2,$$

where $$n\sigma_0=(-1)^\alpha a_m^2 d_m^2 \qquad (2m+\alpha+\beta=n-1).$$

since $$y'(x)=n\sigma_0x^{n-1}+\cdots=(1-x)^\alpha(1+x)^\beta u^2(x).$$

Hence

$$y(x)=\frac{n\sigma_0(-1)^\alpha}{d_m^2}\int_{-1}^{x}(1-x)^\alpha(1+x)^\beta P_m^2(x)\,dx, \tag{5}$$

$$y(1)=\frac{2^{2m+\alpha+\beta+1}\,|\,\sigma_0\,|\,m!\,(m+\alpha+\beta)!\,(m+\alpha)!\,(m+\beta)!}{\{(2m+\alpha+\beta)!\}^2}.* \tag{6}$$

For large values of n we find, using Stirling's formula,

$$y(1)\sim\frac{\pi\,|\,\sigma_0\,|\,n}{2^{n-1}}. \tag{7}$$

This problem of Tchebycheff may be generalized as follows:

Find the minimal deviation from zero in the interval $(-1, +1)$ of a polynomial

$$y(x)=\sum_{i=0}^{n}\sigma_i x^{n-i}$$

which is monotonic of order $h+1$ in this interval, its coefficient σ_l being given $(0\leq l\leq n)$.

A polynomial is said to be monotonic of order $h+1$ on a given interval if its $h+1$ first derivatives do not change sign in this interval.† Supposing that $n\to\infty$, while l and h are finite we have ‡

$$\begin{aligned} y(1)&\sim\frac{\pi\,|\,\sigma_{2k}\,|\,n^{h-k+1}k!}{2^{n-2k-1}h!}, && (l=2k),\\ y(1)&\sim\frac{\pi\,|\,\sigma_{2k+1}\,|\,n^{h-k+1}k!}{2^{n-2k-2}h!\,[h+(h^2+1)^{1/2}]}, && (l=2k+1). \end{aligned} \tag{8}$$

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* Pòlya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. II, Ss. 292-293.

† S. Bernstein, "Sur les polynomes multiplement monotones qui s'écartent le moins de zéro," *Comptes Rendus*, t. 185 (1927), p. 247.

‡ J. Geronimus, "Sur le polynome multiplement monotone qui s'écarte le moins de zéro dont un coefficient est donné," *Bulletin de l'Académie des Sciences de l'URSS, Classe des Sciences Physico-Mathématiques*, N 4 (1929), pp. 388-389.

For $h = 0$, i. e. for ordinary monotonic polynomials, these two formulae may be combined into one:

$$(9) \qquad y(1) \sim \frac{\pi \mid \sigma_l \mid \nu!}{2^{n-l-1} \cdot n^{\nu-1}}, \qquad \nu = [l/2],$$

($[m]$ denotes the integral part of m).

The problem of Tchebycheff may be generalized in another way:

Find the least deviation from zero in the interval $(-1, +1)$ *of a polynomial*

$$y(x) = \sigma_0 x^n + \sigma_1 x^{n-1} + \cdots + \sigma_l x^{n-l} + \cdots + \sigma_n,$$

which is monotonic of order $h+1$ *in this interval, its first* $l+1$ *coefficients* $\sigma_0, \sigma_1, \cdots, \sigma_l$ *being given.*

The writer solved this problem for $l = 1$, i. e. supposing that σ_0 and σ_1 are given. Assuming that $n \to \infty$, while h is finite, we have *

$$(10) \qquad \begin{aligned} y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{h+1}}{2^{n-1} h!} \{1 + (\sigma_1/\sigma_0 - h)^2\}, & \mid \sigma_1/\sigma_0 - h \mid \leq 1, \\ y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{h+1}}{2^{n-2} h!} \cdot \mid \sigma_1/\sigma_0 - h \mid, & \mid \sigma_1/\sigma_0 - h \mid \geq 1. \end{aligned}$$

For the particular case $h = 0$, S. Bernstein gives the solution of this problem for all finite values of l, supposing that the given coefficients $\sigma_0, \sigma_1, \cdots, \sigma_l$ are of the same order of magnitude: †

$$y(1) \sim \frac{\pi \mid \sigma_0 \mid n^{l/2+1}}{2^{n-1}(l/2)!} \qquad (l \text{ even}),$$

$$(11) \qquad \left.\begin{aligned} y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{(l+1)/2}}{2^{n-1}[(l-1)/2]!}\{1 + (\sigma_1/\sigma_0)^2\}, & \mid \sigma_1 \mid \leq \mid \sigma_0 \mid \\ y(1) &\sim \frac{\pi \mid \sigma_1 \mid n^{(l+1)/2}}{2^{n-2}[(l-1)/2]!} & \mid \sigma_1 \mid \geq \mid \sigma_0 \mid \end{aligned}\right\} (l \text{ odd}).$$

We may also modify Tchebycheff's problem in the following manner:

Find the minimal deviation from zero in the interval $(-1, +1)$ *of a monotonic polynomial*

$$y(x) = \sum_{i=0}^{n} \sigma_i x^{n-i}$$

its coefficients $\sigma_{n-1}, \sigma_{n-2}, \cdots, \sigma_{n-k}$ *being given.*

* [illegible] de zéro, dont les deux premiers coefficients sont donnés," *Comptes Rendus de l'Académie des Sciences de l'URSS* (1928), pp. 189-490.

† S. Bernstein, "Zusatz zum vorangehenden Artikel der Herren W. Břečka und [illegible]

Supposing that $n \to \infty$ while k is finite and assuming that all given coefficients are of the same order of magnitude, we obtain *

$$(12) \qquad y(1) \sim \frac{2\pi\sigma_{n-1}}{n}\left[\frac{(\nu+1)!}{2^{\nu}[(\nu/2)!]^2}\right]^2,$$

where $\nu = k - 1$, if k is odd and $\nu = k - 2$, if k is even.

2. *On the ratio of two integrals taken between the same limits.*†

Find the extreme values of the ratio

$$(13) \qquad \int_{-1}^{1} y(x)u(x)\,dx : \int_{-1}^{1} y(x)v(x)\,dx,$$

where $y(x)$ is a polynomial of degree $\leq n$; this polynomial and the function $v(x)$ are not negative for $-1 \leq x \leq 1$.

It is easy to show that $y(x)$ must be of the form †

$$(14) \qquad \begin{aligned} &y(x) = (1-x)^{\alpha}(1+x)^{\beta}Z^2(x), \\ &Z(x) = \sum_{i=0}^{m-1} a_i x^i, \qquad (\alpha, \beta = 0, 1;\ \alpha + \beta + 2m - 2 = n). \end{aligned}$$

Putting, with Tchebycheff,

$$(15) \qquad (1-x)^{\alpha}(1+x)^{\beta}u(x) = \theta_0(x), \quad (1-x)^{\alpha}(1+x)^{\beta}v(x) = \theta(x),$$

we must find the extreme values of the integral

$$\int_{-1}^{1} \theta_0(x)Z^2(x)\,dx,$$

being given the value of another integral

$$\int_{-1}^{1} \theta(x)Z^2(x)\,dx = 1, \qquad [\theta(x) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1].$$

Applying the classical method of Analysis, we find the conditions of extremum

$$(16) \qquad \int_{-1}^{1} Z(x)[\theta_0(x) - \lambda\theta(x)]x^k\,dx = 0, \qquad (k = 0, 1, 2, \cdots, m-1).$$

Hence the new parameter λ represents the required extremum

$$(17) \qquad \frac{\int_{-1}^{1} \theta_0(x)Z^2(x)\,dx}{\int_{-1}^{1} \theta(x)Z^2(x)\,dx} = \lambda.$$

* W. Břečka und J. Geronimus, "Ueber das monotone Polynom, welches die minimale Abweichung von Null hat, wenn die Werte seiner ersten Ableitungen gegeben sind," *Mathematische Annalen*, Bd. 102 (1929), S. 514.

† Tchebycheff, *Oeuvres*, T. II, pp. 377-402.

To derive an equation for λ, introduce the quantities

$$(18) \qquad \int_{-1}^{1} [\theta_0(x) - \lambda\theta(x)] x^k dx = c_k, \qquad (k = 0, 1, 2, \cdots),$$

and rewrite (16) in the following form:

$$(19) \qquad \sum_{i=0}^{m-1} a_i c_{s+i} = 0, \qquad (s = 0, 1, 2, \cdots, m-1).$$

It is clear that the determinant

$$(20) \qquad \| c_{ik} \| = 0, \qquad (i, k = 0, 1, \cdots, m-1), \quad c_{ik} = c_{i+k}$$

vanishes, for otherwise we would have $a_0 = a_1 = \cdots = a_{m-1} = 0$. Thus we see that

$$(21) \qquad \lambda_2 \leq \frac{\int_{-1}^{1} \theta_0(x) Z^2(x) dx}{\int_{-1}^{1} \theta(x) Z^2(x) dx} \leq \lambda_1,$$

where λ_1 is the largest and λ_2 is the smallest root of the equation (20). *Particular case:*

$$(22) \qquad \theta_0(x) = (ax+b)^r \theta(x), \qquad (r \text{ positive integer}).$$

Here we can solve our problem without using the equation (20). The conditions (16) of extremum are

$$(23) \qquad \int_{-1}^{1} Z(x)[(ax+b)^r - \lambda]\theta(x) x^k dx = 0, \quad (k = 0, 1, 2, \cdots, m-1),$$

and show that

$$(24) \qquad Z(x)[(ax+b)^r - \lambda] = \sum_{s=0}^{r-1} b_s \phi_{m+s}(x),$$

where the orthogonal and normal polynomials $\{\phi_i(x)\}$ correspond to the interval $(-1, +1)$ with the characteristic function $\theta(x)$. Hence

$$(25) \qquad Z(x) = \frac{\sum_{s=0}^{r-1} b_s \phi_{m+s}(x)}{(ax+b)^r - \mu^r}, \qquad (\mu^r = \lambda).$$

Since $Z(x)$ is a polynomial, we have necessarily

$$(26) \qquad \sum_{s=0}^{r-1} b_s \phi_{m+s}(\nu_k) = 0, \qquad (k = 1, 2, \cdots, r),$$

$$\nu_k = (\mu e^{2\pi i k/r} - b)/a.$$

Thus we get finally (since not all b_s are 0)

$$\mu_2{}^r \leqq \frac{\int_{-1}^{1} \theta(x)\,(ax+b)^r Z^2(x)\,dx}{\int_{-1}^{1} \theta(x) Z^2(x)\,dx} \leq \mu_1{}^r \tag{27}$$

where μ_1 and μ_2 are resp. the largest and the smallest roots of the equation

$$\begin{vmatrix} \phi_m(\nu_1) & \phi_{m+1}(\nu_1) & \cdots & \phi_{m+r-1}(\nu_1) \\ \phi_m(\nu_2) & \phi_{m+1}(\nu_2) & \cdots & \phi_{m+r-1}(\nu_2) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \phi_m(\nu_r) & \phi_{m+1}(\nu_r) & \cdots & \phi_{m+r-1}(\nu_r) \end{vmatrix} = 0. \tag{28}$$

The case $\theta(x)=(1-x)^h$, $\theta_0(x)=(1-x)^{h+1}$ has been discussed by S. Bernstein, who found *

$$\mu_2 \sim 2u/m^2, \qquad (m\to\infty), \tag{29}$$

where u is the smallest root of Bessel's function

$$\mathcal{J}_h(2u^{1/2})=(u^{1/2})^h \sum_{s=0}^{\infty} \frac{(-1)^r u^r}{r!\,(h+r)!}\ . \tag{30}$$

In case $\theta(x)=(1-x)^h$, $\theta_0(x)=(1-x)^{h+2}$ we find † that μ_1 is the largest and μ_2 is the smallest root of the equation

$$P_m(1-\mu)P_{m+1}(1+\mu)-P_{m+1}(1-\mu)P_m(1+\mu)=0, \tag{31}$$

$P_k(x)$ being the normalized Jacobi polynomial corresponding to the interval $(-1,\,+1)$ with the characteristic function $p(x)=(1-x)^h$. It follows, that

$$\mu_2 \sim 2z_0/m^2, \qquad (m\to\infty), \tag{32}$$

where z_0 is the smallest positive root of the integral function

$$F(z)=\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\,(h+k)!\,(h+2k+1)!}\ . \tag{33}$$

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* S. Bernstein, "Sur les polynomes multiplement monotones," *Communications de la Société Mathématique de Kharkow*, IV série, t. I (1927), p. 9.

† W. Břečka und J. Geronimus, "Ueber die monotone Polynome, welche die minimale Abweichung von Null haben," *Mathematische Zeitschrift*, Bd. 30 (1929), Ss. 366-369.

3. *On extreme values of sums involving a polynomial and its derivatives.* The following problem has been solved by Tchebycheff *:

Find the extreme values of the sum

$$\sum_{i=1}^{n} F(x, y, y', y'', \cdots)_{x=x_i} \tag{34}$$

at certain given points $x_1, x_2, \cdots, x_n$, where F is a given polynomial in $x, y, y', \cdots$.

We shall consider a particular case of this problem:

Find the minimum of the sum

$$L = \sum_{i=1}^{n} \theta(x_i) [y(x_i) - f(x_i)]^2, \tag{35}$$

where $\theta(x_i) > 0$, $(i = 1, 2, \cdots, n)$, $f(x)$ is a given polynomial of degree $r \leq n-1$ and

$$y(x) = \sum_{s=0}^{m} A_s x^s, \qquad m < r, \tag{36}$$

the coefficients $A_{i_1}, A_{i_2}, \cdots, A_{i_\nu}$ being given.

Here again we use the orthogonal and normal Tchebycheff polynomials determined by

$$\sum_{i=1}^{n} \theta(x_i) \phi_k(x_i) \phi_s(x_i) = \begin{cases} 0, & k \neq s, \\ 1, & k = s \leq n-1. \end{cases} \tag{37}$$

{They necessarily exist as has been shown by Tchebycheff}. Putting

$$\begin{gathered} y(x) = \sum_{k=0}^{m} a_k \phi_k(x), \qquad f(x) = \sum_{k=0}^{r} b_k \phi_k(x), \\ y(x) - \sum_{k=0}^{m} b_k \phi_k(x) = \sum_{k=0}^{m} c_k \phi_k(x) = y_1(x), \end{gathered} \tag{38}$$

we see that we are to minimize the sum

$$L_1 = L - \sum_{k=m+1}^{r} b_k^2 = \sum_{k=0}^{m} c_k^2 \tag{39}$$

under conditions

$$\sum_{k=0}^{m} c_k \phi_k^{(i_s)}(0) = i_s!\, A_{i_s} - \sum_{k=0}^{m} b_k \phi_k^{(i_s)}(0) = a_s, \qquad (s = 1, 2, \cdots, \nu). \tag{40}$$

Using the classical method we obtain the conditions of extremum

* P. Tchebycheff, "Des maxima et minima des sommes composées des valeurs d'une fonction entière et de ses dérivées," *Oeuvres*, t. II, pp. 3-40; *Journal des Mathématiques pures et appliquées*, II série, t. XIII (1868), p. 9-42.

$$c_k = \sum_{s=1}^{\nu} \lambda_s \phi_k^{(i_s)}(0), \qquad (k = 0, 1, 2, \cdots, m), \tag{41}$$

whence

$$L_1 = \sum_{s=1}^{\nu} \lambda_s a_s. \tag{42}$$

Further we have

$$a_r = \sum_{s=1}^{\nu} \lambda_s a_{sr}, \qquad (r = 1, 2, \cdots, \nu), \tag{43}$$

where we have put

$$a_{kr} = \sum_{j=0}^{m} \phi_j^{(i_k)}(0)\, \phi_j^{(i_r)}(0), \qquad (k, r = 1, 2, \cdots, \nu). \tag{44}$$

From (42) and (43) we see that L may be found from the equation

$$\begin{vmatrix} L - \sum_{k=m+1}^{r} b_k^2 & a_1 & a_2 & \cdots & a_\nu \\ a_1 & a_{11} & a_{21} & \cdots & a_{\nu 1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_\nu & a_{1\nu} & a_{2\nu} & \cdots & a_{\nu\nu} \end{vmatrix} = 0. \tag{45}$$

The polynomial $y(x)$ for which this minimum is attained may be found from the equation

$$\begin{vmatrix} y(x) - \sum_{k=0}^{m} b_k \phi_k(x) & R_1(x) & R_2(x) & \cdots & R_\nu(x) \\ a_1 & a_{11} & a_{21} & \cdots & a_{\nu 1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_\nu & a_{1\nu} & a_{2\nu} & \cdots & a_{\nu\nu} \end{vmatrix} = 0, \tag{46}$$

where

$$R_s(x) = \sum_{r=0}^{m} \phi_r(x) \phi_r^{(i_s)}(0), \qquad (s = 1, 2, \cdots, \nu). \tag{47}$$

In particular, if $\nu = 1$ and the coefficient A_i is given then

$$L = \sum_{k=m+1}^{r} b_k^2 + \frac{\{i!\, A_i - \sum_{k=0}^{m} b_k \phi_k^{(i)}(0)\}^2}{\sum_{k=0}^{m} \{\phi_k^{(i)}(0)\}^2}, \tag{48}$$

and

$$y(x) = \sum_{k=0}^{m} b_k \phi_k(x) + \frac{i!\, A_i - \sum_{k=0}^{m} b_k \phi_k^{(i)}(0)}{\sum_{k=0}^{m} \{\phi_k^{(i)}(0)\}^2} \cdot \sum_{k=0}^{m} \phi_k^{(i)}(0) \phi_k(x). \tag{49}$$

KHARKOW (UKRAINA).

THREE NOTES ON CHARACTERISTIC EXPONENTS AND EQUATIONS OF VARIATION IN CELESTIAL MECHANICS.

By Aurel Wintner.

I. Upon the Characteristic Exponents of the Celestial Mechanics.

In the following there is given a proof of a theorem, first enunciated by Poincaré,* a satisfactory proof for which is not to be found in the works of Poincaré nor in the later literature.† In particular, the theorems of Poincaré upon the position of the characteristic exponents of a dynamical system with two degrees of freedom, for instance in the case of the restricted problem of three bodies,‡ are proven, probably for the first time.

If $A(t) = \| a_{jk}(t) \|$ is a matrix of continuous functions which are defined for $-\infty < t < +\infty$, then the system of differential equations

$$\mathring{x}_j = \sum_{k=1}^{N} a_{jk}(t) x_k \qquad (j = 1, 2, \cdots, N) \tag{1}$$

possesses in the interval $-\infty < t < +\infty$ one and only one solution $x_j = x_j(t)$ which satisfies the N initial conditions $(x_j(t))_{t=0} = x_j(0)$. The matrix $X(t) = \| x_{jk}(t) \|$ of N linearly independent solutions

$$(2) \quad x_1 = x_{m1}(t), \quad x_2 = x_{m2}(t), \cdots, \quad x_N = x_{mN}(t); \qquad (m = 1, 2, \cdots, N)$$

* H. Poincaré, *Méthodes nouvelles de la Mécanique Céleste*, Vol. 1, p. 192.

† Poincaré proceeds (*loc. cit.*) on the assumption that if $x_j = x_j(t)$ is a solution of the differential system (1) [cf. above] and the coefficients of the differential system possess the period T, there must exist a constant matrix $\| \alpha_{jk} \|$ so that

$$(*) \qquad x_j(t+T) = \sum \alpha_{jk} x_k(t)$$

identically in t. It is clear that (*) is not valid [even if one interprets (*) symbolically and conceives of the $x_j(t)$ as vectors, namely as different solutions of (1)].

‡ *Loc. cit.*, Vol. 3, p. 343-344.

is called a *fundamental matrix* of (1). The determinant of the matrix $X(t)$ which satisfies the Jacobian identity

$$\det X(t) = \det X(0) \cdot \exp \int_0^t - \sum_{k=1}^{N} a_{kk}(\tau)\, d\tau, \tag{3}$$

obviously either vanishes identically in t, or vanishes for no value of t. The fundamental matrices are accordingly characterized by

$$\det X(0) \neq 0. \tag{4}$$

If $X(t)$ is a fundamental matrix, then a matrix $Y(t)$ is then and only then a fundamental matrix, if there exists a non-singular matrix K with constant elements, for which

$$Y(t) = KX(t). \tag{5}$$

The matrix K is obviously uniquely determined by the matrices $X(t)$ and $Y(t)$ (Principle of Superposition).

We shall place $N = 2n$ and shall assume the existence of a quadratic form H in the variables x_j (the coefficients of which are given functions of t) of such a character that the system (1) can be written in the canonical form

$$\dot{x}_{2i-1} = \partial H/\partial x_{2i}, \quad \dot{x}_{2i} = -\partial H/\partial x_{2i-1} \qquad (i = 1, 2, \cdots, n = N/2). \tag{6}$$

In order that this be possible, it is necessary and sufficient that the $4n^2$ elements $a_{jk}(t)$ of the *coefficient matrix* $A(t)$ satisfy the following $2n^2 - n$ conditions arising from the peculiar symmetry of the Hamiltonian equations:

$$\begin{aligned} a_{2i-1\,2h} &= a_{2h-1\,2i}, \\ a_{2i\,2h-1} &= a_{2h\,2i-1}, \\ a_{2i-1\,2h-1} &= -a_{2h\,2i}, \end{aligned} \qquad (h, i = 1, 2, \cdots, n). \tag{7}$$

We note in particular that if the matrix $A(t)$ satisfies the above conditions for Hamiltonian symmetry we have $a_{2i2i} = -a_{2i-1\,2i-1}$, so that the sum under the integral sign in (3) vanishes identically and $\det X(t)$ is independent of t. In addition if $x_j = x^{(1)}{}_j(t)$, $x_j = x^{(2)}{}_j(t)$ are two arbitrary solutions of (1) the determinant sum

$$c = \sum_{i=1}^{n} \begin{vmatrix} x^{(1)}{}_{2i-1}(t) & x^{(1)}{}_{2i}(t) \\ x^{(2)}{}_{2i-1}(t) & x^{(2)}{}_{2i}(t) \end{vmatrix} \tag{8}$$

is, as was proven by Poincaré,* independent of t. In order to demonstrate this it is only necessary to calculate the time derivative $\dot{c}$ of (8) which, as follows from (1) and (7), is identically zero.

* *Loc. cit.*, Vol. 1, p. 166-167.

It is accordingly possible to introduce, for a given fundamental matrix $X(t) = \| x_{jk}(t) \|$, by means of the definition

$$(8) \qquad C_X = \| c_{jk} \|; \qquad (j, k = 1, 2, \cdots, 2n-1, 2n),$$

$$c_{jk} = \sum_{i=1}^{n} \begin{vmatrix} x_{j\,2i-1}(t) & x_{j\,2i}(t) \\ x_{k\,2i-1}(t) & x_{k\,2i}(t) \end{vmatrix} \equiv \sum_{i=1}^{n} \begin{vmatrix} x_{j\,2i-1}(0) & x_{j\,2i}(0) \\ x_{k\,2i-1}(0) & x_{k\,2i}(0) \end{vmatrix}$$

a constant matrix of $4n^2$ elements. This matrix is skew-symmetric (but not necessarily real) and will be called the *commutator matrix* of the fundamental matrix $X(t)$. It follows readily from (8) that we have

$$(9) \qquad \det C_X = [\det X(0)]^2.$$

The elements of the commutator matrix of the fundamental matrix KX in (5) may be calculated from (8) and one obtains for the general element of C_{KX}

$$(10) \qquad \sum_{p=1}^{2n} \sum_{q=1}^{2n} l_{jp} c_{pq} l_{kq}$$

in which we understand $K = \| l_{jk} \|$.

We shall now suppose that the matrix $A(t)$ of the coefficients of (1) is periodic and not a constant matrix; hence it possesses a primitive period T such that

$$(11) \qquad a_{jk}(t+T) = a_{jk}(t); \qquad (j, k = 1, 2, \cdots, N).$$

It follows from (11) that the matrix $X(t+T)$ is, simultaneously with $X(t)$, a matrix of solutions of (1) and, from the superposition principle, that there exists for every fundamental matrix $X(t)$ one and only one constant matrix Γ_X for which

$$(12) \qquad X(t+T) = \Gamma_X X(t).$$

This matrix Γ_X will be called the *characteristic matrix* of $X(t)$. It is clear, from (8) and (12), that we have

$$(13) \qquad C_{LX} = C_X \text{ where } L = \Gamma_X,$$

and for the fundamental matrix (5) by means of (12), that

$$(14) \qquad KX(t+T) = K\Gamma_X X(t) = K\Gamma_X K^{-1} \cdot KX(t).$$

One can obviously combine (10) and (14) into the following pair of formulae:

$$(15) \qquad C_{KX} = KC_X K', \qquad \Gamma_{KX} = K\Gamma_X K^{-1}.$$

In these equations the accent is used to denote the transposed matrix and

K is any non-singular, constant matrix so that (15), as follows from the superposition principle (5), exhibits the connection between the various commutator and characteristic matrices of the different fundamental matrices of (1).

It follows from (4), (9) and (12) that for every fundamental matrix $X(t)$ we have

$$(16)\qquad \det C_X \neq 0, \qquad \det \Gamma_X \neq 0.$$

From the second formula of (15) it is clear that the equation of the $2n$-th degree

$$(17)\qquad \det(\lambda E - \Gamma_X) = 0,$$

together with the elementary divisors, is invariant if one introduces a new fundamental matrix $X(t)$, so that one may speak simply of the invariants (i. e. characteristic constants and elementary divisors) of the characteristic group belonging to the periodic coefficient matrix $A(t)$. From the first equation of (15) we obtain because of (13)

$$(18)\qquad C_X = \Gamma_X C_X \Gamma'_X.$$

Since C_X and Γ_X are non-singular [cf. (16)] we may write this equation in the form

$$(19)\qquad \Gamma'_X = C^{-1}{}_X \Gamma^{-1}{}_X C_X.$$

Consequently Γ'_X and $\Gamma^{-1}{}_X$ and therefore Γ_X and $\Gamma^{-1}{}_X$ possess the same characteristic constants. Now the characteristic constants of the reciprocal matrix of a matrix M are always the reciprocals of the characteristic constants of the matrix M. *We therefore conclude that* (17) *is a reciprocal equation,* i. e., by a suitable choice of the notation for the characteristic constants of the characteristic group we may write

$$(20)\qquad \lambda_{2i-1}\lambda_{2i} = 1, \qquad (i = 1, 2, \cdots, n).$$

We may infer that if the coefficient matrix $A(t)$ is real, the coefficients of the algebraic equation (17) are evidently real and therefore complex characteristic numbers λ_j of the characteristic group can occur only in conjugate pairs.

If now we introduce in place of the "multipliers" λ_j the "characteristic exponents" ρ_j defined by the equations

$$(21)\qquad \rho_j = -(-1)^{1/2}(T/2\pi)\log \lambda_j; \qquad (j = 1, 2, \cdots, 2n-1, 2n),$$

(which are only determined mod 1), equations (20) are equivalent to

$$(22)\qquad \rho_{2i-1} + \rho_{2i} = 0; \qquad (i = 1, 2, \cdots, n).$$

It is desirable to emphasize at this point that the characteristic group becomes illusory if (11) is fulfilled for every T, or in other words, if the coefficient matrix A of (1) is a constant matrix (this case has previously been excluded). If T is arbitrary, the characteristic matrix Γ_X of a given fundamental matrix $X(t)$ is in no case determined by (12) and indeed, because T in (12) is arbitrary, a continuum of different determinations is possible. It would be accordingly useless to seek for a direct connection, by means of continuity considerations between the characteristic constants of a constant matrix A, and the characteristic constants of the characteristic group which are only defined for the non-constant, periodic coefficient matrices (11). Such attempts in the literature, for example in the case of the small periodic orbits about the Lagrangian libration points of the restricted problem of three bodies, have led to various misunderstandings. On the contrary certain continuity considerations of Liapounoff * can readily be justified in the following manner: For a coefficient matrix A, independent of t, we introduce in the system (1) a new independent variable t defined by †

$$(21') \qquad \tau = -(-1)^{1/2} \log t.$$

The Hamiltonian character of the differential equations obviously remains unchanged and the coefficient matrix of the new differential system, provided that A is not the zero matrix, is a periodic function of t possessing the primitive period 2π. One perceives quite readily that the invariants of the characteristic group of the new differential system coincide, up to the conformal transformation (21), with the invariants of the original differential system from which it follows that the characteristic constants ω_j of a constant matrix A satisfying the conditions (7) are connected not by (20) but by the relation ‡

* A. Liapounoff, *Annales de Toulouse* (2), 1907, p. 413. The continuity considerations of Liapounoff (in the application of the method of Liapounoff) are not permissible for a periodic solution of a dynamical problem since in this case there always occurs a multiple root in the characteristic equations. This arises from (20) and the well known fact that for the equations of variations belonging to a periodic solution (which is not independent of t) at least one root of the characteristic equation is unity.

† The transformation (21′) is nothing more than the transformation of Euler employed to transform a differential system (1) with constant coefficient matrix A (which is therefore not of the Fuchsian type) into a differential system of the Fuchsian type.

‡ The theorem (22) has been proven (under an unessential [illegible]) in a direct manner by an application of the method of Liapounoff. Cf. G. D. Birkhoff, "Dynamical Systems," *American Mathematical Society Colloquium Publications*, Vol. 9, pp. 77-78.

$$(22') \qquad \omega_{2i-1} + \omega_{2i} = 0; \qquad (i = 1, 2, \cdots, n).$$

We proceed now to the canonical equations of motion

$$(23) \qquad \dot{y}_{2i-1} = \partial F/\partial y_{2i}, \qquad \dot{y}_{2i} = -\partial F/\partial y_{2i-1}; \qquad (i = 1, 2, \cdots, n)$$

in which we shall assume that F is independent of t and possesses continuous second derivatives with respect to the arguments. We consider a one-parametric sheaf of curves

$$(24) \qquad y_j = \bar{y}_j(t;\epsilon); \qquad (j = 1, 2, \cdots, 2n-1, 2n)$$

possessing continuous second derivatives with respect to the variables t and ϵ, the member of the sheaf corresponding to $\epsilon = 0$ being a solution

$$(25) \qquad y_j = y_j^0(t) \equiv \bar{y}_j(t;0)$$

of (23), the sheaf being otherwise perfectly arbitrary.

If we introduce the notation

$$\delta = (\partial/\partial\epsilon)_{\epsilon=0}, \qquad \delta^2 = \tfrac{1}{2}(\partial^2/\partial\epsilon^2)_{\epsilon=0}$$

in connection with the sheaf (24), the equations

$$(26) \qquad \delta\dot{y}_{2i-1} = \delta(\partial F/\partial y_{2i}), \qquad \delta\dot{y}_{2i} = -\delta(\partial F/\partial y_{2i-1})$$

are called the variational equations of (23) belonging to the solution (25). If we now write

$$(27) \qquad x_j = x_j(t) = \delta y_j \equiv (\partial\bar{y}_j(t;\epsilon)/\partial\epsilon)_{\epsilon=0}$$

and

$$(28) \qquad \delta^2 F = \sum_{j=1}^{2n} \sum_{k=1}^{2n} F^0_{y_j y_k}(t) x_j x_k,$$

where we understand

$$(29) \qquad F^0_{y_j y_k}(t) = F_{y_j y_k}(y_1^0(t), \cdots, y^0_{2n}(t)), \qquad F_{y_j y_k} = \partial^2 F/\partial y_j \partial y_k$$

and if we identify H with (28), equation (26) can be written in the form of (6). The variations (27) accordingly satisfy a differential system (1), the coefficients of which, as follows from the second part of (29), satisfy the conditions (7), and are uniquely determined by (29) and the initial solution (25). The coefficients of (1) are independent of the arbitrary sheaf (24) and if (25) is a periodic solution, it follows from (29) that (11) is also fulfilled.

II. Upon the Characteristic Exponents in the Strömgrenian Groups of Periodic Orbits.

In this note the theory of the characteristic exponents * is applied to the so-called groups † of periodic orbits. The final purpose is the treatment of a question which Strömgren and Burrau called to my attention, some time ago. Strömgren had found, by means of mechanical quadratures and harmonic analysis, in the Copenhagen group n of periodic solutions of the restricted problem of three bodies, an orbit ‡ which, while it is a simple orbit (i. e. a curve of Jordan), is nevertheless the limiting position of double orbits of the group each of which possesses two circuits before re-entering into itself. Burrau and Strömgren inquired if it would not be possible to find an analytical condition for such a coalescence of two circuits. This problem is simply the converse of the one treated by Poincaré § in his theory of the second "genre." The answer to the problem of Strömgren and Burrau obtained by the method of the present note has led to the conjecture that the results of Poincaré ¶ concerning the existence of his periodic solutions of the second "genre" cannot be correct without some additional restrictions. It has been easy to find an example, mentioned at the end of this note, showing that the existence statements of Poincaré ‖ (for which no satisfactory proof was given) are not valid even for the periodic groups of the restricted problem of three bodies.

We shall denote by

$$\xi = \xi^0(t), \qquad \eta = \eta^0(t) \tag{1}$$

a given solution of the differential equations

$$\ddot{\xi} - 2\dot{\eta} = \Omega_\xi(\xi, \eta), \qquad \ddot{\eta} + 2\dot{\xi} = \Omega_\eta(\xi, \eta); \qquad (\Omega_\zeta = \partial\Omega/\partial\zeta) \tag{2}$$

of the restricted problem of three bodies and by

$$\xi = \bar{\xi}(t, \epsilon), \quad \eta = \bar{\eta}(t, \epsilon); \qquad \bar{\xi}(t, 0) = \xi^0(t), \quad \bar{\eta}(t, 0) = \eta^0(t) \tag{3}$$

any sheaf of curves (fulfilling the usual differentiability conditions) which

* H. Poincaré, *Les méthodes nouvelles de la Mécanique Céleste*, Vol. 3 (1899), Chap. XXVIII, and Chap. XXXI.

† E. Strömgren, "Forms of Periodic Motion in the Restricted Problem etc.," *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, Nr. 39 (1922).

‡ Cf., for instance, E. Strömgren, *loc. cit.* †, p. [illegible].

§ Poincaré, *loc. cit.* *, p. 201 etc.

¶ Poincaré, *loc. cit.* *, p. 226 etc. or p. 331 etc.

‖ H. Poincaré, *loc. cit.* *, previous reference and also p. 351 or pp. 355-356.

for the value $\epsilon = 0$ of the parameter is identical with (1), without necessarily being a solution of (2) for $\epsilon \neq 0$. If one places, for any given function G or F of ξ and η,

$$\delta F^0(t) = [(\partial/\partial\epsilon) F(\bar{\xi}(t,\epsilon), \bar{\eta}(t,\epsilon))]_{\epsilon=0}, \qquad G^0(t) = G(\xi^0(t), \eta^0(t))$$

and employs the abbreviations $\Omega_{\xi\eta} = \partial^2\Omega/\partial\xi\partial\eta, \cdots$ and

$$(4) \qquad \begin{aligned} u = u(t) &= [(\partial/\partial\epsilon)\bar{\xi}(t,\epsilon)]_{\epsilon=0} = \delta\xi^0(t), \\ v = v(t) &= [(\partial/\partial\epsilon)\bar{\eta}(t,\epsilon)]_{\epsilon=0} = \delta\eta^0(t), \end{aligned}$$

the so-called variational equations of (2), belonging to (1), are

$$(5) \quad \ddot{u} - 2\dot{v} = \Omega^0{}_{\xi\xi}(t)u + \Omega^0{}_{\xi\eta}(t)v, \quad \ddot{v} - 2\dot{u} = \Omega^0{}_{\eta\xi}(t)u + \Omega^0{}_{\eta\eta}(t)v.$$

The coefficients of these linear differential equations, belonging to the given solution (1) of (2), are independent of the special choice of the sheaf (3) so that any sheaf yields by means of (4) a solution of (5). The coefficients of the linear differential equations (5) have obviously the period T if the given solution (1) of (2) has the period T, as will be supposed in the following. We shall exclude the trivial case where (1) is one of the five equilibrium solutions of Lagrange, that is we shall suppose that (1) is not independent of t, so that

$$(6) \qquad u = \dot{\xi}^0(t), \qquad v = \dot{\eta}^0(t),$$

which obviously* constitute a solution of (5), are not identically zero. According to the classical theory of homogeneous linear differential equations with periodic coefficients there exists a uniquely determined real quartic equation

$$(7) \qquad \det(\lambda E - \Gamma) = 0, \quad \text{where} \quad \det \Gamma \neq 0,$$

the roots λ of which are characterized by the fact that (5) possesses at least one not identically vanishing solution with the multiplicative property

$$(8) \qquad u(t+T) = \lambda u(t), \qquad v(t+T) = \lambda v(t).$$

The condition (8) can be written in the form

$$(9) \qquad u(t) = \phi(t) \exp 2\pi i\rho t/T, \qquad v(t) = \psi(t) \exp 2\pi i\rho t/T,$$
$$\phi(t+T) = \phi(t), \qquad \psi(t+T) = \psi(t),$$

where

$$(10) \qquad \lambda = \exp 2\pi i\rho/T.$$

---

* If we define the sheaf (3) with the use of (1) as $\bar{\xi}(t,\epsilon) = \xi^0(t+\epsilon)$, $\bar{\eta}(t,\epsilon) = \eta^0(t+\epsilon)$ the Jacobian rule (4) yields for (5) the solution (6).

If the four roots λ_j of the quartic equation (7) are distinct or at least if the corresponding elementary divisors are simple, equations (8) yield four linearly independent solutions of (5) and therefore also the general solution of (5). If not all elementary divisors are simple, there can not exist four linearly independent solutions with the multiplicative property (9) and the general solution will contain secular terms.

Since (1) has the period T and is not independent of t it follows from (6) and (8) that at least one of the four roots λ_j of the characteristic equation (7) must be equal to one, say

(11) $$\lambda_1 = 1.$$

Furthermore it follows from (6) that not all coefficients of the linear differential equations (5) can be constant; otherwise the periodic solution (6) of (5) and therefore also the functions (1) would have the form

$$a + b\cos 2\pi(t - t_0)/T; \qquad (b \neq 0),$$

whereas the non-linear differential equations (2) of the restricted problem of three bodies do not possess a solution (1) of this elementary character. Since not all coefficients of (5) are independent of t it follows from a general theorem * on dynamical systems that the quartic equation (7) is a reciprocal one, that is we have $\lambda_1\lambda_2 = 1$, $\lambda_3\lambda_4 = 1$, or, according to (11),

(12) $$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = \lambda^0, \quad \lambda_4 = 1/\lambda^0.$$

λ^0 is here a number which is not zero † and which can be real or complex; if it is complex it must be of modulus unity inasmuch as the complex roots of the real quartic equation can occur only in conjugate pairs. The number λ^0 which lies either on the real axis or on the boundary of the unit circle determines ‡ (in the sense of the characteristic exponents) the stability character of the given solution (1) or (2); cf. (9), (10), (12). The two § possible domains $\lambda^0 > 0$, $\lambda^0 < 0$ of instability are joined with the domain $|\lambda^0| = 1$ of stability at the two indifferent points $\lambda^0 = 1$, $\lambda^0 = -1$ respectively.¶ The indifferent case $\lambda^0 = -1$ is of special importance in what follows. The proofs are also valid for a periodic ejection solution (1)

* Cf. p. 608.

† Otherwise (7) would yield $\det \Gamma = 0$ whereas Γ is, according to (7), a non-singular matrix.

‡ The stability number λ^0 of (1) may be calculated, according to the method of [illegible]

inasmuch as the above mentioned theorems hold also in the regularizing variables of Thiele.

We now suppose that we have instead of one periodic solution (1) an analytic sheaf *

$$(13)\qquad \xi=\xi^0(t;C),\qquad \eta=\eta^0(t;C)$$

of solutions of (2) with the period $T=T(C)$ so that (1) is contained in (13) for a special value of C. The sheaf (12) is, in the sense of Strömgren,† a "group" of periodic solutions (the mass ratio μ having an arbitrarily *fixed* value). The number λ^0 (or the corresponding characteristic exponent ρ^0) which defines the stability character ‡ of the path (1) is also a function of the parameter C of the group:

$$(13')\qquad \lambda^0=\lambda^0(C)=\exp 2\pi i\,\rho^0(C)/T(C)\qquad [\text{cf. } (10),\ (12)].$$

In order to apply the stability function (13′) of the group (13) to the problem of multiple paths mentioned in the introduction, we consider in the domain of the group parameter C a point $C=C_0$ having the property that the *primitive* (that is smallest) period of the solution (13) is for $C<C_0$ equal to $T(C)$ but for $C=C_0$ equal to $T(C_0)/p$ where p is a positive integer different from 1. In other words the path is for $C<C_0$ a "p-fold" orbit which shrinks for $C=C_0$ to a "simple" orbit, the primitive period being p-times smaller than $T(C)$ if $C=C_0$ (the integer p is in the case mentioned in the introduction equal to 2). We shall call $C=C_0$ a shrinking orbit of the order p. If we write

$$(14)\qquad T_0=T(C_0),\qquad \lambda_0=\lambda^0(C_0)$$

the primitive period of (13) for $C=C_0$ is T_0/p and

$$(15)\qquad \xi=\xi_0(t)=\xi^0(t/p;C_0),\qquad \eta=\eta_0(t)=\eta^0(t/p;C_0)$$

is a solution of (2) with the primitive period T_0. If we identify (1) with

* For the dynamical meaning of the parameter C of the sheaf cf. G. Herglotz, *Seeliger-Festschrift* (1924), pp. 197-199.

† Cf., for instance, *loc. cit.*

‡ It is necessary to refer the stability character of (13) to the characteristic exponent λ^0 alone, independent of the question whether (5) possesses a secular solution or not. Without this convention no group for which the period is not independent of the group parameter C would possess a stable domain. In order to show this it is only necessary to define the sheaf (3) with the use of (13) as $\bar\xi(t,\epsilon)=\xi^0(t;C+\epsilon)$, $\bar\eta(t,\epsilon)=\eta^0(t;C+\epsilon)$. The rule (4) yields in the case, if C is not a stationary point of the function $T=T(C)$, a secular solution of (5) [one need only differentiate the Fourier series of (13) term by term]. This secular solution, together with the periodic solution (6), furnishes two solutions of (5) belonging to $\lambda_1=\lambda_2=1$.

(15) the rule (6) yields for (5) the particular solution

$$u = \dot{\xi}_0(t), \qquad v = \dot{\eta}_0(t) \tag{16}$$

where

$$\dot{\xi}_0(t + T_0) = \dot{\xi}_0(t), \qquad \dot{\eta}_0(t + T_0) = \dot{\eta}_0(t) \tag{17}$$

and T_0 is the *primitive* period of (16). If

$$u = u_j(t), \qquad v = v_j(t); \qquad (j = 1, 2, 3, 4) \tag{18}$$

is any system of linear independent solutions of (5) the solution (16) may be represented in the form

$$\dot{\xi}_0(t) = \sum_{j=1}^{4} c_j u_j(t), \qquad \dot{\eta}_0(t) = \sum_{j=1}^{4} c_j v_j(t). \tag{19}$$

It has been pointed out that for the multiplicative condition (8) one can choose (18) in such a manner, that for a *fixed* value of j the solution $u = u_j(t)$, $v = v_j(t)$ either fulfills the two conditions

$$u_j(t + T_0/p) = \lambda_j u_j(t), \qquad v_j(t + T_0/p) = \lambda_j v_j(t) \tag{20}$$

or it contains a secular term. If $u = u_j(t)$, $v = v_j(t)$ contains a secular term the corresponding coefficient c_j must vanish inasmuch as the superposition (19) of (18) is periodic. It therefore follows from (20) that

$$c_j u_j(t + T_0/p) = c_j \lambda_j u_j(t), \quad c_j v_j(t + T_0/p) = c_j \lambda_j v_j(t) \tag{21}$$

holds for all four values of j whereas (20) was valid only for such values of j for which $u = u_j(t)$, $v = v_j(t)$ do not contain secular terms. From (12) and (13′) follows

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = \lambda_0, \quad \lambda_4 = 1/\lambda_0 \tag{22}$$

and the equations (21) yield by p-fold iteration

$$c_j u_j(t + T_0) = c_j \lambda_j^p u_j(t), \quad c_j v_j(t + T_0) = c_j \lambda_j^p v_j(t). \tag{23}$$

Since the four solutions (18) are linearly independent we obtain from (17), (19) and (23) the four conditions $c_j(\lambda_j^p - 1) = 0$ which can be written, according to (22), in the form

$$c_3(\lambda_0^p - 1) = 0, \qquad c_4(\lambda_0^{-p} - 1) = 0. \tag{24}$$

[illegible]

[illegible]

[illegible]

[illegible] (16) [illegible]

the *primitive* period of (16) being, according to (15), equal to $T_0 > T_0/p$. From the assumption that λ_0 is not a *primitive* p-th root of unity one obtains in the same manner the same contradiction. We therefore have the following theorem:

In order that $C = C_0$ should be a shrinking point of the order p, it is necessary that the stability function (13′) *of the group should be for $C = C_0$ a p-th primitive root of unity. In particular for the case $p = 2$ mentioned in the introduction we have the necessary condition*

$$\lambda^0(C_0) = -1. \tag{25}$$

It may be pointed out that the p-fold iteration of the general solution of the variational equations used above is essentially the same as the topological iteration, initiated by Poincaré and Levi-Civita, and developed by Birkhoff in his researches on the corresponding surface transformations associated with dynamical systems of two degrees of freedom.

The question now arises whether the necessary condition just given is also a sufficient one; for instance, in the case $p = 2$, whether all paths $C = C_0$, for which the stability function (13′) fulfills the condition (25), must be simple limiting positions of double paths of the given group. The answer would be an affirmative one if the existence statements of Poincaré concerning his second "genre" were correct. However the conclusions of Poincaré are not satisfactory inasmuch as the main difficulty of the question, namely the explicit compatibility and reality discussion of certain finite non-linear equations of condition ("Verzweigungsgleichungen") is not treated in all its details. It it possible to give an example which shows that also the final result is wrong, i. e. that the second "genre" need not exist under the condition stated by Poincaré and that the necessary condition (25) is not a sufficient one.

If one treats in particular the group g of Strömgren, which originates with the moon orbits, in an analytical manner it is easy to show that the orbits are, for a sufficiently small *fixed* value of the mass ratio μ, simple orbits (curves of Jordan), insofar as the orbit does not lie too close to the Hecuba gap. In addition, if the mass ratio is sufficiently small, the group represents, exclusive of the immediate neighborhood of the Hecuba gap, a regular sheaf (without branch points with respect to the Jacobian constant C). Nevertheless there exist in the domain between the moon orbits and the Hecuba gap, namely in the vicinity of the Hestia commensurability, two values of C for which $\lambda^0(C) = -1$. This follows readily by a combination of previous results of Birkhoff, von Zeipel, and of the writer.—For details cf. a paper which will be published in the *Mathematische Zeitschrift*.

III. Upon the Equation of Jacobi for Dynamical Systems with Two Degrees of Freedom.

In this note the method of reduction of the equations of variation for conservative dynamical systems of two degrees of freedom which Hill * employed in his treatment of the motion of the lunar perigee (in which he initiated the use of infinite determinants), is placed upon an analytical basis. In particular certain unsatisfactory points in Poincaré's † treatment of the equation of normal displacement (which will be calculated explicitly) are removed. There is given a proof of the isoenergetic equivalence of the reduced equation of the *second* order (equation of Jacobi) and the original equations of the *fourth* order, and in the course of the proof it will become clear how such a paradoxical fact is possible. For the special case where the variations are those of a periodic solution, one obtains the corresponding equivalence proof for the characteristic exponents and it is then easy to see precisely why the principle of Maupertuis ‡ can be employed to determine the pair of non-trivial characteristic exponents (in so far as the periodic solution is not an equilibrium solution).

The differential equations of the most general conservative dynamical system with two degrees of freedom can be reduced to the form §

$$(1) \quad \ddot{x} - 2\lambda(x, y)\dot{y} = \Omega_x(x, y), \quad \ddot{y} + 2\lambda(x, y)\dot{x} = \Omega_y(x, y), \quad (\cdot = d/dt),$$

where $\lambda = \lambda(x, y)$ and $\Omega = \Omega(x, y)$ are given functions of x and y. Let

$$(2) \qquad x = x^0(t), \qquad y = y^0(t)$$

be a given solution of (1). We shall use the abbreviations

* G. W. Hill, *Collected Mathematical Works*, Vol. 1 (1905), p. 244 ff. Cf. also G. H. Darwin, *Scientific Papers*, Vol. 4 (1911), p. 27 ff.

† H. Poincaré, *Méthodes nouvelles de la Mécanique Céleste*, Vol. 3 (1899), Chap. XXIX. In this respect cf. also a note of mine to appear in the volume for 1930 of the *Berichte der mathematisch-physischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig*, [illegible] to the literature concerning the relation of the dynamical equation of Jacobi to the calculus of variations.

‡ [illegible]

$$(3)\qquad \begin{aligned} &F^0(t)=F(x^0(t),y^0(t)), \qquad F_x{}^0(t)=F_x(x^0(t),\ y^0(t)),\\ &F^0{}_{xy}(t)=F_{xy}(x^0(t),y^0(t)),\cdots;\ F_x=\partial F/\partial x,\ F_{xy}=\partial^2F/\partial x\partial y,\cdots, \end{aligned}$$

$F=F(x,y)$ being either the function $\lambda=\lambda(x,y)$ or the function $\Omega=\Omega(x,y)$.

A pair of functions

$$(4)\qquad u=u(t), \qquad v=v(t)$$

is called a variation belonging to the solution (2) if it satisfies the homogeneous differential equations

$$(5)\qquad \begin{aligned} \ddot{u}-2\lambda^0(t)\dot{v}&=[\Omega^0{}_{xx}(t)+2\lambda_x{}^0(t)\dot{y}^0(t)]u+[\Omega^0{}_{xy}(t)+2\lambda_y{}^0(t)\dot{y}^0(t)]v,\\ \ddot{v}+2\lambda^0(t)\dot{u}&=[\Omega^0{}_{yx}(t)-2\lambda_x{}^0(t)\dot{x}^0(t)]u+[\Omega^0{}_{yy}(t)-2\lambda_y{}^0(t)\dot{x}^0(t)]v. \end{aligned}$$

The coefficients of this linear differential system of the fourth order are given functions of t, which are defined by (2) and (3). The differential system (5) possesses the homogeneous linear integral

$$(6)\qquad \dot{x}^0(t)\dot{u}+\dot{y}^0(t)\dot{v}-\Omega_x{}^0(t)u-\Omega_y{}^0(t)v=c,$$

the time derivative of (6) being, by means of (5), identically zero. The differential equations (5) are necessary and sufficient in order that

$$(4')\qquad x=x^0+u, \qquad y=y^0+v \qquad\qquad [\text{cf. } (2)]$$

should be, up to terms of second order in u, v, a solution of (1). We have, therefore, $\mu=\delta x$, $v=\delta y$ and (6) can be derived in a *formal* manner from the vis viva integral

$$(7)\qquad \tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-\Omega(x,y)=C$$

of (1), if one places $c=\delta C$. We shall call, therefore, a variation (4), i. e. a solution of (5) then and only then an isoenergetic or Maupertuisian variation if the integration constant c in (6), determined by the four initial values of the solution (4) of (5), vanishes so that we have, for all values of t,

$$(8)\qquad \dot{x}^0(t)\dot{u}(t)+\dot{y}^0(t)\dot{v}(t)-\Omega_x{}^0(t)u(t)-\Omega_y{}^0(t)v(t)=0.$$

If one introduces (2) in (1) and in (7) and differentiates (1) and (7) with respect to t, one obtains exactly the relations (5) and (8), where

$$(9)\qquad u=\dot{x}^0(t), \qquad v=\dot{y}^0(t),$$

i. e. (9) is an isoenergetic variation of (2). We shall now suppose that the given initial solution (2) of (1) is not an equilibrium solution, i. e. the solution (9) of (5) is not identically zero. It is then possible to choose the origin $t=0$ of the t-axis in such a manner that for the given solution (2) of (1)

$$(10) \qquad x^0(0) \neq 0, \quad y^0(0) \neq 0, \qquad \dot{x}^0(t) \neq 0, \quad \dot{y}^0(0) \neq 0$$

(it is of course possible that the solution (2) lies on a straight line of the (x, y)-plane, but one can rotate the coördinate system in such a manner that this straight line should not be parallel to either of the two coördinate axes). The solution (4) of (5), defined by the four initial values

$$(11) \qquad u(0), \quad v(0), \qquad \dot{u}(0), \quad \dot{v}(0),$$

is then and only then an isoenergetic variation if

$$(8') \qquad \dot{x}^0(0)\dot{u}(0) + \dot{y}^0(0)\dot{v}(0) - \Omega_x{}^0(0)u(0) - \Omega_y{}^0(0)v(0) = 0,$$

inasmuch as (6) is an integral of (5). It follows from (10) and (8′) that the manifold of the isoenergetic variations depends on three of the four arbitrary constants (11).

The projection of the variation $u = \delta x$, $v = \delta y$ on the orientated normal of the curve (2), belonging to a fixed value of t, is

$$(12) \qquad \zeta(t) = \frac{-\dot{y}^0(t)}{\sqrt{\dot{x}^0(t)^2 + \dot{y}^0(t)^2}}\, u(t) + \frac{\dot{x}^0(t)}{\sqrt{\dot{x}^0(t)^2 + \dot{y}^0(t)^2}}\, v(t).$$

We put

$$(13) \qquad \vartheta = \vartheta(t) = \begin{vmatrix} \dot{x}^0(t) & \dot{y}^0(t) \\ u(t) & v(t) \end{vmatrix}$$

so that

$$(14) \qquad \vartheta = [(\dot{x}^0)^2 + (\dot{y}^0)^2]^{1/2}\, \zeta.$$

We shall call a given function of t then and only then an isoenergetic normal displacement of (2) if there exists at least one isoenergetic variation (4) by means of which the given function may be represented in the form (12). We want to show that the isoenergetic normal displacements of (2) can be characterised as the solutions of a linear differential equation of the second order (equation of Jacobi). First of all we notice the Lagrangian relation

$$\begin{aligned} &[X_2V_0 - Y_2U_0 + X_1V_1 - Y_1U_1][X_1X_2 + Y_1Y_2] \\ &- [U_1X_1 + V_1Y_1 - U_0X_2 - V_0Y_2][X_1Y_2 - X_2Y_1] \\ &+ [U_1Y_2 - V_1X_2][X_1{}^2 + Y_1{}^2] - [V_0X_1 - U_0Y_1][X_2{}^2 + Y_2{}^2] = 0 \end{aligned}$$

(used also by Poincaré) which is identically fulfilled for the ten independent variables. In particular we have

$$(15) \qquad \begin{aligned} &\dot{\vartheta}[\dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\ddot{y}^0(t)] \\ &- [\dot{u}\dot{x}^0(t) + \dot{v}\dot{y}^0(t) - u\ddot{x}^0(t) - v\ddot{y}^0(t)][\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)] \\ &+ [\dot{u}\ddot{y}^0(t) - \dot{v}\ddot{x}^0(t)][\dot{x}^0(t)^2 + \dot{y}^0(t)^2] - \vartheta[\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] = 0 \end{aligned}$$

if we use the abbreviated notation (13).

Let (4) be an isoenergetic variation so that (5) and (8) are fulfilled. From (1), (2), (3) follows

$$(16) \qquad \Omega_x{}^0(t) = \ddot{x}^0(t) - 2\lambda^0(t)\dot{y}^0(t), \quad \Omega_y{}^0(t) = \ddot{y}^0(t) + 2\lambda^0(t)\dot{x}^0(t)$$

and therefore from (8)

$$\dot{x}^0(t)\dot{u} + \dot{y}^0(t)\dot{v} - [\ddot{x}^0(t) - 2\lambda^0(t)\dot{y}^0(t)]u - [\ddot{y}^0(t) + 2\lambda^0(t)\dot{x}^0(t)]v = 0$$

i. e. by means of (13)

$$(17) \qquad -2\lambda_0(t)\vartheta = \ddot{x}^0(t)u + \ddot{y}^0(t)v - \dot{x}^0(t)\dot{u} - \dot{y}^0(t)\dot{v}.$$

The definition (13) of ϑ yields

$$(18) \qquad \ddot{\vartheta} + 2[\dot{u}\ddot{y}^0(t) - \dot{v}\ddot{x}^0(t)] = v\dddot{x}^0(t) - u\dddot{y}^0(t) + \ddot{v}\dot{x}^0(t) - \ddot{u}\dot{y}^0(t)$$

and from (16) and (4) follows

$$\begin{aligned}
\dddot{x}^0(t) &= 2\lambda^0(t)\ddot{y}^0(t) + [\Omega^0{}_{xx}(t) + 2\lambda_x{}^0(t)\dot{y}^0(t)]\dot{x}^0(t) \\
&\qquad + [\Omega^0{}_{xy}(t) + 2\lambda_y{}^0(t)\dot{y}^0(t)]\dot{y}^0(t), \\
\dddot{y}^0(t) &= -2\lambda^0(t)\ddot{x}^0(t) + [\Omega^0{}_{yx}(t) - 2\lambda_x{}^0(t)\dot{x}^0(t)]\dot{x}^0(t) \\
&\qquad + [\Omega^0{}_{yy}(t) - 2\lambda_y{}^0(t)\dot{x}^0(t)]\,\dot{y}^0(t), \\
\ddot{v} &= -2\lambda^0(t)\dot{u} + [\Omega^0{}_{yx}(t) - 2\lambda_x{}^0(t)\dot{x}^0(t)]u \\
&\qquad + [\Omega^0{}_{yy}(t) - 2\lambda_y{}^0(t)\dot{x}^0(t)]v, \\
\ddot{u} &= 2\lambda^0(t)\dot{v} + [\Omega^0{}_{xx}(t) + 2\lambda_x{}^0(t)\dot{y}^0(t)]u \\
&\qquad + [\Omega^0{}_{xy}(t) + 2\lambda_y{}^0(t)\dot{y}^0(t)]v.
\end{aligned}$$

If one multiplies these four equations by v, $-u$, $\dot{x}^0(t)$, $-\dot{y}^0(t)$ respectively and adds the products, one obtains

$$\begin{aligned}
&v\dddot{x}^0(t) - u\dddot{y}^0(t) + \ddot{v}\dot{x}^0(t) - \ddot{u}\dot{y}^0(t) \\
&= 2\lambda_0(t)[\ddot{x}^0(t)u + \ddot{y}^0(t)v - \dot{x}^0(t)\dot{u} - \dot{y}^0(t)\dot{v}] \\
&\quad + [\Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t) - 2\{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}][\dot{x}^0(t)v - \dot{y}^0(t)u],
\end{aligned}$$

i. e. by means of (18), (17) and (13)

$$(19) \qquad \ddot{\vartheta} + 2[\dot{u}\ddot{y}^0(t) - \dot{v}\ddot{x}^0(t)] = L(t)\vartheta,$$

where

$$(20) \qquad L(t) = -4\lambda^0(t)^2 + \Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t) - \{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}.$$

Introducing (16) and (19) in (15) one obtains

$$\begin{aligned}
&\dot{\vartheta}[\dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\ddot{y}^0(t)] + 2\lambda^0(t)[\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)]\vartheta \\
&\qquad + \tfrac{1}{2}[-\ddot{\vartheta} + L(t)][\dot{x}^0(t)^2 + \dot{y}^0(t)^2] - \vartheta[\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] = 0,
\end{aligned}$$

i. e.

(21) $$\alpha(t)\ddot{\vartheta} + \beta(t)\dot{\vartheta} + \gamma(t)\vartheta = 0,$$

where [cf. (20)]

(22) $$\begin{aligned} \alpha(t) &= -\tfrac{1}{2}[\dot{x}^0(t)^2 + \dot{y}^0(t)^2], \\ \beta(t) &= \dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\ddot{y}^0(t), \\ \gamma(t) &= 2\lambda^0(t)[\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)] - [\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] \\ &\quad + \tfrac{1}{2}[\dot{x}^0(t)^2 + \dot{y}^0(t)^2][-4\lambda^0(t)^2 + \Omega^0_{xx}(t) + \Omega^0_{yy}(t) \\ &\quad - \{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}]. \end{aligned}$$

If one places (14) in (21) there follows finally the self-adjoint equation

(23) $$\ddot{\zeta} + \nu(t)\zeta = 0,$$

where

(24) $$\nu(t)$$

is a given function of t, uniquely determined by (22), i. e. any isoenergetic normal dispalcement (2) is a solution of the differential equation (23). We shall now show that any solution of the differential equation (23) is an isoenergetic normal displacement of (2), in other words that there exists for any solution $\vartheta = \vartheta(t)$ of (21) at least one pair (4) of functions $u(t)$, $v(t)$ so that for the three functions $\vartheta(t)$, $u(t)$, $v(t)$ the conditions (5), (8) and (13) are fulfilled.

The arbitrarily given solution $\vartheta = \vartheta(t)$ of (21) is characterized by its initial values

(25) $$\vartheta(0) \quad \text{and} \quad \dot{\vartheta}(0).$$

We determine four initial values (11) in the following manner. One of these, for instance $u(0)$, shall be chosen arbitrarily. The three other numbers, namely $v(0)$, $\dot{u}(0)$, $\dot{v}(0)$, shall fulfill the three conditions

(26, 1) $$0 = [-\vartheta(0) - \dot{y}^0(0)u(0)] + \dot{x}^0(0)v(0),$$

(26, 2) $$0 = [-\dot{\vartheta}(0) - \ddot{y}(0)u(0)] + \ddot{x}^0(0)v(0) - \dot{y}^0(0)\dot{u}(0) + \dot{x}^0(0)\dot{v}(0),$$

(26, 3) $$0 = [0 - \Omega_x{}^0(0)u(0)] - \Omega_y{}^0(0)v(0) + \dot{x}^0(0)\dot{u}(0) + \dot{y}^0(0)\dot{v}(0),$$

which always determine the three constants uniquely, the determinant being $-\dot{x}^0(0)[\dot{x}^0(0)^2 + \dot{y}^0(0)^2] \neq 0$ [cf. (10)]. Let

(27) $$u = u^*(t), \qquad v = v^*(t)$$

[illegible] (11). The integration constant c [cf. (6)] for the solution (27) is by means of (26, 3) equal to zero, i. e. (27) is an isoenergetic variation and therefore

$$(28) \qquad \vartheta^*(t) = \begin{vmatrix} \dot{x}^0(t) & \dot{y}^0(t) \\ u^*(t) & v^*(t) \end{vmatrix}$$

a solution of (21). Furthermore it follows from (26, 1), (26, 2) and (28)

$$(25') \qquad \vartheta(0) = \vartheta^*(0), \qquad \dot{\vartheta}(0) = \dot{\vartheta}^*(0)$$

i. e. the given solution $\vartheta(t)$ of (21) has the same initial values as the solution (28) and therefore $\vartheta(t) \equiv \vartheta^*(t)$ so that the arbitrarily given solution $\vartheta(t)$ may be represented with the use of an isoenergetic variation (4). This completes the demonstration of the theorem that a function $f(t)$ of t is then and only then an isoenergetic normal displacement of (2) if $\zeta = f(t)$ fulfills the differential equation (23).

The differential equation (23) is of the second order whereas there exists a three parametric series of isoenergetic variations (4) [cf. p. 619]. It follows that the isoenergetic normal representation (12) of a given solution $\zeta(t)$ of (23) is not uniquely determined inasmuch as it contains one arbitrary parameter [cf. also p. 621, where it was possible to choose one of the four integration constants (11) in an arbitrary manner]. The reason of this circumstance is obviously the following one. (9) and therefore also

$$(29) \qquad u = a\dot{x}^0(t), \qquad v = a\dot{y}^0(t) \qquad\qquad (a = \text{constant} \neq 0)$$

is a solution of (5) and (8) which does not vanish identically [cf. p. 618] whereas the solution of (21) defined by (13) and (29) vanishes identically for any value of the integration constant a. We infer that the equation (21) or (23) is illusory if the given solution (2) of (1) is independent of t. It is of course allowed that both functions (9) should simultaneously vanish for certain isolated values of t but the limit values $\zeta(t+0)$, $\zeta(t-0)$ of (14) will exist also for these values of t.

If the function $\lambda(x, y)$ introduced by the Coriolis forces is identically zero the three functions $\lambda^0(t)$, $\lambda_x{}^0(t)$, $\lambda_y{}^0(t)$ will be zero for all values of t and (20) becomes

$$(20') \qquad L(t) = \Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t).$$

However this is not the case in the restricted problem of three bodies. For this problem is $\lambda(x, y) \equiv 1$ if x and y denote Cartesian coördinates and t denotes the time, so that one must add to (20′) the term

$$(20'') \qquad -4.$$

If one uses the variables of Thiele, then $\lambda_x{}^0(t)$ and $\lambda_y{}^0(t)$ are also not identically zero.

APPENDIX.

On a Theorem in the Pfaffian Dynamics of Birkhoff.

The conservative Hamiltonian principle

$$\delta \int \left\{ \sum_{i=1}^{n} p_i \dot{q}_i - H(p_1, \cdots, q_n) \right\} dt = 0 \text{ or } \dot{q}_i = \partial H/\partial p_i, \ -\dot{p}_i = \partial H/\partial q_i, \quad (i = 1, \cdots, n)$$

which is rather unsymmetrical with respect to q_i and p_i has been generalized by Birkhoff * to the "Pfaffian" principle

$$(1) \qquad \delta \int \left\{ \sum_{i=1}^{2n} R_i dx_i - H dt \right\} = 0, \quad \text{i. e.} \quad \delta \int \left\{ \sum_{i=1}^{2n} R_i \dot{x}_i - H \right\} dt = 0$$

or

$$(2) \qquad \sum_{j=1}^{2n} (\partial R_i/\partial x_j - \partial R_j/\partial x_i) \dot{x}_j = \partial H/\partial x_i; \qquad (i = 1, 2, \cdots, 2n-1, 2n)$$

where

$$(3) \qquad H = H(x_1, \cdots, x_{2n}); \qquad R_i = R_i(x_1, \cdots, x_{2n})$$

are $2n + 1$ given functions of the $2n$ coördinates of the phase space and

$$(4) \qquad \det (\partial R_i/\partial x_j - \partial R_j/\partial x_i) \neq 0.$$

The equations (2) which possess the integral $H =$ const. are, according to (1), invariant for *any* transformation of the coördinates. The conservative Hamiltonian systems are special cases of the symmetrical Pfaffian problem (2), the condition (4) being for

$$(5) \qquad x_i = q_i, \quad x_{i+n} = p_i, \quad R_i = x_{i+n}, \quad R_{i+n} = 0; \qquad (i = 1, 2, \cdots, n)$$

obviously fulfilled. Birkhoff has shown that the majority of the essential properties of the Hamiltonian systems hold also for the Pfaffian case. For the purposes of Birkhoff a theorem concerning the characteristic exponents of the variational equations belonging to a given periodic solution

$$(6) \qquad x_i = x_i(t); \qquad (i = 1, 2, \cdots, 2n-1, 2n)$$

of (2) is of particular importance. This theorem has been demonstrated

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* G. D. Birkhoff, "Dynamical Systems," *American Mathematical Society Colloquium Publications*, Vol. 9 (1927), p. 55, p. 89 etc. Cf. also L. Feraud, "On Birkhoff's Pfaffian Mechanics," *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 817 etc.

by Birkhoff * only under certain restrictions. In the present note a simple proof is given which holds without any particular assumptions, showing that the theorem of Birkhoff is a general property of all conservative Pfaffian systems.

The variational equations belonging to the given periodic solution (6) of (2), namely the $2n$ linear differential equations

$$(7)\qquad \sum_{j=1}^{2n}(\partial R_i/\partial x_j-\partial R_j/\partial x_i)\dot{\xi}_j + \sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j-\partial^2 R_j/\partial x_l\partial x_i)\xi_l\dot{x}_j=\sum_{j=1}^{2n}\partial^2 H/\partial x_j\partial x_i\xi_j,$$
$$(i=1,2,\cdots,2n-1,2n),$$

can be written in the form

$$(8)\qquad \dot{\xi}_i=\sum_{j=1}^{2n}a_{ij}(t)\xi_j;\qquad (j=1,2,\cdots,2n-1,2n)$$

inasmuch as the "velocities" $\mathring{x}_j$ and the partial derivatives occuring in (7) are, by virtue of (6) and (3), given functions of t (the determinant of the coefficients

$$(9)\qquad \partial R_i/\partial x_j-\partial R_j/\partial x_i=s_{ij}(t)$$

is, according to (4), different from zero). If one uses instead of ξ the letter η and interchanges i and j equations (7) can be written in the form

$$(10)\qquad \sum_{i=1}^{2n}(\partial R_i/\partial x_j-\partial R_j/\partial x_i)\dot{\eta}_i + \sum_{i=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j-\partial^2 R_j/\partial x_l\partial x_i)\eta_l\mathring{x}_i=-\sum_{i=1}\partial^2 H/\partial x_i\partial x_j\eta_i,$$
$$(j=1,2,\cdots,2n-1,2n).$$

Furthermore, the expression

$$(11)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j-\partial^2 R_j/\partial x_l\partial x_i)(\xi_j\eta_i\mathring{x}_l-\xi_l\eta_i\mathring{x}_j-\xi_j\eta_l\mathring{x}_i)$$

is the sum six of six trilinear forms three of which are the negatives of the three others. Multiplying (7) by η_i and (10) by ξ_j one obtains $4n$ relations and on adding these it follows that the expression

$$(12)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}(\partial R_i/\partial x_j-\partial R_j/\partial x_i)(\dot{\xi}_j\eta_i+\dot{\eta}_i\xi_j) + \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j-\partial^2 R_j/\partial x_l\partial x_i)(\xi_l\eta_i\mathring{x}_j+\xi_j\eta_l\mathring{x}_i)$$

* *Loc. cit.*, p. 90-91.

also vanishes. The sum

$$(13)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}(\partial R_i/\partial x_j - \partial R_j/\partial x_i)(\dot{\xi}_j\eta_i + \dot{\eta}_i\xi_j)$$
$$+\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)\xi_j\eta_i\dot{x}_l$$

of the two vanishing expressions (11), (12) is, according to (9), simply the time derivative of the bilinear covariant

$$(14)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n} s_{ij}(t)\xi_j(t)\eta_i(t),$$

associated with the Pfaffian

$$(1')\qquad \sum_{i=1}^{2n} R_i dx_i - H dt.$$

Since the expression (13) is zero the value of (14) is, for two arbitrary solutions ξ, η of the equations of variations belonging to (6), independent of t.

It obviously follows, in the same manner as above, that the characteristic equation associated with (7) or (8) is always a reciprocal one. This is the theorem of Birkhoff in its generalized form. The skew symmetrical, non-singular bilinear form (14) is, according to (5) and (9), the Pfaffian generalization of the bilinear differential invariant of Poincaré used above which corresponds to the integral invariant

$$\int\int \sum_{i=1}^{n} \delta p_i \delta q_i$$

of the Hamiltonian systems. For the integral invariants of the Pfaffian systems cf., for instance, the paper of Féraud, *loc. cit.*

THE JOHNS HOPKINS UNIVERSITY.

THE EQUATION OF STABILITY OF PERIODIC ORBITS OF THE RESTRICTED PROBLEM OF THREE BODIES IN THIELE'S REGULARISING COÖRDINATES.

By Jenny E. Rosenthal.*

The harmonic analysis of periodic orbits calculated by mechanical quadrature at the Copenhagen Observatory is available in print in some of its main features; † the rest has not yet been published. Since the harmonic analysis has been done mostly for ejection orbits the Fourier series also have been calculated in Thiele's variables. A determination of the stability character of the orbits in the Copenhagen material would make it necessary, therefore, to write the Jacobi second order differential equation explicitly in terms of Thiele's variables. This equation then determines the characteristic exponents. The preceding article by A. Wintner ‡ includes a study of the Jacobi differential equation for any mechanical problem with two degrees of freedom. This article has also a discussion of the connection between this second order equation and the original fourth order variational equations. On the basis of formulas given there I have expressed the Jacobi second order differential equation of the restricted problem of three bodies for an arbitrary value of the mass ratio in terms of Thiele's regularising variable. The regularising transformation in the general case was first given by Burrau § (Thiele has published his regularising transformation only in the case of two equal masses.)

The regularised differential equations are as follows:

$$(1)\qquad E''-\lambda F'=\partial\Omega/\partial E;\quad F''+\lambda E'=\partial\Omega/\partial F;\quad (E')^2+(F')^2-2\Omega=0,$$

* National Research Fellow.

† E. Strömgren, *Tre Aartier Celest Mekanik paa Københavns Observatorium*, pp. 46-86, Copenhagen 1923.

‡ A Wintner, "Three Notes on Characteristic Exponents and Equations of Variation in Celestial Mechanics," *American Journal of Mathematics*, Vol. 53 (1931), p. 605.

§ C. Burrau, "Über einige in Aussicht genommene Berechnungen betreffend einen Spezialfall des Dreikörper-Problems," *Vierteljahresschrift der Astronomischen Gesellschaft*, Vol. 41 (1906), pp. 261 ff. Cf. also J. Fischer-Petersen, "Über unendlich kleine periodische Bahnen um die Massenpunkte im problème restreint," *Astronomische Nachrichten*, Vol. 200 (1915), pp. 387-388. [Publikationer fra Köbenhavns Observatorium, No. 22].

where

$$(2) \qquad 2\lambda = \cos 2iF - \cos 2E,$$

$$(3) \qquad \Omega = (1/16)(\cos 4iF - \cos 4E) - (k/4)(\cos 2iF - \cos 2E) + 8 \cos iF + (1/4)(1 - 2\mu)(\cos E \cos 3iF - \cos 3E \cos iF - 32 \cos E).$$

Here k is the energy constant, μ the mass ratio, E and F the cartesian coördinates in the plane of Thiele's variables and the prime represents differentiation with respect to Thiele's time variable ψ (and not with respect to the ordinary time t).

Let

$$(4) \qquad E = E(\psi), \; F = F(\psi)$$

be a given solution of (1); and let δE and δF be such functions of ψ that the equations:

$$(5) \qquad E = E(\psi) + \delta E; \; F = F(\psi) + \delta F$$

represent a virtual motion infinitesimally close to the given orbit (4) in such a manner that this variation, i. e., the transition from (4) to (5), be isoenergetic. This assumption, namely $\delta k = 0$, is necessitated by the third equation (1). In Wintner's article it is generally shown that the non trivial (i. e., the non identically vanishing) characteristic exponent (in other words, the stability character) is determined in any case by such an isoenergetic transformation ($\delta k = 0$). The direction cosines of the normal of orbit (4) in the E, F plane are:

$$(6) \qquad -F'[(E')^2 + (F')^2]^{-\frac{1}{2}}, \quad +E'[(E')^2 + (F')^2]^{-\frac{1}{2}}.$$

The normal displacement is therefore

$$(7) \qquad -F'[(E')^2 + (F')^2]^{-\frac{1}{2}}\delta E + E'[(E')^2 + (F')^2]^{-\frac{1}{2}}\delta F,$$

or

$$(8) \qquad \theta[(E')^2 + (F')^2]^{-\frac{1}{2}} \text{ where } \theta = E'\delta F - F'\delta E.$$

satisfies the following differential equation

$$(9) \qquad \alpha(\psi)\theta'' + \beta(\psi)\theta' + \gamma(\psi)\theta = 0.$$

Here

$$\alpha(\psi) = [(E')^2 + (F')^2],$$
$$\beta(\psi) = b_1 E' + b_2 F',$$
$$\gamma(\psi) = c_0 + c_1 E' + c_2 F' + [(E')^2 + (F')^2][c_3 + c_4 E' + c_5 F'],$$

where

$$b_1 = b_{10} + b_{11}(1-2\mu), \qquad b_2 = b_{20} + b_{21}(1-2\mu),$$
$$c_0 = c_{00} + c_{01}(1-2\mu) + c_{02}(1-2\mu)^2,$$
$$c_1 = c_{10} + c_{11}(1-2\mu), \qquad c_2 = c_{20} + c_{21}(1-2\mu),$$
$$c_3 = c_{30} + c_{31}(1-2\mu), \qquad c_4 = c_{40}, \qquad c_5 = c_{50},$$

and

$$b_{10} = (1/2)(-\sin 4E + 2k \sin 2E),$$
$$b_{11} = (1/2)(\sin E \cos 3iF - 3 \sin 3E \cos iF - 32 \cos E),$$
$$b_{20} = (i/2)(\sin 4iF - 2k \sin 2iF + 32 \sin iF),$$
$$b_{21} = (i/2)(3 \cos E \sin 3iF - \cos 3E \sin iF),$$

$$c_{00} = (1/16)\ [(\cos 8iF - \cos 8E) - 4k\ (\cos 6iF - \cos 6E) + 64 \cos 5iF + 4k^2\ (\cos 4iF - \cos 4E) - 64(1+2k) \cos 3iF + (1024 + 4k) \cos 2iF - 4k \cos 2E + 128k \cos iF - 1024],$$

$$c_{01} = (i/8)\ [-32 \cos 5E + 64(1+k) \cos 3E - 64k \cos E - 3\ (\cos 7E \cos iF - \cos E \cos 7iF) + 6k\ (\cos 5E \cos iF - \cos E \cos 5iF) + (\cos 5E \cos 3iF - \cos 3E \cos 5iF) - 2k\ (\cos 3E \cos iF - \cos E \cos 3\, iF) - 32 \cos 3E \cos 2iF + 96 \cos E \cos 4iF - 96 \cos E \cos 2iF],$$

$$c_{02} = (1/16)\ [5\ (\cos 6iF - \cos 6E) - 64 \cos 3iF + 5 \cos 2iF - 1029 \cos 2E + 4(\cos 2E \cos 6iF - \cos 6E \cos 2iF) - 6(\cos 2E \cos 4iF - \cos 4E \cos 2iF) + 192 \cos 2E \cos iF + 64 \cos 2E \cos 3iF - 192 \cos 4E \cos iF],$$

$$c_{10} = (3i/4)\ [\sin 6iF - 2k \sin 4iF + 32 \sin 3iF + \sin 2iF - 32 \sin iF - 2 \cos 2E \sin 4iF + 4k \cos 2E \sin 2iF - 64 \cos 2E \sin iF],$$

$$c_{11} = (3i/4)\ [\cos 5E \sin iF - 4 \cos 3E \sin 3iF + \cos 3E \sin iF + 3 \cos E \sin 5iF - 3 \cos E \sin 3iF + 4 \cos E \sin iF],$$

$$c_{20} = (3/4)\ [-\sin 6E + 2k \sin 4E - \sin 2E + 2 \sin 4E \cos 2iF - 4k \sin 2E \cos 2iF],$$

$$c_{21} = (3/4)\ [-32 \sin 3E + 32 \sin E - 3 \sin 5E \cos iF + 4 \sin 3E \cos 3iF + 3 \sin 3E \cos iF - \sin E \cos 5iF - \sin E \cos 3iF + 64 \sin E \cos 2iF - 4 \sin E \cos iF],$$

$$c_{30} = (1/2)\ [10 + 3\ (\cos 4iF + \cos 4E) + 2k\ (\cos 2iF + \cos 2E) - 16 \cos iF - 20 \cos 2iF \cos 2E],$$

$$c_{31} = -2 \cos E \cos 3iF - 2 \cos 3E \cos iF - 8 \cos E,$$
$$c_{40} = -2i \sin 2iF,$$
$$c_{50} = -2 \sin 2E,$$

In these expressions for the three coefficients, E, F, E' and F' are to be considered as given functions of ψ on account of (4). The function (4) [and its derivatives] are given numerically by Strömgren (*loc. cit.*) in the form of Fourier series. Substitution of Strömgren's expressions for E and F in the above given expressions for α, β and γ determines the differential equation (9) which corresponds to the given orbit. The further treatment of (9) is done by known methods, for example by Hill's method.

In researches at the Copenhagen Observatory Burrau * has treated such virtual displacements. A similar arrangement of the calculations would probably make the seemingly complicated formulas given above applicable for numerical purposes. Numerical calculations are made more practicable by the fact that in a given stage of a given group it is often possible to predict what terms will make a contribution which is numerically negligible. It may be stressed that the above expressions for α, β, γ are valid without the neglect of any terms; the length of the expressions seems to be unavoidable. If the Fourier series of solution (4) are known analytically so that the Fourier coefficients and the period are given power series of Jacobi's constant k, it is simple to determine in advance the preponderant terms. Such convergent analytical expressions are known, for example,† in the neighborhood of the masses for groups f and g and at a large distance from the masses for groups l and m; and it is easy to obtain corresponding developments for the libration groups, a, b, c, d, e in the neighborhood of the corresponding libration point. If we are concerned with periodic solutions (of these groups) whose range is neither quite in the neighborhood of the masses or the libration points nor in the neighborhood of the infinitely distant point, the analytical developments are not applicable, so that we have to use the numerical Fourier series given by E. Strömgren (*loc. cit.*). This is also the case for groups k, n, etc., for which there is no analytical theory available at present. For the rôle of differential equation (9) of the characteristic exponent in the theory of the Strömgren groups see Wintner (*loc. cit.*). From the expressions

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* C. Burrau, "Recherches numériques concernant les solutions périodiques d'un cas spécial du problème des trois corps (Deuxième mémoire)," *Astronomische Nachrichten*, [illegible]

† For reference cf. E. Strömgren, *loc. cit.*, pp. 70-[illegible]. Cf. also E. Strömgren, "Forms of Periodic Motion in the Restricted Problem etc.," *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, No. 39.

above it is clear that the formulas are considerably simplified in the symmetrical case ($\mu = 1/2$) since then all terms with $(1 - 2\mu)$ and $(1 - 2\mu)^2$ as factors drop out. It may be mentioned here that, of course, the characteristic exponents can be determined by the same method in the system of variables x, y, t (instead of E, F, ψ), and in that case the expressions for α, β, γ are not so lengthy.* But in the system x, y, t one cannot treat for instance even the essential stages of the group, such as the passing of a group through an ejection orbit. Moreover, Strömgren's harmonic analysis (*loc. cit.*) has always been done in the system E, F, ψ and possible sources of error would arise in reducing Strömgren's Fourier series (as long as they do not belong to the ejection orbit) to the x, y, t system. (It would also be necessary then to determine each time the period *referred to* t which would mean considerable additional numerical work for single orbits and might give rise to new sources of error since the connection between t and ψ is given explicitly only by a differential equation.†) Hence there is apparently no way of avoiding the above expressions for α, β, γ in problems connected with characteristic exponents of the Copenhagen groups [compare also Wintner (*loc. cit.*)].

In conclusion I wish to thank Dr. A. Wintner for his suggestions and advice.

THE JOHNS HOPKINS UNIVERSITY.

* A. Wintner, "Über die Jacobische Differentialgleichung des restringierten Dreikörperproblems," *Sitzungsberichte der mathematisch-physikalischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig*, Vol. 82 (1930), pp. 345-354.

† Cf. A. Wintner, "Über eine Revision der Sortentheorie des restringierten Dreikörperproblems," *ibid.*, p. 46.

ALGEBRAS OF CERTAIN DOUBLY TRANSITIVE GROUPS.

By R. D. Carmichael.

A class of finite algebras $A[p^n]$ is defined directly (§ 1) by means of doubly transitive groups of prime-power degree p^n and order $p^n(p^n-1)$ and is shown (§ 1) to be equivalent to a class of finite algebras defined by Dickson in 1905 (Göttingen Nachrichten, 1905). The set of all linear transformations on the marks of an $A[p^n]$ induces on those marks a group which is conjugate to that by which the algebra is defined (§ 2). Three forms are given (§ 3) to the (only partially solved) problem of determining all algebras $A[p^n]$, one of them being of fundamental importance in the investigation of the group of isomorphisms of an Abelian group of order p^n and type $(1, 1, \cdots, 1)$. This problem deserves further attention. Two algebras $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic (§ 4) when and only when their multiplicative groups are simply isomorphic. The integral elements of an $A[p^n]$ form (§ 5) a Galois field $GF[p]$. The algebras $A[p^n]$ are capable (§ 6) of various analytical representations, including as a special case that employed by Dickson. A large class of doubly transitive groups of degree p^n and order $p^n(p^n-1)$ is exhibited (§ 7) and these groups are employed (§ 8) in the rapid construction of a large class of algebras $A[p^n]$, closely related to those determined by Dickson by other methods.

1. *Construction of Algebras $A[s]$.* Let G be a doubly transitive group of degree ρ and order $\rho(\rho-1)$. Then it is well known that ρ is a prime-power p^n $(n \geqq 1)$, that such a doubly transitive group G contains a single subgroup H of order p^n, that this Sylow subgroup H is Abelian and of type $(1, 1, \cdots, 1)$, that H contains all the elements of G (p^n-1 in number) each of which displaces all the symbols permuted by G, that H is self-conjugate in G, and that every element in G and not in H is a regular permutation on just p^n-1 symbols.

Let $a_0, a_1, a_2, \cdots, a_{s-1}$, where $s = p^n$, be the p^n symbols permuted by G. Then H permutes these symbols among themselves according to a regular group, as is well known and may be readily shown from the fact that H consists of the identity and p^n-1 elements each of which permutes all the symbols. Then there is one and just one element h_i of H which replaces a_0 by a_i.

Let us denote by M the subgroup of order p^n-1 in G each element

of which leaves a_0 fixed. It is a regular group on $a_1, a_2, \cdots, a_{s-1}$. Hence there is one and just one element m_i of M which replaces a_1 by a_i. It is evident that $m_i^{-1}h_1m_i = h_i$.

By means of these properties of G we shall define an algebra $A[p^n]$. Let the p^n symbols or marks of this algebra be denoted by $u_0, u_1, u_2, \cdots, u_{s-1}$. We introduce a law of addition for the marks u_i of this algebra in the following manner: The sum $u_i + u_j$ is the mark u_k $(u_i + u_j = u_k)$ where k is such that $h_ih_j = h_k$ in the group G. Then, in particular, $u_i + u_0 = u_i$ for every mark u_i.

For the purpose of defining a law of multiplication for the marks u_i, exclusive of the zero-mark u_0, we employ the elements of the subgroup M of G. We write $u_iu_j = u_l$ $(i > 0,\ j > 0)$ where l is defined by the relation $m_l^{-1} = m_i^{-1}m_j^{-1}$ or $m_l = m_jm_i$. [At this point it would seem more natural to take $u_iu_j = u_\lambda$ where $m_\lambda = m_im_j$; but this would give $(b + c)a = ba + ca$ instead of the relation $a(b + c) = ab + ac$, presently to be established; and the latter relation is slightly more natural from the point of view of the algebras.] We define the products u_iu_0 and u_0u_i by the requirement that each of them shall have the value u_0.

With the named laws of addition and multiplication the marks u_0, u_1, $\cdots, u_{s-1}$ constitute an algebra of the type defined by Dickson * in 1905, as we shall now show. Dickson subjects his algebras to nine postulates. The first four of these postulates require merely that all the marks of the algebra shall form a group under addition. The next four postulates require merely that all the marks, exclusive of the zero-mark, shall form a group under multiplication. The remaining postulate asserts that if a, b, c are elements of the algebra then $a(b + c) = ab + ac$. With these postulates in hand it is easy to show (cf. Dickson, l. c.) that the additive group and the multiplicative group have the properties already employed. Therefore, in order to show that the algebras here defined are identical with those of Dickson, it is sufficient to prove that his last postulate is verified.

As expressed in terms of the u's we have then to establish the following relation:

$$u_i(u_\rho + u_\sigma) = u_iu_\rho + u_iu_\sigma. \tag{1.1}$$

This is immediately verified if any one of the subscripts i, ρ, σ is zero. Then for the further argument suppose that each of them is greater than zero. Since (1.1) involves two operations it is convenient to reduce the relation to be proved to a corresponding relation among the elements of H and M

* Dickson, *Göttinger Nachrichten*, 1905.

since they are all subject to the single rule of combination in G. From the definitions of addition and multiplication we have the following propositions:

$$u_\rho + u_\sigma = u_\tau \quad \text{if} \quad h_\rho h_\sigma = h_\tau, \qquad u_i u_\rho = u_\lambda \quad \text{if} \quad m_\rho m_i = m_\lambda,$$
$$u_i u_\sigma = u_\nu \quad \text{if} \quad m_\sigma m_i = m_\nu, \qquad u_i u_\tau = u_\alpha \quad \text{if} \quad m_\tau m_i = m_\alpha,$$
$$u_\lambda + u_\nu = u_\beta \quad \text{if} \quad h_\lambda h_\nu = h_\beta.$$

In order to establish the required relation (1.1) it is necessary and sufficient to show that $\alpha = \beta$. Now we have

$$\begin{aligned} h_\beta &= h_\lambda h_\nu = m_\lambda^{-1} h_1 m_\lambda \cdot m_\nu^{-1} h_1 m_\nu \\ &= (m_\rho m_i)^{-1} h_1 (m_\rho m_i) \cdot (m_\sigma m_i)^{-1} h_1 (m_\sigma m_i) \\ &= m_i^{-1} h_\rho h_\sigma m_i = m_i^{-1} h_\tau m_i \\ &= m_i^{-1} m_\tau^{-1} h_1 m_\tau m_i = (m_\tau m_i)^{-1} h_1 (m_\tau m_i) \\ &= m_\alpha^{-1} h_1 m_\alpha = h_\alpha. \end{aligned}$$

Since $h_\beta = h_\alpha$ it follows that $\alpha = \beta$ and hence that (1.1) is established.

From the foregoing analysis it follows that every doubly transitive group of prime-power degree p^n $(n \geqq 1)$ and order $p^n(p^n - 1)$ may be employed for the definition of an algebra $A[p^n]$. In the next section we consider the converse problem.

2. *Linear Transformations in an* $A[s]$. If β is any given one of the marks $u_0, u_1, \cdots, u_{s-1}$ of an $A[s]$, $s = p^n$, and if x is a variable running over the marks of the algebra then $x + \beta$ is a new variable x' running over the marks of the algebra. Thus we have the transformation $x' = x + \beta$ corresponding to the addition of β on the right to all the marks of the algebra. By varying β we obtain the p^n transformations corresponding to the additive group of the algebra.

More generally the set of all transformations

$$x' = \alpha x + \beta, \tag{2.1}$$

where α and β run independently over all the marks of the algebra except that α remains different from the zero mark u_0, constitutes a group K, as one may readily verify by means of the stated properties of $A[s]$. Its order is $s(s-1)$. Each element of K permutes the marks of $A[s]$ among themselves; thus K gives rise to a permutation group K_1 of degree s on the marks $u_0, u_1, \cdots, u_{s-1}$. If a and b are any two distinct marks of $A[s]$ then the marks u_0, u_1, as values of x, are replaced by the marks a, b respectively, as values of x', by the transformation

$$x' = (b - a)x + a.$$

Therefore K_1 is a doubly transitive group of degree s and order $s(s-1)$.

From this it follows that *the number s of marks in an algebra $A[s]$* (satisfying Dickson's postulates) *is necessarily of the form $s = p^n$*, a result proved by Dickson (*l. c.*) directly from his postulates.

Those transformations (2.1) in which $\alpha = u_1$ correspond to the additive group of the algebra, since they may be written in the form $x' = x + \beta$. [Thence follow readily the known essential properties of the additive group of the algebras $A[s]$.] Those transformations (2.1) in which $\beta = u_0$ may be written in the form $x' = \alpha x$. They correspond to the multiplicative group of the algebra, the element $x' = \alpha x$ corresponding to multiplication on the right by α. This multiplicative group induces on the non-zero marks of the algebra a regular permutation group of order $p^n - 1$.

Let M denote the permutation group on the marks $u_1, u_2, \cdots, u_{s-1}$ induced by the multiplicative group of the algebra and denote by m_i the element of M induced by the transformation $x' = u_i x$, $u_i \neq u_0$. Let H denote the permutation group on the marks $u_0, u_1, \cdots, u_{s-1}$ induced by the additive group of the algebra and denote by h_i the element of H induced by the transformation $x' = x + u_i$. Let G be the group generated by H and M. Then h_i replaces $x = u_0$ by $x' = u_i$ and m_i replaces $x = u_1$ by $x' = u_i$; whence it follows that $h_i = m_i^{-1} h_1 m_i$. Then $u_i + u_j = u_k$ where k is such that $h_k = h_i h_j$, while $u_i u_j = u_l$ where l is such that $m_l^{-1} = m_i^{-1} m_j^{-1}$.

From these results it follows that the group G to which the algebra leads by use of (2.1) may in turn be employed as in § 1 to recover the algebra itself. Therefore, *every possible algebra $A[s]$, satisfying Dickson's postulates, is an algebra $A[p^n]$ defined as in § 1 by means of a doubly transitive group of prime-power degree p^n $(n \geqq 1)$ and order $p^n(p^n - 1)$ while conversely such a doubly transitive group is induced by the totality of transformations of the form* (2.1) *on the marks of such an algebra.*

3. *Three Equivalent Forms of an Unsolved Problem.* From the theorem just stated it follows that the problem of constructing all algebras $A[p^n]$ is equivalent to the problem of constructing all doubly transitive groups of degree p^n and order $p^n(p^n - 1)$. With respect to these latter groups it is not difficult to establish the following theorem:

Every doubly transitive group G of degree p^n and order $p^n(p^n - 1)$ is contained in the holomorph of the Abelian group H of order p^n and type $(1, 1, \cdots, 1)$ when that holomorph is written in the usual way as a permutation group. Moreover, the regular subgroup M of G, consisting of those elements which leave one symbol fixed, is contained in the group I of isomorphisms of H.

It is easy also to establish the following theorem:

For every regular group I_1 *of degree and order* p^n-1 *contained in the group* I *of isomorphisms of an Abelian group of order* p^n *and type* $(1, 1, \cdots, 1)$ *there exists one and just one doubly transitive group* G *of degree* p^n *and order* $p^n(p^n-1)$ *containing* I_1 *as a subgroup.*

From these results follows readily the theorem (proved otherwise by Dickson, *l. c.*) that *the multiplicative group of an algebra* $A[p^n]$ *is simply isomorphic with a regular subgroup* I_1 *of degree* p^n-1 *contained in the group* I *of isomorphisms (with itself) of an Abelian group* H *of order* p^n *and type* $(1, 1, \cdots, 1)$ *when* I *is represented in the usual way as a permutation group on the elements of* H *exclusive of the identity.*

These considerations lead to the formulation of the following three problems:

1. To construct all the regular subgroups I_1 of degree p^n-1 contained in the group I of isomorphisms (with itself) of an Abelian group H of order p^n and type $(1, 1, \cdots, 1)$ when I is represented in the usual way as a permutation group on the elements of H exclusive of the identity.
2. To construct all doubly transitive groups of degree p^n and order $p^n(p^n-1)$.
3. To construct all algebras $A[p^n]$ subject to the postulates of Dickson (*l. c.*).

From foregoing results it follows that the solution of any one of these problems carries with it the solution of the other two. It appears that, up to the present, no one of these problems has been completely solved. The work of Dickson (*l. c.*) is the most important which has yet been done in this direction. His most comprehensive results, however, are based on an empirical (unproved) proposition; though this proposition is a remarkable one if true (and he has verified it in a wide range of cases), nevertheless no one else (so far as I know) has given it further consideration. It appears that the problem here formulated is a difficult one; its importance is indicated by the three-fold formulation and the variety of connections which it is thus shown to have. It deserves further attention.

For every value of p^n, as is well known, there exists a cyclic I_1; every Abelian I_1 is cyclic. When this I_1 is employed, the resulting algebra $A[p^n]$ is the Galois field $GF[p^n]$. But, so far as I am aware, no direct proof has been given of the existence of this cyclic I_1; that is to say, the known proofs appear all to be based on the (previously proved) existence of the $GF[p^n]$

or on what is essentially equivalent to that, such existence being established by methods which are not directly group-theoretic in character. It is not satisfactory thus to be driven outside the domain of direct group-theoretic considerations to establish the existence of this cyclic I_1. Emphasis is here put upon this particular problem in the hope of directing to it the attention of other investigators.

4. *Simple Isomorphism of Algebras* $A[p^n]$. Two algebras $A_1[p^n]$ and $A_2[p^n]$ will be called simply isomorphic if each element of A_1 may be made to correspond uniquely to an element of A_2 in such a way that each element of A_2 is the correspondent of a single element of A_1 while moreover the sum [product] of any two elements in A_1 corresponds to the sum [product] of the corresponding two elements of A_2. It will be said that two simply isomorphic algebras are identical. Any two algebras $A[p^n]$ are evidently such that their additive groups are simply isomorphic. An obvious necessary condition for the algebras to be simply isomorphic is that their multiplicative groups of order p^n-1 shall be simply isomorphic. We shall show that this condition is also sufficient.

If the multiplicative groups of $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic then the doubly transitive groups of degree p^n and order $p^n(p^n-1)$, to which they lead by the method of § 2, have simply isomorphic regular subgroups of degree p^n-1, as is seen from the named isomorphism of the multiplicative groups of the algebras. Hence these two doubly transitive groups are conjugate. Now, on recovering the algebras from these conjugate groups, by the method of § 1, we exhibit the algebras themselves as simply isomorphic.

Thus we have the following theorem:

Two algebras $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic when and only when their multiplicative groups are simply isomorphic.

5. *Integral Elements of an Algebra* $A[p^n]$. Denote the elements of an $A[p^n]$, as before, by the symbols $u_0, u_1, \cdots, u_{s-1}$. An element of the form $u_1+u_1+\cdots+u_1$ will be called an integral element; the other elements are said to be non-integral. From the properties of the additive group H of the algebra it follows that there are just p integral elements of the algebra. When there is no danger of confusion these may be denoted by $0, 1, \cdots, p-1$, where 0 and 1 denote the elements u_0 and u_1 respectively. Addition and multiplication of the integral elements are equivalent to ordinary addition and multiplication followed by a reduction modulo p. Hence the integral elements of $A[p^n]$ form a sub-algebra which is simply isomorphic with $GF[p]$.

In particualr, an algebra $A[p]$ consists entirely of integral elements and is the $GF[p]$.

It is well known that $GF[p^n]$ contains a subfield $GF[p^k]$ when and only when k is a factor of n. If an $A[p^n]$ contains a sub-algebra $A[p^k]$, then the multiplicative group of order $p^k - 1$ of the latter must be a subgroup of the multiplicative group of order $p^n - 1$ of the former: hence $p^k - 1$ must be a factor of $p^n - 1$, whence it follows that k is a factor of n.

6. *Analytical Representation of Algebras* $A[p^n]$. Let us write $n = k\nu$ where k and ν are positive integers (either or both of which may be unity). We now denote the p^n elements of an algebra $A[p^n]$ by $(a_1, a_2, \cdots, a_k)$ where the a's run independently over the marks of the $GF[p^\nu]$. In view of the properties of the additive group H of the algebra it is evident that we may take for the rule of addition in the algebra that expressed by the formula

$$(6.1) \qquad (a_1, a_2, \cdots, a_k) + (b_1, b_2, \cdots, b_k) = (a_1 + b_1, \cdots, a_k + b_k).$$

Then $(0, 0, \cdots, 0)$ is the zero element of the algebra. The product of the zero element by any other element (in either order) is the zero element. It remains to define a suitable rule of multiplication for the non-zero elements of the algebra.

The multiplication of the non-zero elements is according to a group M which permutes these non-zero elements according to a regular group contained in the group I of isomorphisms of H with itself. Moreover (§ 2) the group of linear transformations in the algebra permutes the marks of the algebra according to a doubly transitive group of degree p^n and order $p^n(p^n - 1)$. From these facts and from the analytical representation T of the group I given in an earlier memoir * it follows that if

$$(6.2) \qquad (a_1, a_2, \cdots, a_k) \cdot (x_1, x_2, \cdots, x_k) = (x_1', x_2', \cdots, x_k')$$

then we have

$$(6.3) \qquad x_i' = \sum_{t=1}^{\nu} \sum_{j=1}^{k} a_{ijt} x_j^{p^{n-t}}, \qquad (i = 1, 2, \cdots, k),$$

where the coefficients a_{ijt} are marks of $GF[p^\nu]$ which depend on $(a_1, a_2, \cdots, a_k)$ but are independent of $(x_1, x_2, \cdots, x_k)$. Consequently the multiplicative group of the algebra may be defined by means of a transformation group whose elements have the foregoing form. A necessary and sufficient condition on these transformations is that they shall permute the non-zero marks of [illegible] according to a regular group.

* Carmichael, *American Journal of Mathematics*, Vol. 52 (1930), pp. 745-7[illegible] (see § 8).

When $\nu=1$ the transformation group is linear and we have the form of analytical representation employed by Dickson (*l. c.*). When $k=1$ we have the other extreme case of the foregoing transformations. In this case we have $\nu=n$ and the marks of the algebra are the symbols (a) where a runs over the marks of $GF[p^n]$. The rule of addition in the algebra, namely, $(a)+(b)=(a+b)$, coincides with the rule of addition in $GF[p^n]$. For the product $(a)(x)$ we have $(f(a,x))$ where $f(a,x)$ has the form

$$(6.4) \qquad f(a,x)=\sum_{t=1}^{n} a_t x^{p^{n-t}}.$$

Therefore we may write

$$(6.5) \qquad (\alpha_i)(x)=(f(\alpha_i,x))=(\sum_{t=1}^{n} a_t^{(i)} x^{p^{n-t}}), \qquad (i=0,1,\cdots,p^n-2),$$

where the α's are the non-zero marks of $GF[p^n]$ and the $a_t^{(i)}$ are marks of $GF[p^n]$ to be suitably determined. We take (1) to be the unit element in the algebra. Then we have

$$a_1^{(i)}+a_2^{(i)}+\cdots+a_n^{(i)}=\alpha_i.$$

We have also $(0)(x)=(f(0,x))=(0)$.

It thus appears that every algebra $A[p^n]$ may be represented analytically by means of $GF[p^n]$. As already indicated, the problem of determining all such algebras has not yet been completely solved. In § 8 we shall employ the method just indicated to set forth the analytical representations of each of a large class of algebras $A[p^n]$.

It is convenient to close this section with the statement of three propositions whose proofs will be omitted. If p is an odd prime the multiplicative group M of an algebra $A[p^n]$ contains just one element of order 2 (perhaps most readily proved by aid of (6.3) with $\nu=1$). In the multiplicative group M of an algebra $A[p^n]$ the Sylow subgroups of odd order are cyclic and those of even order are either cyclic or of the sole non-cyclic type containing a single element of order 2. If M contains a non-cyclic Sylow subgroup of order 2^a then this Sylow subgroup contains at least three subgroups of each of the orders $2^2, 2^3, \cdots, 2^{a-1}$.

7. *Certain Doubly Transitive Groups of Degree p^n.* In proceeding to construct algebras $A[p^n]$ it is convenient first to consider certain doubly transitive groups which are representable by means of transformations of the form

$$(7.1) \qquad x'=ax^{p^t}+b, \qquad a\neq 0,$$

where a and b belong to $GF[p^n]$, t belongs to the set $0,1,\cdots,n-1$ and

x and x' are variables running over the marks of $GF[p^n]$. The permutation groups involved are those according to which the marks of $GF[p^n]$ are permuted by the named transformation groups. When a, b, t range respectively over all the elements on which they may range we have a doubly transitive group of degree p^n and order $p^n(p^n-1)n$. The transformations

$$(7.2) \qquad x' = x + 1, \qquad x' = \omega x, \qquad x' = x^{p^\alpha},$$

where ω is a primitive mark of the field and α is a factor n, generate a group of order $p^n(p^n-1)n/\alpha$ whose elements are all the elements of the form (7.1) with the further restriction that t shall be a multiple of α. This group induces on the marks of the field a doubly transitive group of degree p^n and order $p^n(p^n-1)n/\alpha$. When $\alpha = n$ this is the sole doubly transitive group of degree p^n and order $p^n(p^n-1)$ whose regular subgroups of order p^n-1 are cyclic.

By way of digression it may be pointed out that if we adjoin to the generators (7.2) the transformation $x' = 1/x$ then we are led to a group of order

$$(p^n+1)p^n(p^n-1)n/\alpha$$

which permutes ∞ and the marks of $GF[p^n]$ according to a triply transitive group of degree p^n+1. When p is odd each of these groups contains transformations whose determinants are not squares; then the elements whose determinants are squares constitute a subgroup of index 2 which is doubly transitive on the symbols involved.

We shall now determine all the doubly transitive groups G of degree p^n and order $p^n(p^n-1)$ contained in the group induced by the $p^n(p^n-1)n$ transformations (7.1). Such a group contains a regular permutation group M of degree and order p^n-1 on the non-zero marks of the field. There is one and just one group G in which the corresponding group M is cyclic; it is generated by the first two elements in (7.2). Henceforth let M be non-cyclic. It is obvious that the transformation group T, by which M is induced, consists of transformations of the form

$$(7.3) \qquad x' = a_i x^{p^{t_i}}, \qquad (i = 1, 2, \cdots, p^n-1, \quad 0 \leqq t_i < n).$$

Such a transformation replaces the mark $x = 1$ by the mark $x' = a_i$. Since M is regular on the non-zero marks of $GF[p^n]$ it follows that the coefficients a_i are in some order the non-zero marks of the field (without repetition or omission).

Now the totality of linear transformations in T constitutes a subgroup

of T; and this subgroup is contained in the cyclic group generated by the transformation S,

$$S: \qquad x' = \omega x,$$

where ω is a primitive mark of $GF[p^n]$. Then there exists a least positive integer σ such that this linear subgroup is generated by S^σ. It is clear that σ is a factor of the order $p^n - 1$ of S. If $\sigma = 1$ we have a cyclic group T, a case which we have already excluded; therefore, $\sigma > 1$. Then some of the exponents t_i are positive.

Let t be the least positive value of t_i appearing in the transformations (7. 3) of T; and let the transformation U,

$$U: \qquad x' = ax^{p^t},$$

be one of the transformations in which $t_i = t$. By taking successive powers of U we obtain transformations with the exponents $t, 2t, 3t, \cdots$ on p. Since these are to be reduced modulo n (on account of the equation $u^{p^n} = u^{p^0}$ for marks of $GF[p^n]$) it follows that t is a factor of n. Moreover, since t is the least positive value of an exponent t_i, each t_i must be a multiple of t; whence one concludes that the exponents t_i are $t, 2t, 3t, \cdots$. If T_1 and T_2 are two transformations in T with the same value of the exponent t_i, then $T_1^{-1}T_2$ is a linear transformation and hence is in $\{S^\sigma\}$. Therefore all the transformations in T having a given value of t_i are products of the form T_1S_1 where S_1 is in $\{S^\sigma\}$. Therefore T is generated by S^σ and U. The smallest positive value of λ such that U^λ is in $\{S^\sigma\}$ is $\lambda = n/t$. Since T and $\{S^\sigma\}$ are of orders $p^n - 1$ and $(p^n - 1)/\sigma$ it follows that $\sigma = n/t$ and hence that σ is a factor of n.

We have now to determine the further conditions on σ, t and a such that the group $\{S^\sigma, U\}$ shall indeed induce a permutation group of the type prescribed for M. If d is the greatest divisor of σ such that a is a d-th power of a mark in $GF[p^n]$, then every coefficient in the transformations belonging to $\{S^\sigma, U\}$ is a d-th power. Since d is a factor of $p^n - 1$ and every mark of $GF[p^n]$ occurs among these coefficients it follows that $d = 1$. Therefore if γ is such that $a = \omega^\gamma$ we must have γ prime to σ. We may now combine the transformation U with an appropriate power of S^σ so that in the resulting transformation $U_{l,t}$ of the form U (with the same value of t) we shall have the corresponding coefficient of the form ω^l where $0 < l < \sigma$ and l is prime to σ. Then we have

$$U_{l,t}: \qquad x' = \omega^l x^{p^t}, \quad 0 < l < \sigma, \quad l \text{ prime to } \sigma, \quad \sigma t = n.$$

Then $\{S^\sigma, U\} \equiv \{S^\sigma, U_{l,t}\}$.

The λ-th power of $U_{l,t}$ may be written in the form

$$U^{\lambda}_{l,t}: \qquad x' = \omega^{l(1+p^t+p^{2t}+\cdots+p^{(\lambda-1)t})}x^{p^t}.$$

The least positive value of λ for which this is in $\{S^\sigma\}$ is $\lambda = n/t = \sigma$. In order that the induced permutation group M shall be regular it is further necessary that the least value of λ for which

$$1 + p^t + p^{2t} + \cdots + p^{(\lambda-1)t}$$

shall be a multiple of σ is $\lambda = \sigma$, since otherwise at least one mark of $GF[p^n]$ would occur as a coefficient in two transformations belonging to $\{S^\sigma, U_{l,t}\}$.

When the necessary conditions now obtained are satisfied we shall easily show that $\{S^\sigma, U_{l,t}\}$ permutes the non-zero marks of $GF[p^n]$ according to a regular permutation group M. The coefficients in the transformations belonging to $\{S^\sigma, U_{l,t}\}$ are the marks

$$\omega^{k\sigma}\cdot\omega^{l(1+p^t+p^{2t}+\cdots+p^{(\lambda-1)t})}, \quad [k = 1, 2, \cdots, (p^n - 1)/\sigma, \lambda = 1, 2, \cdots, \sigma - 1],$$

together with the σ-th power marks appearing as coefficients in the transformations of $\{S^\sigma\}$. No two of these coefficients are equal since the second exponent on ω in the foregoing expressions is not a multiple of σ and no two such exponents have their difference a multiple of σ. Therefore no two transformations in $\{S^\sigma, U_{l,t}\}$ have the same coefficient and hence that group replaces the value of 1 of x by every non-zero mark of the field; whence it follows that $\{S^\sigma, U_{l,t}\}$ induces a regular permutation group M on the non-zero marks of the field.

Since $\sigma > 1$ it is easy to verify that the group $\{S^\sigma, U_{l,t}\}$ is non-Abelian; for the equation $S^{-\sigma}U_{l,t}S^\sigma = U_{l,t}$ would imply that $\sigma\,(p^t - 1) \equiv 0 \bmod p^n - 1$, and this is impossible since

$$\sigma(p^t - 1) < (p^\sigma - 1)(p^t - 1) = (p^{n/t} - 1)(p^t - 1) < p^n - 1 \quad \text{when} \quad \sigma < n.$$

Among the results established by the foregoing argument we have the following theorem (and thence the easily established corollaries):

THEOREM. *Every non-cyclic group T which is contained in the group whose elements are the transformations*

$$x' = ax^{p^t} + b, \qquad (a \neq 0, \quad t = 0, 1, \cdots, n - 1),$$

where a and b are marks of $GF[p^n]$, subject to the condition that T shall be of order $p^n - 1$ and shall permute according to a regular permutation group M the non-zero marks of $GF[p^n]$, is a non-Abelian group $\{S^\sigma, U_{l,t}\}$

where σ $(\sigma > 1)$ *is a common factor of* n *and* $p^n - 1$ *such that* $\lambda = \sigma$ *is the least value of* λ *for which* $1 + p^t + p^{2t} + \cdots + p^{(\lambda-1)t}$ *is divisible by* σ, *where* $t = n/\sigma$; *and every such group* $\{S^\sigma, U_{l,t}\}$ *is such a group* T.

COROLLARY I. *If the elements of* $\{S^\sigma, U_{l,t}\}$ *are the transformations*

$$x' = a_i p^{ti}, \qquad (i = 1, 2, \cdots, p^n - 1),$$

then the transformations

$$x' = a_i p^{ti} + b_i, \qquad (i = 1, 2, \cdots, p^n - 1),$$

where for each value of i *the symbol* b_i *runs over all the marks of* $GF[p^n]$, *induce a doubly transitive group of degree* p^n *and order* $p^n(p^n - 1)$ *on the marks of* $GF[p^n]$ *in which* M *is the largest subgroup each element of which leaves zero fixed.*

COROLLARY II. *If* $n = \sigma t$, $\sigma > 1$, *and* p *is a prime of the form* $\sigma z + 1$ *then there exists a doubly transitive group of degree* p^n *and order* $p^n(p^n - 1)$ *whose regular subgroups of order* $p^n - 1$ *are non-Abelian.*

COROLLARY III. *Whenever* n *and* $p^n - 1$ *are not relatively prime there exist* at least two doubly transitive groups of degree* p^n *and order* $p^n(p^n - 1)$.

From the last corollary it follows that there are at least two distinct doubly transitive groups of degree p^2 and order $p^2(p^2 - 1)$ for every odd prime p. In no case does the theorem assert the existence of more than two such groups when $n = 2$. When $p^n = 3^2$ there are just two such groups. But when $p^n = 5^2$ or $p^n = 7^2$ there are three (and just three) such doubly transitive groups.

By aid of the foregoing theorem one may establish the following three theorems which we state without proof:

When p is an odd prime and n is an even integer there exist two triply transitive groups of degree $p^n + 1$ and order $(p^n + 1)p^n(p^n - 1)$. In one of these the regular subgroups of degree and order $p^n - 1$ are cyclic; in the other these subgroups are non-Abelian and contain cyclic subgroups of index 2.

The only triply transitive groups of degree $p^n + 1$ and order $(p^n + 1)p^n \times (p^n - 1)$ contained as subgroups in the triply transitive group of order

* This is in contradiction with a conjecture of Burnside [*Messenger of Mathematics*, Vol. 25 (1896), pp. 147-153; see also the footnote on p. 184 of the second edition of his *Theory of Groups*] to the effect that, with an exception in the case when $p^n = 3^2$, there is always one and just one doubly transitive group of degree p^n and order $p^n(p^n - 1)$, and in particular with the cases $n = 2$ and $n = 3$ in which he offered a supposed proof of the incorrect conclusion.

$(p^n+1)p^n(p^n-1)n$, described earlier in this section, are (1) one (a well known case) in which the regular subgroup of order p^n-1 is cyclic (existent for every p^n) and (2) the additional group described in the foregoing paragraph for the case when p is odd and n is even.

For every positive integer L three exists a prime p and a positive integer n such that the number of doubly transitive groups of degree p^n and order $p^n(p^n-1)$ is greater than L.

8. *The Algebras* $A_{\sigma,l}[p^n]$. In the main theorem of the preceding section and its first corollary we have a means of defining an important class of algebras $A[p^n]$. We denote their elements by (a) where a runs over the marks of $GF[p^n]$ and where (0) and (1) are to be the zero and unit elements of the algebra respectively. Addition is defined by the relation $(a)+(b)=(a+b)$. For the product $(a_i)(x)$ we take the element $(a_i x^{p^{l_i}})$, where the symbols are those of the theorem cited and its first corollary. Such an algebra will be called an algebra $A_{\sigma,l}[p^n]$, where σ and l are defined as in the theorem cited. They do not include all the algebras $A[p^n]$ as is shown by an examination of the three algebras $A[5^2]$ or the three algebras $A[7^2]$.

To construct three algebras $A[5^2]$ we proceed as follows. In the first place we have $GF[5^2]$ as one of the algebras. A second one is $A_{2,1}[5^2]$. To construct a third algebra $A[5^2]$ we observe that the transformations

$$x'=x+1, \qquad x'=\omega^8 x, \qquad x'=\omega^{12}x^5+\omega^{21}x,$$

where ω is a primitive mark of $GF[5^2]$ satisfying the relation $\omega^2=\omega+3$, permute the marks of this Galois field according to a doubly transitive group of degree 25 and order $25\cdot 24$. If an $A[5^2]$ is formed from this group by the method of § 1 it will be different from the other two $A[5^2]$ already described in this paragraph. It may be shown that the three $A[5^2]$ thus exhibited are all the possible algebras $A[5^2]$.

It seems probable that the algebras $A_{\sigma,l}[p^n]$ are contained among the algebras otherwise constructed by Dickson (*l. c.*); but I did not seek to verify this proposition.

From the main theorem of the preceding section it follows that σ is a factor of

$$1+p^t+p^{2t}+\cdot\cdot\cdot+p^{(\sigma-1)t}=(p^n-1)/(p^t-1)$$

and hence it is a factor of

$$\frac{p^n-1}{p^t-1}\cdot\frac{p^t-1}{p-1}\quad\frac{p^n-1}{p-1}.$$

Therefore the order of the group $\{S^\sigma\}$ is a multiple of $p-1$ and hence that

group contains a cyclic subgroup of order $p-1$ and therefore contains all the transformations of the form $x' = \alpha x$ where α is an integral mark of $GF[p^n]$. Therefore in $A_{\sigma,\iota}[p^n]$ we have $(\alpha)(x) = (\alpha x)$ where (x) is any element of the algebra and (α) is an integral element. But $(a_i)(\alpha) = (a_i \alpha) = (\alpha a_i)$ since $\alpha^p = \alpha$. Therefore an integral element of $A_{\sigma,\iota}[p^n]$ is permutable under multiplication with every element of the algebra.

That this property of permutability of integral elements with all elements is not common to all algebras $A[p^n]$ may be seen from an examination of the last $A[5^2]$ defined earlier in this section and in fact from a consideration of the elements $x' = \omega^{12}x^5 + \omega^{21}x$ and $x' = \omega^{16}x$ in the underlying group.

Let k be any factor of n, and consider the subset (r) of elements in an algebra $A_{\sigma,\iota}[p^n]$ where r runs over the marks of the subfield $GF[p^k]$ contained in $GF[p^n]$. Under addition these elements (r) obviously form a group of order p^k. Moreover, the product of two of these elements (r) is an element of this set. Therefore the p^k elements named form a sub-algebra $A[p^k]$. From this it follows readily that $A_{\sigma,\iota}[p^n]$ contains a sub-algebra $A[p^k]$ when and only when k is a factor of n.

By means of the algebras $A_{\sigma,\iota}[p^n]$ one may easily prove the following proposition: For every positive integer L there exists a prime p and a positive integer n such that the number of algebras $A[p^n]$ is greater than L.

CAYLEY DIAGRAMS ON THE ANCHOR RING.

By R. P. Baker.

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1. Maschke* determined the Cayley color groups representable in the plane, postulating independence of generators. These are all half regular in Archimedes' sense. (There are the same number of polygons of the same orders at each vertex). Adding this as a postulated property this paper extends the enumeration to connectivities two and three.

2. On account of the importance in analysis of the 'groups of genus p,' I add them to the lists. Given classically with 'schraffirte" diagrams there is a corresponding Cayley diagram the method of construction being indicated by the plate in Burnside's Theory of Groups, Ch. XIX. The generators of these groups are not independent but connected by a relation of the form $ABC\cdots K=1$. They do not therefore occur in Maschke's list. Adding them and two 'general' Cayley diagrams for the groups of order four on the tetrahedron it appears that all the half regular and regular bodies except two are the basis for Cayley diagrams.

In the list the net is specified by the orders of the polygons at a vertex.

	Net	Order	Generators	Group	Maschke's figure
1.	3.3.3	4	2.2.2	G_4	(general)
		4	4.2	C_4	(general)
2.	4.4.4	8	4.2	$G_8{}^4$	2
			4.2	Abelian	2a
			2.2.2	Abelian	16
3.	3.3.3.3	6	3.2.2	G_6	$(p=0)$
4.	3.3.3.3.3	12	3.3.2	$G_{12}{}^4$	$(p=0)$
5.	5.5.5	20		None	
I	3.6.6	12	3.2	$G_{12}{}^4$	3
II	3.8.8	24	3.2	$G_{24}{}^4$	6
II				$G_{12}{}^4C_2$	6a
III	3.10.10	60	3.2	$G_{60}{}^5$	9
IV	4.6.6	24	4.2	$G_{24}{}^4$	5
			2.2.2	$G_{24}{}^4$	17
V	4.4.n	2n	n.2	Dihedral	2
			2.2.2	Dihedral	16
VI	5.6.6	60	5.2	$G_{60}{}^5$	8
VII	3.4.4.4	24	4.3	$G_{24}{}^4$	7
VIII	3.4.3.4	12	3.3	$G_{12}{}^4$	4
IX	3.5.3.5	30		None	

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* *American Journal of Mathematics*, Vol. 18 (1896), p. 156.

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	Net	Order	Generators	Group	Maschke's figure
X	3.3.3.n	2n	n.2.2	Dihedral	$(p=0)$
XI	3.3.3.3.4	24	2.3.4	Octahedral	$(p=0)$
XII	3.3.3.3.5	60	2.3.5	Icosahedral	$(p=0)$
XIII	4.6.8	48	2.2.2	Extended octahedral	18
XIV	4.6.10	120	2.2.2	Extended icosahedral	
XV	3.4.3.4.5	60	3.5	$G_{60}{}^5$	10

3. Extending the problem to higher connectivities we are confronted by an embarassment of riches.

$G_{24}{}^4$ with three generators of order two has a half regular representation on the anchor ring and 72 other representations, mostly bizarre.

The Abelian G_9 $(3,3)$ has representations not half regular, on the projective plane and on the anchor ring.

Taking the elements as $(1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta, \beta^2, \alpha\beta^2, \alpha^2\beta^2)$ and numbering from $1\cdots 9$ the polygons are:

(125697); (1364), (4587), (2398); (123), (456), (789), (147), (258), (369)

and in the second case,

(145693), (458936); (2365), (4786), (1287); (123), (147), (258), (896).

The diagrams are not perspicuous and lack the symmetry which is the essential characteristic of a group.

These facts together with the number of half regular diagrams discovered here justify the restricting hypothesis that the net is half regular.

4. For half regular nets the extended Euler equation becomes

$$\Sigma(l-2)/2l = 1 + (k-3)/V$$

where the l's are the orders of the polygons at each vertex, k is the connectivity and V the number of vertices. This is a necessary but not a sufficient condition. A sufficient condition is I believe not known, but we may note two types of failure. $l_1 = 5$, $l_2 = 5$, $l_3 = 10$, $k = 3$ satisfy the condition but no net can be constructed. The statement does not allow us to bound a single pentagon. This may be called failure 'im kleinen.'

$l = 3, 4, 4$ $V = 6$, $k = 1$ belongs to the triangular prism but $l = 3, 4, 4$ $V = 3$, $k = 2$ also satisfies the equation and there is no such figure in the projective plane. This may be called a failure 'im grössen.'

If a solution $(l_1, l_2, \cdots l_n)$ k, V exists and also a solution $(l_1, l_2, \cdots l_n)$ k', V' which is possible if $(k-3)V' = (k'-3)V$ then if V' is a multiple of V and the first solution has a net the second has, but not conversely. For $k = 2$, $k' = 1$ the construction can be carried out by Klein's

double representation of the projective plane and joining up the two nets. In the higher cases we must combine this method with that of connecting overlying Riemann surfaces by branch cuts following Dyck.*

5. The enumeration of the diagrams possible in the projective plane can be made speedily as we have the list for $k=1$ and can pick out those which can be bisected by a simple closed curve not passing through a vertex. One condition for bisection is that it must be possible to arrange the points on the sphere so as to exhibit central symmetry. This is met in all cases. A further condition is that the opposite edges must be of like color. No. 17 fails here. Still further the arrows must not conflict with the reversal of the Möbius indicatrix. This means that the arrows if any must be alternately clock and counterclock. This cuts out Nos. 3.4.5.6.7.8.9.10. In No. 2, the even prisms lose the independence of generators on bisection.

No. 6a gives G_{12}^4 and the three color group 18 gives G_{24}^4. The extended icosahedron group (not drawn in Maschke's paper) G_{60}^5 with three generators of order two.

These three groups are the subjects of extension in the extended groups of Maschke's list.

CAYLEY DIAGRAMS IN THE PROJECTIVE PLANE (Plate I).

(Half regular).

Net	Order	Generators	Group
3.8.8	12	3.2	G_{12}^4
4.6.8	24	2.2.2	G_{24}^4
4.6.10	60	2.2.2	G_{60}^5

6. For the anchor ring, by dissection and development to a parallelogram, and then by indefinite repetition in the plane, the problem is reduced to one of homogeneous assemblages in the plane. If we consider such a development colored and arrowed on one of the nets and being a Cayley diagram for a group, certain of the relations can be determined by inspection of any small region. Such are the order of the generators and the definition of the intermediate polygons. These we may call 'im kleinen' while the others which demand a knowledge of the 'cut' that is of the fundamental region are properly called 'im grössen' relations.

The latter will in general contain one or more arbitrary integers due to the fact that V disappear from the Euler equation or, geometrically, or

* *Mathematische Annalen*, Vol. 17 (1880), p. 473.

pressed, to the arbitrary nature of the fundamental region. The heuristic program is then to determine (1) all possible half regular nets (2) all consistent colorings for each net (3) all consistent arrowings.

Then for each case we have the 'im kleinen' relations and can discuss the possible 'im grössen.'

There are some cases where the 'im kleinen' determine the 'im grössen' and one or more integers disappear from the expression for the order. In the more general case it is possible to classify the points into sets invariant under a subgroup called by Burnside, in his discussion of groups of genus one, the group of translations. The word is hardly well chosen here for often some elements are moved in opposite senses a phenomenon which can be associated with the idea of an electrolytic vector, but more complicated cases exist where generalized 'ballet' seems needed.

Falling back on the technical language of group theory this subgroup is a maximal self conjugate abelian subgroup and the order of the quotient group gives the number of classes of points.

7. *The half regular nets on the anchor ring.* The equation

$$\sum_{i=1}^{i=n} (l_i - 2)/2l_i = 1$$

has 17 solutions.

$n=3$ (6, 6, 6), (12, 12, 3), (8, 8, 4), (10, 5, 5), (42, 7, 3), (24, 8, 3), (18, 9, 3), (15, 10, 3), (20, 5, 4), (12, 6, 4),

$n=4$ (4, 4, 4, 4), (6, 3, 6, 3), (6, 4, 3, 4), (12, 4, 3, 3),

$n=5$ (4, 4, 3, 3, 3), (6, 3, 3, 3, 3),

$n=6$ (3, 3, 3, 3, 3, 3).

We note first that any solution with two odd numbers and one even fails 'im kleinen': we cannot properly bound an odd polygon. Similarly two different evens and one odd fails to properly bound the odd. The case (12, 4, 3, 3) leads eventually to either two twelves or two fours at a vertex.

This leaves ten cases:

(6, 6, 6), (12, 12, 3), (8, 8, 4), (12, 6, 4), (4, 4, 4, 4)
(6, 3, 6, 3), (6, 4, 3, 4), (4, 3, 4, 3, 3), (3, 3, 3, 3, 3, 3).

For those with 4 or 5 polygons the cyclic order given is the only possible.

The duals of these nets give all the 'space fillers' for the plane. Those with all evens are the basis for the 'schraffirte' diagrams of the groups of genus one.

Metric solutions can be constructed in several ways employing only the draftsman's triangles and one opening of the compasses.

8. Before proceeding to the detailed enumeration of cases certain general features of the configurations may profitably be discussed, and an explanation given of the tabular entry of the final results. Going from the table to the picture, the net and the 'im kleinen' relations enable us to construct the homogeneous assemblage colored and arrowed. I use throughout the following symbols for the generators and colors

α	————————————	blue
β	— · — · — · — ·	red
γ	· · · · · · · · · · · ·	green
δ	· · · — · · · — · · · —	yellow

In the table 'con.' means that the arrows are all of the same sense, while 'alt.' implies that half of them are of each sense. In one case (41) reference to the picture shows the exact sense, there being two possible alternate arrangements of which only one is possible for us. In 5 and 5′ two arrangements are possible. The choice of generators X, Y for the abelian subgroup is necessarily arbitrary. It is always possible to arrange a fundamental region as a parallelogram (or rhombus if X and Y are conjugate) with the identity at each vertex. These cases are listed as (a). There are other possibilities however. If we take X for some point as leading to the right and of order k and Y leading downwards from the same point and the t-th row down parallel to the X line as the first row with the identity, at the point r steps from the Y line so that $Y^tX^r = 1$ there will in general be relations between k, t and r. Figure 1 shows a colored net with the net of X, Y, superimposed.

To determine these we consider points not on the XY net. Under the operations X, Y these points in general move in different directions but since each aggregate of points one of each class which neighbor or form a molecule must be gathered together again at the identity there are other relations of the type $Y^\tau X^\rho = 1$ but not always identical with the first.

The totality of these relations, not usually independent, must be satisfied and restrict the arithmetic nature of the order and the structure of the group.

One special case may be reported in full. The net 8.8.4 may be colored with the squares of one color α and of one sense and the other lines with a second color β. This gives the 'im kleinen' relations $\alpha^4 = \beta^2 = (\alpha\beta)^4 = 1$ and X, Y may be taken as $(\alpha^2\beta)$ and $(\alpha\beta\alpha)$ respectively for these operators carry into themselves points of each class and are commutative and generate a self conjugate subgroup. Moreover it is a maximal group of this kind for all other operations carry points of one class to points of another.

X and Y being conjugate we have $X^k = Y^k = 1$. The 'molecule' may

be the four points of a quadrilateral and as drawn X carries (1) to the right (2) up, (3) left and (4) down. Y carries (1) down (2) right (3) up and (4) left. The relation $Y^tX^r = 1$ for (1) becomes for (2) $X^tY^{-r} = 1$ and for (3) $Y^{-t}X^{-r} = 1$ and for (4) $X^{-t}Y^r = 1$. Of these the first two are independent giving $X^{r^2+t^2} = Y^{r^2+t^2} = 1$ and $r^2 + t^2$ must contain k as a factor. Now if r and t are relatively prime we may determine p and q so that

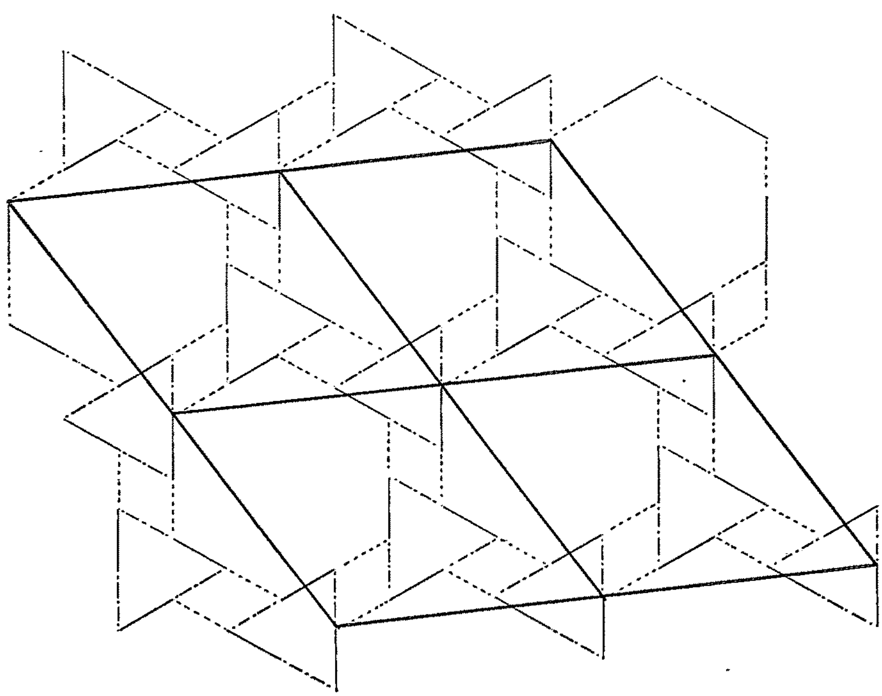


FIG. 1

$YX^s = 1$ which means that the identity occurs in the first row down amending our first statement. $s = pr - qt$.

If r and t have a common divisor m we obtain similarly a relation of the form $Y^mX^{mr} = 1$, which is taken as a standard form for representations of type b. In this case the order of the abelian subgroup is $m^2(r^2 + 1)$ and of the whole group $4m^2(r^2 + 1)$, the quotient group being C_4. This is determined in the usual manner by setting up a table for the group with $[X, Y]$ in the first line.

The groups whose numbers are given with a * are groups of genus one

with one generator removed. The order is given in this reduced form and differs apparently from Burnside's order but is arithmetically equivalent. The reduction is essential for the draftsman. The groups then occur in sets whose orders are square multiples of a primitive $m = 1$ and whose developments are geometrical repetitions of this in square array. The two forms given by Dyck (*loc. cit.*) are special forms for $r = 0$ and $r = 1$ (or 2). Since the form (a) cannot in every case be obtained from the form (b) by placing $r = 0$ I list them separately in all cases.

In the case of groups of genus one of class I Burnside gives the order as $2(ab' - a'b)$ with $(ab' - a'b) \neq 0$ for my postulates this must be amended by saying $(ab' - a'b)$ is not a prime for in this case a second generator loses its independence.

As a final check on the correctness of the pictures Maschke's theorem (2) has been used. This is equivalent to demanding that if the points be numbered and the operators expressed as substitutions the group generated is regular.

9. The picture of the development being established the next proceeding is to place it on the anchor ring. This is arbitrary in several ways. In the first place X and Y may run round either of two non-homologous circuits and may start at a point of any class. Secondly the whole picture is subject to continuous deformation.

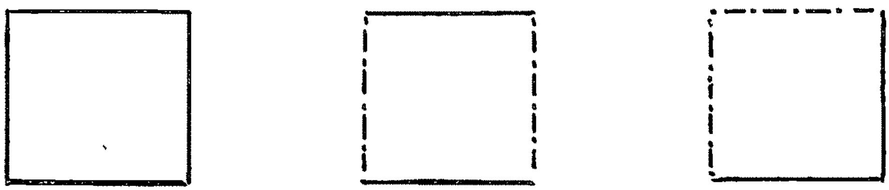
There is at least one way of fixing the position of the points on mathematical principles. If we accept the drawing of the net as standard and the proportions of the anchor ring are agreed on the fundamental parallelogram may be treated as belonging to an elliptic function, transferred to a two-sheeted Riemann surface and thence conformally to the anchor ring. The required calculations are obviously rather severe and the result might be aesthetically more satisfactory and might not. Maschke did not use the conformal stereographic projection and my experiments with that method confirm his judgment. I have only tried to attain perspicuity.

In the drawings the left hand figure can be considered at the front of the anchor ring seen from the outside and the right hand one as the back seen from the inside so that homothetic points on the edges agree. This causes an apparent reversal of arrows on homothetic cycles which cut out 'flachenstücken' but not of cycles encircling the hole. This method seems psychologically preferable to the plan adopted with coins.

10. *Proceeding to the detailed enumeration.*

The net (4.4.4.4).

The only possible colorings for a square are I all α's, II alternate α and β, III two adjacent α's and two adjacent β's.



I $\alpha\alpha$ II $\alpha\beta$ III $\overline{\alpha\beta}$

FIG. 2

For the discussion of the possible sets of four squares at a corner these types may be denoted by $\alpha\alpha$, $\alpha\beta$, $\overline{\alpha\beta}$ respectively.

If there are four colors $\alpha\alpha$ and $\overline{\alpha\beta}$ are impossible and four squares at a corner must be of types $(\alpha\beta \mid \alpha\gamma)/(\beta\delta \mid \gamma\delta)$, so that the horizontal lines read $\beta\gamma\beta\gamma\beta\gamma\cdots$ and the vertical $\alpha\delta\alpha\delta\cdots$. The operators necessarily of period two are by this coloring compelled to have α, δ each permutable with β and γ. $(\alpha\delta)$ and $(\beta\gamma)$ are not necessarily conjugate and for the type (a) the fundamental region is a $2k \times 2l$ rectangle with $(\alpha\delta)^k = (\beta\gamma)^l = 1$ as the 'im grössen' relations. For the type (b) the (t, r) technique gives $Y^tX^r = Y^{-t}X^r = 1$ whence $X^{2r} = Y^{2t} = 1$ and the identity occurs at two corners and the midpoint of the opposite side of a $2r \times t$ rectangle. There are four classes of points, a molecule may be taken as the four points on any square while the quotient group works out as G_4. If the independence of operators is to be kept k, l must be greater than unity. In type (b) there are no restrictions on t, r. A possible amendment to the general forms in the cases where the integers entering take the values 1 or 2 is left to the reader. These two cases are listed as 1(a) and 1(b).

If there are three colors.

The squares of type $\overline{\alpha\beta}$ cannot be used, they demand two colors. Taking first the case where all the squares are $\alpha\beta$, the four at a corner must be equivalent to $(\alpha\beta \mid \alpha\beta)/(\alpha\gamma \mid \alpha\gamma)$. The α color must run through. This gives the 'im kleinen' relations $\alpha^k = \beta^2 = \gamma^2 = 1$.

If the arrows concur α permutes with $(\beta\gamma)$ and these can be used as X and Y. The (t, r) argument yields the same result as in 1(b) and the two cases are listed as 2(a), 2(b).

If the arrows alternate the situation is different, a general r is possible, the cases are 3(a), 3(b). In both (2) and (3) the quotient group is C_2.

If with three colors any square is of type $\alpha\alpha$ the arrangement at a corner is $(\alpha\alpha \mid \alpha\beta)/(\alpha\gamma \mid \beta\gamma)$ which is a self perpetuating system.

If the arrows are all counterclock an internal square reads $\alpha\beta\alpha\beta = 1$ which is also the reading of an edge. The group is of order 16, the 'im kleinen' relations determining the 'im grössen.' This occurs also if the arrows alternate. The cases (4) and (5) exhibit G_{16} III vii (Burnside's enumeration) and the abelian $(4, 2, 2)$.

If there are two colors:

A square of the $\alpha\alpha$ type must meet a $\beta\beta$ square at each end of one of its diagonals and at each corner there is the arrangement $(\alpha\alpha \mid \alpha\beta)/(\alpha\beta \mid \beta\beta)$ which is self perpetuating.

If each set of arrows is counterclock the 'im kleinen' relations are $\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$ and $(\alpha^3\beta)$ and $(\alpha\beta^3)$ serve as $X, Y \ldots$

The (t, r) method gives the common order of these as $m(r^2 + 1)$ provided $r \neq 0$ and for $r = 0$ the (b) case includes the (a). For graphical distinctions however I list two cases 6(a) and 6(b). These are groups of genus one.

If the arrows of each set alternate there arises the abelian group of order 16 $(4, 4)$, while if one set concur and the other alternates the group G_{16} III viii the 'im kleinen' dominating (7) and (8).

With two colors and all squares of the second type the α and β lines intersect, a self perpetuating situation. The general case has a group of order mtr without restriction save $m, t, r > 1$. In the special case the order is mn. Both groups are abelian the quotient group being the identity. 9(a), 9(b).

If the arrows of one set concur and the others alternate the groups are of doubled order but the rest is similar. 10(a), 10(b).

If both sets of arrows alternate the quotient group is G_4 the rest of the situation similar with groups of fourfold order. 11(a), 11(b).

All the squares may be of the third type $\overline{\alpha\beta}$, the color lines being corrugated. With concurrent arrows there are two cases 12(a), 12(b).

If the arrows concur for one set and alternate for the other we have both $\alpha^2 = \beta^2$ and $\alpha^2 = \beta^{-2}$. The group is the quaternion group. (13).

If both sets alternate we have the abelian group [4, 2]. (14).

[illegible]
A square of type III implies a whole diagonal row of the same type and if a square of type II joins it there must be a whole diagonal row of the same [illegible] the row of III's.

Taking first the case of concurrent arrows in both sets and considering two columns the III square gives $\alpha^2 = \beta^2$ and the II square $\alpha\beta = \beta\alpha$. The identity recurs across the diagonal of the two III squares. It is then only necessary to consider these two columns. The results depend on the sequence of the types in the columns. This is the same for each and we can assume that it starts with a III.

Let this sequence be $(\mathrm{III})^{k_1}\ (\mathrm{II})^{l_1}\ (\mathrm{III})^{k_2}\ (\mathrm{II})^{l_2}\ (\mathrm{III})^{k_3}\ (\mathrm{II})^{l_3}\cdots$.

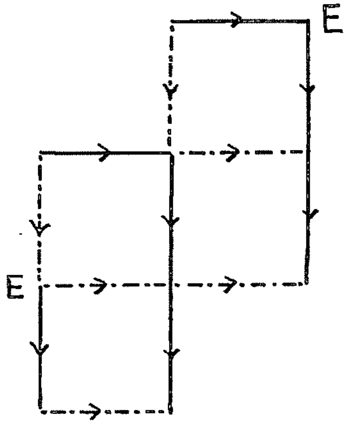


FIG. 3

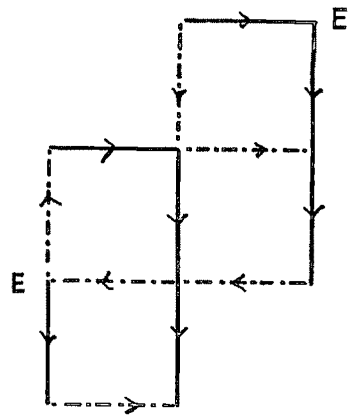


FIG. 4

The total number in the sequence must be $2k$ the order of α and β. Now the III's change the color on the right edge and the II's continue them so that if the number of III's is odd the last segment on either the right edge or the mid line reads $\alpha^{2k-1}\alpha^{2k}$ and the configuration is non-oriented. If the number of III's is even there are two cases.

If the number of β's is odd the identity does not recur till the second column is filled and the whole could be rearranged as a single column of squares. If the number of β's is even the identity recurs at the bottom of the right edge and there is a two column set (15a), (15b).

The number of β's is odd or even with $\Sigma(k_i/2) + \Sigma l_i + \Sigma'(k_i j_i) - \Sigma''(k_i l_j)$ where in Σ' $j > i$ and in Σ'' $j \geq i$.

If the arrows for α concur and those for β alternate one III square gives $\alpha^2 = \beta^2$ and the other $\alpha^2\beta^2 = 1$ and the II squares $\beta\alpha\beta = \alpha$. The group is the quaternion group in two column form for the sequence III, II, III, II and in one column for III, II, II, III. (16), (17).

If the arrows both alternate the III squares read $\alpha^2 = \beta^2$ and the II squares $(\alpha\beta)^2 = 1$. In this case one can prove $\alpha^8 = 1$ but not $\alpha^4 = 1$. There

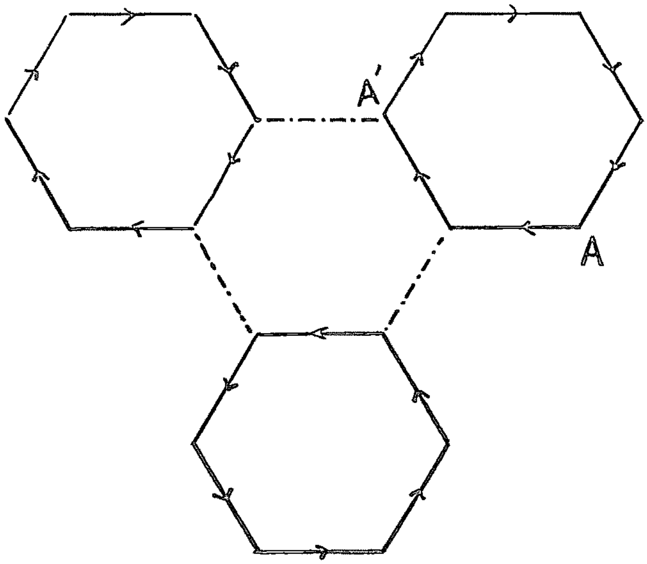
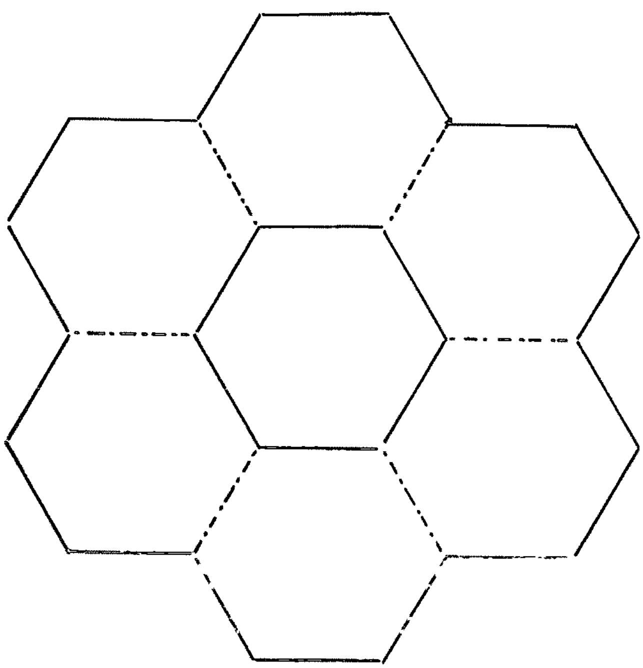


FIG. 5

No hexagon of type I can join one of type III or IV or V.

If I joins a VI it is surrounded by VI's and there is a cycle of 18 α's whereas $\alpha^6 = 1$.



is a group of order 16 with four columns or rearranged as a 4×4 square with cycles III, II, III, II in both columns and rows. (20).

If $\alpha^4 = 1$ there are two and one column representations of the abelian [4, 2]. (18), (19).

The net (6.6.6).

There are six types of hexagon available.

I α^6, II $(\alpha\beta)^3$, III $(\alpha\beta\gamma)^2$, IV $\alpha\beta\gamma\alpha\gamma\beta$, V $(\alpha^2\beta)^2$, VI $\alpha^3\beta\alpha\beta$.

Type I may be surrounded by type II and with concurrent arrows gives rise to two cases (25a) and (25b), groups of genus one.

If the arrows alternate there is a contradiction. Somewhere on the diagram, of the three I's neighboring a II two must be clock and the third counterclock. The II hexagon reads $\alpha\beta\alpha\beta\alpha^{-1}\beta = 1$ and starting from A with this operation we reach A′ with the contradiction $\alpha^2 = 1$ for $\alpha^6 = 1$.

If there are no I's but a II, this may be joined by II's of other pairs of colors with two kinds of fundamental region. (24a), (24b).

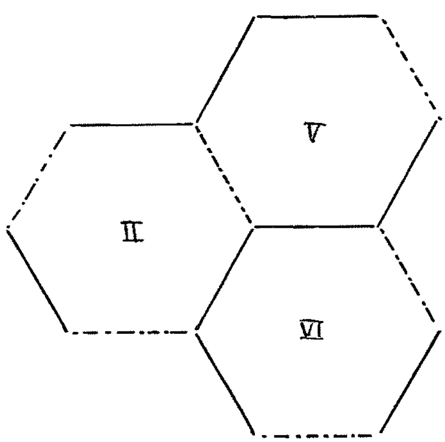


FIG. 7

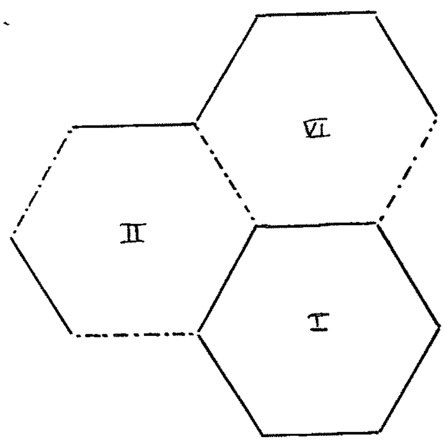


FIG. 8

A II cannot join a III. If a II join a IV it is surrounded by IV's and the outer cycle reads $(\alpha\beta)^9 = 1$, a contradiction.

If a II join a V on the β side it also joins a VI. The relations $(\alpha\beta)^3 = \alpha^2\beta\alpha^2\beta = \alpha^3\beta\alpha\beta = 1$ lead to $\beta = \alpha^2$.

If a II join a VI the common neighbor is a I and this has been disposed of.

Hexagons of type III can exist alone the X, Y not being conjugate. The groups are of genus one and graphically fall into two classes. 21a, 21b.

A III cannot join any one or two colored hexagon. III's and IV's can exist together in alternate rows and as $\beta\gamma$ is of order two there are only two rows. (23). Here the 'im kleinen' relations remove one of the parameters from the order but not both.

Hexagons of type IV can exist alone in two ways 22a, 22b.

Hexagons of type IV cannot be used with V or VI.

All the hexagons may be of type V and there are four cases with arrows concurrent or alternate and two kinds of fundamental region. 26a, 26b, 27a, 27b.

VI's can exist alone but the 'im kleinen' relations again control the size. With concurrent arrows the step from A to B is $\alpha^4\beta$ and also $\beta\alpha^4$. The hexagon gives $\alpha^3\beta\alpha\,\beta = 1$ hence $\alpha^8 = 1$. There is a G_{16} (28) and a G_8 (29) with alternate arrows similarly a G_{16} (30) and G_8 (31). If V and VI both occur a similar argument leads to $\alpha^4 = 1$ for both ways of placing the arrows but the edge of the fundamental region shows that the figure is non-oriented.

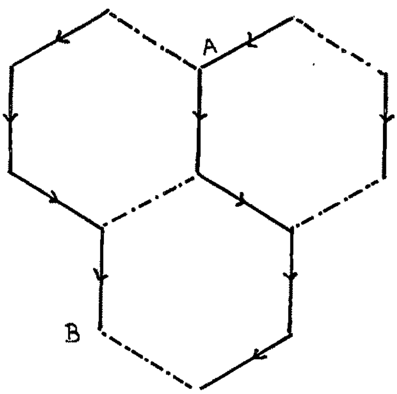


FIG. 9

The net (8.8.4).

With two colors. Squares of type III (see net 4.4.4.4) cannot occur as they demand vertices of the fourth order. The octagons cannot have an even succession of one color for one of the neighbors at the end of the run must be a square and cannot be either I or II. The permissible octagons are then reduced to the list:

I α^8, II $\alpha^3\beta\alpha^3\beta$, III $\alpha^3\beta\alpha\beta\alpha\beta$, IV $\alpha\beta\alpha\beta\alpha\beta\alpha\beta$, V $\alpha^5\beta\alpha\beta$.

From the relations only, the only [illegible] types are I, II with the [illegible] reading $\alpha\beta\alpha\beta$; I, IV the squares reading $\alpha\beta = \beta\alpha$, and II, IV the squares being $\alpha\beta\alpha\beta$. The geometry prevents I and II adjoining. If I, IV join each

type is surrounded by the other and IV must read $(\alpha\beta\alpha^7\beta)^2=1$ while the squares give $\alpha\beta\alpha^7\beta=1$ a contradiction in the diagram if not in the abstract symbols. II and IV cannot join.

The octagons must then be all of one type. They cannot be all I's or all III's. V's can be arranged in a pattern but the octagon reads $\beta\alpha=\alpha^5\beta$ if the squares read $\alpha\beta=\beta\alpha$ and $\alpha^5\beta\alpha=\beta$ if the squares are $\alpha\beta\alpha\beta$. In each case $\alpha^4=1$ contradicts V.

If the octagons are all of type II and arrows concur we have (36) and if they alternate (37) with loss of a parameter in each case.

If the octagons are all IV's and arrows concur $\alpha^4=\beta^2=(\alpha\beta)^4=1$.

The groups are of genus one of order $4m^2(r^2+1)$, or with $r=0$. (32a) and (32b). If the arrows alternate the order is doubled (33a) and (33b).

With three colors. If α, γ occur in one square they occur in all. There are two possible arrangements, homothetic and alternate. (34a), (34b), (35a), (35b). These may be derived from (32) and 33) by replacing the α^4 squares by $\alpha\gamma\alpha\gamma$, a method used by Maschke.

The net (12.12.3).

The triangle must read $\alpha^3=1$, the twelve side either $(\alpha\beta)^6=1$ or $(\alpha\beta\alpha^2\beta)^2=1$ according to the arrows. The concurrent case gives groups of genus one and the alternate case has the same r function (r^2-r+1). (38a), (38b), (39a), (39b).

The net (6.3.6.3).

The triangles must read $\alpha^3=1$, $\beta^3=1$. With concurrent arrows the groups are of genus one. (40a), (40b). With alternate arrows and the particular arrangement of the picture there is a single group of order 24. (41).

The net (6.4.3.4).

The triangle must read $\beta^3=1$ and the hexagons $\alpha^6=1$ for $(\alpha\gamma)^3=1$ leads to a conflict on progressing round a triangle. Concurrent arrows give groups of genus one. Alternation is not possible.

The net (12.6.4).

No polygon can be of one color for the intermediate polygons conflict. With three colors there are two cases. 43a and 43b.

To the list for completeness are added the groups of genus one and the non-oriented groups.

Cayley Diagrams on the Anchor Ring.

Net (4.4.4.4).

No.	Order.	Im kleinen relations.	X	Y		Arrows.
1a	$4kl$	$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1$	$(\alpha\delta)_k$	$(\beta\gamma)_l$	G_4	
1b	$8tr$	α, δ com with β, γ	$(\alpha\delta)_{2r}$	$(\beta\gamma)_{2t} Y^t X^r = 1$	G_4	
2a	$2kl$	$\alpha^k = \beta^2 = \gamma^2 = 1$ α com.	α_k	$(\beta\gamma)_l$	C_2	con.
2b	$4tr$	$\alpha^{2r} = \beta^2 = \gamma^2 = 1$ α com.	α_{2r}	$(\beta\gamma)_{2t} Y^t X^r = 1$	C_2	con.
3a	$2kl$	$\alpha^k = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\alpha\gamma)^2 = 1$	α_k	$(\beta\gamma)_l$	C_2	alt.
3b	$2mtr$	$\alpha^{mr} = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\alpha\gamma)^2 = 1$	α_{mr}	$(\beta\gamma)_{mt} Y^t X^r = 1$	C_2	alt.
4	16	$\alpha^4 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\alpha\gamma)^2 = (\beta\gamma)^2 = 1$				con.
5	16	$\alpha^4 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = 1$ γ com.				alt.
5′	16	$\alpha^4 = \beta^2 = \gamma^2 = 1$ com.				alt.
6a	$4k^2$	$\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$	$(\alpha^3\beta)_k$	$(\alpha\beta^3)_k$	C_4	con. con.
6b	$4m^2(r^2+1)$	$\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$	$(\alpha\beta^3)_{m(r^2+1)}$	$(\alpha^3\beta)_{m(r^2+1)} Y^m X^{mr} = 1$	C_4	con. con.
7	16	$\alpha^4 = \beta^4 = 1$ com.				alt. alt.
8	16	$\alpha^4 = \beta^4 = 1$ $\alpha\beta = \beta\alpha^3$				con. alt.
9a	mn	$\alpha^m = \beta^n = 1$ com.	α_m	β_n	I	con. con.
9b	mtr	$\alpha^{mr} = \beta^{mt} = 1$ com.	α_{mr}	$\beta_{mt} Y^t X^r = 1$	I	con. con.
10a	$2mn$	$\alpha^{2m} = \beta^n = 1$ $\beta\alpha\beta = \alpha$	$(\alpha^2)_m$	β_n	C_2	con. alt.
10b	$4tr$	$\alpha^{4r} = \beta^{2t} = 1$ $\beta\alpha\beta = \alpha$	$(\alpha^2)_{2r}$	$\beta_{2t} Y^t X^r = 1$	C_2	con. alt.
11a	$4mn$	$\alpha^{2m} = \beta^{2n} = (\alpha\beta)^2 = 1$	$(\alpha^2)_m$	$(\beta^2)_n$	G_4	alt. alt.
11b	$8tr$	$\alpha^{4r} = \beta^{4t} = 1$ $\alpha^2 = \beta^2$	$(\alpha^2)_{2r}$	$(\beta^2)_{2t} Y^t X^r = 1$	G_4	alt. alt.
12a	$2k^2$	$\alpha^{2k} = \beta^{2k} = 1$ $\alpha^2 = \beta^2$	$(\alpha\beta)_k$	$(\beta\alpha)_k$	C_2	con. con.
12b	$4k^2$	$\alpha^{4k} = \beta^{4k} = 1$ $\alpha^2 = \beta^2$	$(\alpha\beta)_{2k}$	$(\beta\alpha)_{2k} Y^k X^k = 1$	C_2	con. con.
13	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ $\beta\alpha\beta = \alpha$				con. alt.
14	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ com.				alt. alt.
15a	$4k$	$\alpha^{2k} = \beta^{2k} = 1$ $\alpha^2 = \beta^2$ com.			(2 columns)	con. con.
15b	$4k$	$\alpha^{2k} = \beta^{2k} = 1$ $\alpha^2 = \beta^2$ com.			(1 column)	con. con.
16	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ $\beta\alpha\beta = \alpha$			(2 columns)	con. alt.
17	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ $\beta\alpha\beta = \alpha$			(1 column)	con. alt.
18	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ com.			(2 columns)	alt. alt.
19	8	$\alpha^4 = \beta^4 = 1$ $\alpha^2 = \beta^2$ com.			(1 column)	alt. alt.
20	16	$\alpha^8 = \beta^8 = (\alpha\beta)^2 = 1$				alt. alt.

Net (6.6.6).

No.	Order.	Im kleinen relations.	X	Y	Arrows.
21a*	$2kl$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma)^2 = 1$	$(\alpha\beta)_k$	$(\beta\gamma)_l$	C_2
21b*	$2mtr$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma)^2 = 1$	$(\alpha\beta)_{mr}$	$(\beta\gamma)_{mt}$ $Y^t X^r = 1$	C_2
22a	$4kl$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma\alpha\gamma\beta) = 1$	$[(\alpha\beta)^2]_k$	$(\beta\gamma)_l$	G_4
22b	$8kl$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma\alpha\gamma\beta) = 1$	$[(\alpha\beta)^2]_{2k}$	$(\beta\gamma)_l$ $X^k = Y^2$	G_4
23	$16k$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma)^2 = 1$	$[(\alpha\beta)^2]_{2k}$	$(\beta\gamma)_2$	G_4
24a	$6k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^3 = (\beta\gamma)^3 = (\alpha\gamma)^3 = 1$	$(\alpha\gamma\alpha\beta)_k$	$(\gamma\beta\gamma\alpha)_k$	G_6
24b	$18k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^3 = (\beta\gamma)^3 = (\alpha\gamma)^3 = 1$	$(\alpha\gamma\alpha\beta)_{3k}$	$(\gamma\beta\gamma\alpha)_{3k}$ $X^k = Y^k$	G_6
25a*	$6k^2$	$\alpha^6 = \beta^2 = (\alpha\beta)^3 = 1$	$(\alpha\beta\alpha^2)_k$	$(\alpha^2\beta\alpha)_k$	C_6 con.
25b*	$6m^2(r_+r^2 + 1)$	$\alpha^6 = \beta^2 = (\alpha\beta)^3 = 1$	$(\alpha\beta\alpha^2)_k$	$(\alpha^2\beta\alpha)_k$ $Y^m X^{mr} = 1$	C_6 con.
			$k = m(r^2 + r + 1)$		
26a	$2kl$	$\alpha^{2k} = \beta^2 = 1$ $\alpha^2\beta = \beta\alpha^2$	$(\alpha^2)_k$	$(\beta\alpha)_l$	C_2 con.
26b	$4tr$	$\alpha^{4r} = \beta^2 = 1$ $\alpha^2\beta = \beta\alpha^2$	$(\alpha^2)_{2r}$	$(\beta\alpha)_{2t}$ $Y^t X^r = 1$	C_2 con.
27a	$4kl$	$\alpha^{2k} = \beta^2 = 1$ $(\alpha^2\beta)^2 = 1$	$(\alpha^2)_k$	$(\beta\alpha\beta\alpha)_l$	G_4 alt.
27b	$8tr$	$\alpha^{4r} = \beta^2 = (\alpha^2\beta)^2 = 1$	$(\alpha^2)_{2r}$	$(\beta\alpha\beta\alpha)_{2t}$ $Y^t X^r = 1$	G_4 alt.
28	16	$\alpha^8 = \beta^2 = \alpha^3\beta\alpha^7\beta = 1$			con.
29	8	$\alpha^4 = \beta^2 = \alpha^3\beta\alpha^3\beta = 1$			con.
30	16	$\alpha^8 = \beta^2 = \alpha^3\beta\alpha\beta = 1$			alt.
31	8	$\alpha^4 = \beta^2 = \alpha^3\beta\alpha\beta = 1$			alt.

Net (8.8.4).

No.	Order.	Im kleinen relations.	X	Y	Arrows.
32a*	$4k^2$	$\alpha^4 = \beta^2 = (\alpha\beta)^4 = 1$	$(\alpha^2\beta)_k$	$(\alpha\beta\alpha)_k$	C_4 con.
32b*	$4m^2(r^2 + 1)$	$\alpha^4 = \beta^2 = (\alpha\beta)^4 = 1$	$(\alpha^2\beta)_{m(r^2+1)}$	$(\alpha\beta\alpha)_{m(r^2+1)}$ $Y^m X^{mr} = 1$	C_4 con.
33a	$8k^2$	$\alpha^4 = \beta^2 = (\alpha\beta\alpha^3\beta)^2 = 1$	$(\alpha^2\beta\alpha^2\beta)_k$	$[(\alpha\beta\alpha)^2]_k$	$G_8{}^4$ alt.
33b	$8m^2(r^2 + 1)$	$\alpha^4 = \beta^2 = (\alpha\beta\alpha^3\beta)^2 = 1$	$(\alpha^2\beta\alpha^2\beta)_{m(r^2+1)}$	$[(\alpha\beta\alpha)]_{m(r^2+1)}$ $Y^m X^{mr} = 1$	$G_8{}^4$
34a	$4k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma\beta)^2 = 1$	$(\alpha\gamma\beta)_k$	$(\gamma\beta\alpha)_k$	C_4
34b	$8k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma\beta)^2 = 1$	$(\alpha\gamma\beta)_{2k}$	$(\gamma\beta\alpha)_{2k}$ $X^k = Y^k$	C_4
35a	$4k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^2 = (\alpha\beta)^4 = (\beta\gamma)^4 = 1$	$(\alpha\gamma\beta)_k$	$(\gamma\beta\alpha)_k$	G_4
35b	$8k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^2 = (\alpha\beta)^4 = (\beta\gamma)^4 = 1$	$(\alpha\gamma\beta)_{2k}$	$(\gamma\beta\alpha)_{2k}$ $X^k = Y^k$	G_4
36	$8k$	$\alpha^{4k} = \beta^2 = 1$ com.	α_{4k}	β_2	I con.
37	$8k$	$\alpha^{4k} = \beta^2 = (\alpha\beta)^2 = 1$	α_{4k}		C_2 alt.

No.	Order.	Im kleinen relations.	X	Y		Arrows.
		Net (12.12.3).				
38a	$6k^2$	$\alpha^3 = \beta^2 = (\alpha\beta)^6 = 1$	$(\alpha^2\beta\alpha\beta)_k$	$(\alpha\beta\alpha^2\beta)_k$	C_6	con.
38b	$6m^2(r^2 - r + 1)$	$\alpha^3 = \beta^2 = (\alpha\beta)^6 = 1$	$(\alpha^2\beta\alpha\beta)_k$	$(\alpha\beta\alpha^2\beta)_k Y^m X^{mr} = 1$, $k = m(r^2 - r + 1)$	C_6	con.
39a	$6k^2$	$\alpha^3 = \beta^2 = (\alpha\beta\alpha^2\beta)^3 = 1$	$(\alpha^2\beta\alpha^2\beta)_k$	$(\alpha\beta\alpha\beta)_k$	G_6	alt.
39b	$6m^2(r^2 - r + 1)$	$\alpha^3 = \beta^2 = (\alpha\beta\alpha^2\beta)^3 = 1$	$(\alpha^2\beta\alpha^2\beta)_k$	$(\alpha\beta\alpha\beta)_k Y^m X^{mr} = 1$, $k = m(r^2 - r + 1)$	G_6	alt.
		Net (6.3.6.3).				
40a	$3k^2$	$\alpha^3 = \beta^3 = (\alpha\beta)^3 = 1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k$	C_3	con. con.
40b	$3m^2(r^2 + r + 1)$	$\alpha^3 = \beta^3 = (\alpha\beta)^3 = 1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k Y^m X^{mr} = 1$, $k = m(r^2 + r + 1)$	C_3	con. con.
41	24	$\alpha^3 = \beta^3 = \alpha\beta\alpha^2\beta\alpha\beta^2 = 1$				alt. alt.
		Net (6.4.3.4).				
42a	$6k^2$	$\alpha^6 = \beta^3 = (\alpha\beta)^2 = 1$	$(\alpha^2\beta^2)_k$	$(\alpha^4\beta)_k$	C_6	con. con.
42b	$6m^2(r^2 - r + 1)$	$\alpha^6 = \beta^3 = (\alpha\beta)^2 = 1$	$(\alpha^2\beta^2)_k$	$(\alpha^4\beta)_k Y^m X^{mr} = 1$, $k = m(r^2 - r + 1)$	C_6	con. con.
		Net (12.6.4).				
43a	$12k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^6 = (\alpha\beta)^3 = (\beta\gamma)^2 = 1$	$[(\alpha\beta\gamma)^2]_k$	$[(\beta\alpha\gamma)^2]_k$	G_{12}^5	
43b	$36k^2$	$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^6 = (\alpha\beta)^3 = (\beta\gamma)^2 = 1$	$[(\alpha\beta\gamma)^2]_{3k}$	$[(\beta\alpha\gamma)^2]_{3k} Y^k X^k = 1$	G_{12}^5	

Groups of genus one.

No.	Order.	Im kleinen relations.	X	Y		Arrows.
Ia	$2kl$	$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \alpha\beta\gamma\delta = (\alpha\beta\gamma)^2 = 1$	$(\alpha\delta)_k$	$(\alpha\beta)_l$	C_2	
Ib	$2mtr$	$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \alpha\beta\gamma\delta = (\alpha\beta\gamma)^2 = 1$	$(\alpha\delta)_{mr}$	$(\alpha\beta)_{mt} Y^t X^r = 1$	C_2	
IIa	$3k^2$	$\alpha^3 = \beta^3 = \gamma^2 = \alpha\beta\gamma = (\alpha\beta)^2 = 1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k$	C_3	con. con.
IIb	$3m^2(r^2 - r + 1)$	$\alpha^3 = \beta^3 = \gamma^2 = \alpha\beta\gamma = (\alpha\beta)^2 = 1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k Y^m X^{mr} = 1$, $k = m(r^2 - r + 1)$	C_3	con. con.
IIIa	$4k^2$	$\alpha^2 = \beta^4 = \gamma^4 = \alpha\beta\gamma = (\alpha\beta)^4 = 1$	$(\beta^3\gamma)_k$	$(\beta\gamma^3)_k$	C_4	con. con.
IIIb	$4m^2(r^2 + 1)$	$\alpha^2 = \beta^4 = \gamma^4 = \alpha\beta\gamma = (\alpha\beta)^4 = 1$	$(\beta^3\gamma)_k$	$(\beta\gamma^3)_k Y^m X^{mr} = 1$, $k = m(r^2 + 1)$	C_4	con. con.
IVa	$6k^2$	$\alpha^2 = \beta^3 = \gamma^6 = \alpha\beta\gamma = (\beta\gamma)^2 = 1$	$(\gamma^2\beta^2)_k$	$(\gamma^4\beta)_k$	C_6	con. con.
IVb	$6m^2(r^2 + r + 1)$	$\alpha^2 = \beta^3 = \gamma^6 = \alpha\beta\gamma = (\beta\gamma)^2 = 1$	$(\gamma^2\beta^2)_k$	$(\gamma^4\beta)_k Y^m X^{mr} = 1$, $k = m(r^2 + r + 1)$	C_6	con. con.

Numbers 15, 16, 29, and 31 also occur with a non-oriented figure.

11. There are of course many isomorphisms of the groups though not of the representations. The starred numbers have been mentioned. The possession of the same order, the same self-conjugate abelian subgroup and the same quotient group is not sufficient for isomorphism. For example (2b) and (11b) agree in these respects but if all the integers are equal to two the groups of order 32 are distinct, (11b) has operators of order 8 while (2b) has not.

12. I add a list of groups of low order and the classes in which they occur.

Groups of order 8.

(2)	$C_2 . C_4$	6, 10, 11, 12, 14, 15, 19, 26, 27, 31, 32, 34, 36.
(3)	$C_2 . C_2 . C_2$	2, 3, 21, 22, 34.
(4)	$G_8{}^4$	11, 22, 29, 33, 37.
(5)	Q_8	13, 16, 17.

Groups of order 10.

(1)	$C_2 . C_5$	9.
(2)	$G_{10}{}^5$	10.

Groups of order 12.

(1)	C_{12}	9.
(2)	$C_2 . C_6$	9.
(3)	$G_{12}{}^7 3$	10, 32.
(4)	$G_{12}{}^4$	40.
(5)	$G_{12}{}^5$	2, 3, 21, 27, 43.

Groups of order 14.

(1)	C_{14}	9.
(2)	$G_{14}{}^7$	10.

Groups of order 16.

(2)	$C_8 . C_2$	9, 15, 36.
(3)	$C_4 . C_4$	7, 9.
(4)	$C_4 . C_2 . C_2$	2, 5′.
(5)	$C_2 . C_2 . C_2 . C_2$	1.
([illegible])	[illegible] III [illegible]	10, 12, 15, 20, 30
(7)	" III vii	2, 4.
(8)	" III viii	8, 10.

(9)	Burnside III ix	2, 3, 5, 21, 22, 23, 34, 35.
(10)	" III x	6, 11, 26, 32, 33.
(11)	" VI	absent.
(12)	" VII	11, 21, 37.
(13)	" VIII	11, 27, 28.
(14)	" IX	11.

Groups of order 18.

(1)	C_{18}	9.
(2)	$C_6 . C_3$	9, 12, 42.
(3)	$G_{18}{}^6 1$	2, 3, 10, 25, 26, 38, 39.
(4)	$G_{18}{}^6 2$	21, 24.
(5)	$\alpha^9 = \beta^2 = 1 \quad \alpha\beta = \beta\alpha^8$	absent.

Groups of order 20.

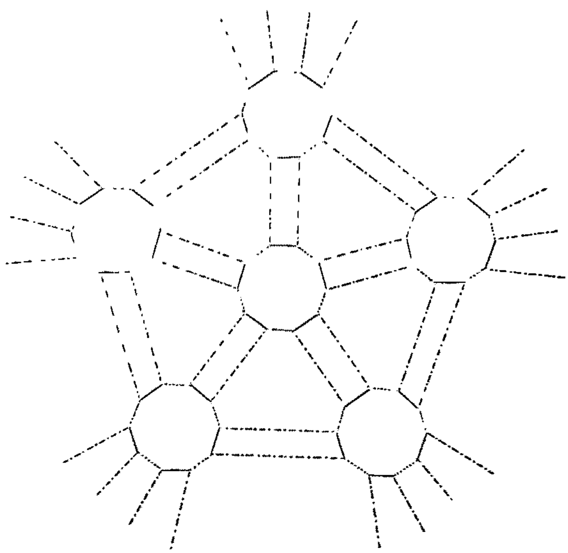
(1)	C_{20}	9.
(2)	$C_{10} . C_2$	2, 9, 10, 15, 26.
(3)	$G_{20}{}^5$	6, 32.
(4)	$G_{20}{}^7$	2, 11, 21, 26, 27.
(5)	$\alpha^5 = \beta^4 = 1 \quad \alpha\beta = \beta\alpha^4$	10.

Groups of order 24.

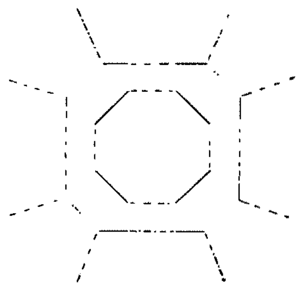
(1)	C_{24}	9.
(2)	$C_{12} . C_2$	9, 10, 15, 22, 36.
(3)	$C_3 . C_2 . C_2 . C_2$	2.
(4)	$G_8{}^4 . C_3$	10, 26.
(5)	$Q_8 . C_3$	2, 10, 26.
(6)	Burnside I	10.
(7)	" II_1	2, 26.
(8)	" II_2	10.
(9)	" III_1	25, 42.
(10)	" III_2	1, 2, 3, 9, 21, 22.
(11)	" IV_1	41.
(12)	" IV_2	absent.
(13)	" V_1	3, 11, 21, 26, 27, 37.
(14)	" V_2	11, 22, 27.
(15)	" $\mathrm{V}_3 = G_{24}{}^4$	24, 39.

The plates give a selection of the drawings of the groups in question. Plate I has the three groups of the projective plane. Plates II, III, IV, V contain groups on the anchor ring and at least one example for every net.

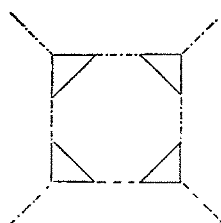
Plate I.



G_{60}^{5}

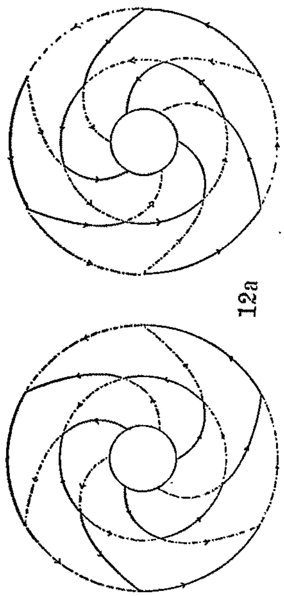


G_{24}^{4}

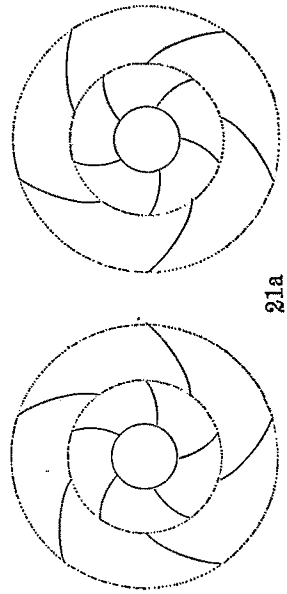


G_{12}^{4}

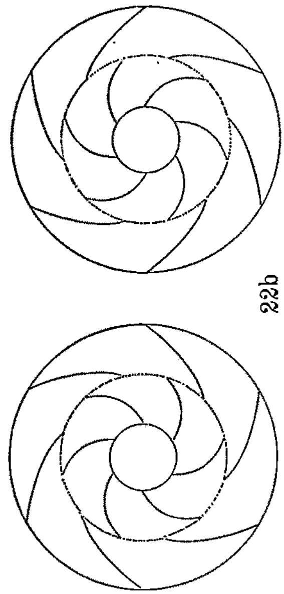
PLATE II.



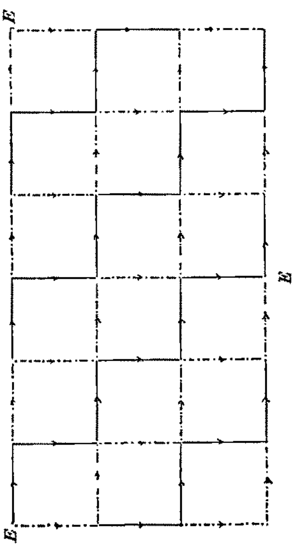
12a



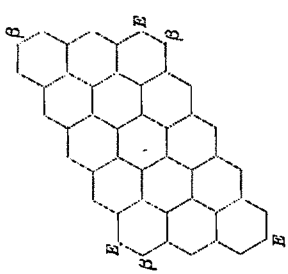
21a



22b

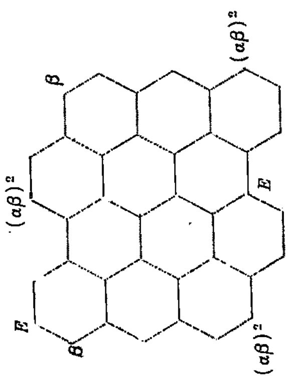


12a $2k^2$, $k=3$ $\alpha^{2k}=\beta^{2k}=1$ $\alpha^2=\beta^2$

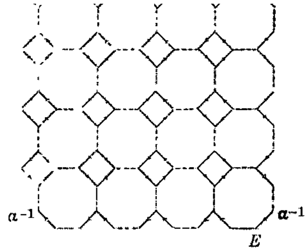


2kl $k=5$ $l=3$ $\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma)^2=E$

21a $(\alpha\beta)_k$, $(\beta\gamma)_l$, G_{30}

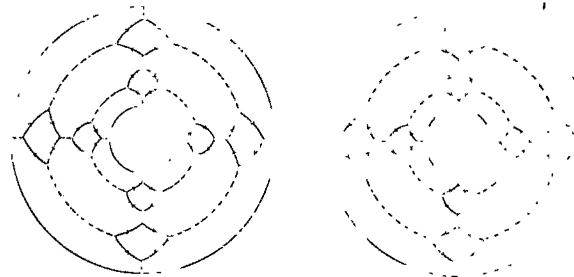


8kl, $k=1$, $l=3$ $\alpha^2=\beta^2=\gamma^2=\alpha\beta\gamma\alpha\gamma\beta=E$

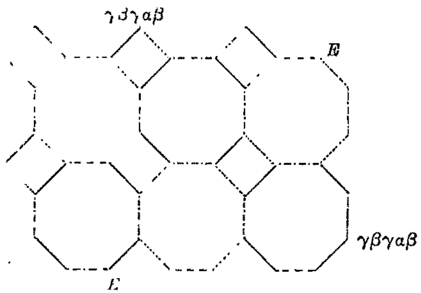


$4k^2 \quad k = 4$

$(\alpha\beta)^4 = E \quad (\alpha^2\beta)_k, \ (\alpha\beta\alpha)_k \quad G\ 64$



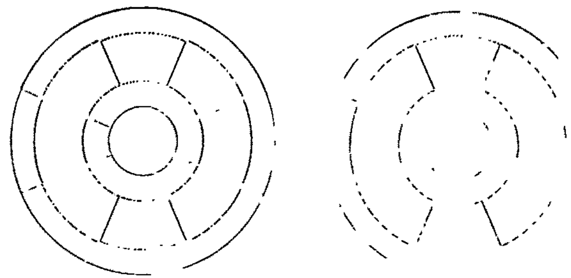
32a



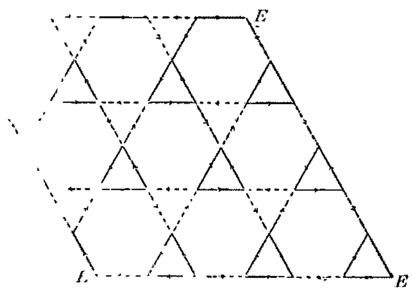
$8k^2, \quad k = 2$

$\gamma^2 = (\alpha\gamma)^2 = (\alpha\beta)^4 = (\beta\gamma)^4 = E$

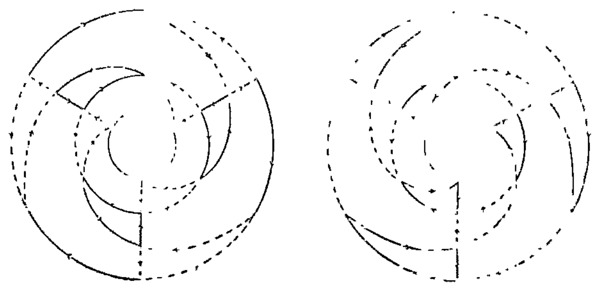
$)_{2k}, \ (\gamma\beta\alpha)_{2k}, \ (\gamma\beta\alpha)^k = (\alpha\gamma\beta)^k \quad G\ 32$



35b



$\beta^3 = (\alpha\beta)^3 = E \quad (\alpha\beta^2)_k, \ (\beta^2\alpha)_k \qquad G\ 27$



10..

PLATE IV.

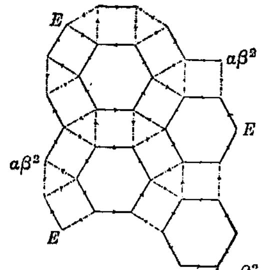
$6k^2 \quad k = 2$

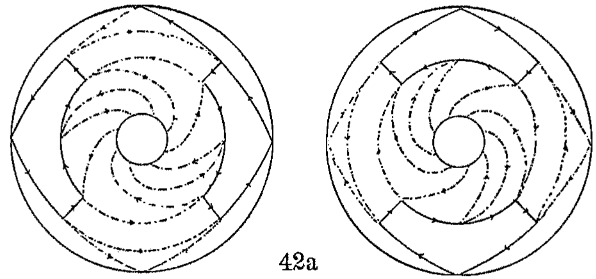
$\alpha^6 = \beta^3 = (\alpha\beta)^2 = E$

$(\alpha^2\beta^2)_k, \ (\alpha^4\beta)_k$

$G\ 24.6.1$

42a





42a

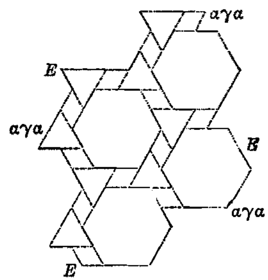
$12k^2 \quad k = 2$

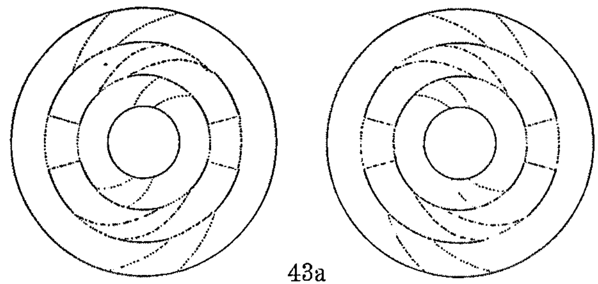
$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^6 = (\alpha\beta)^3 = (\beta\gamma)^2$

$[(\alpha\beta\gamma)^2]_k, \ [(\beta\alpha\gamma)^2]_k$

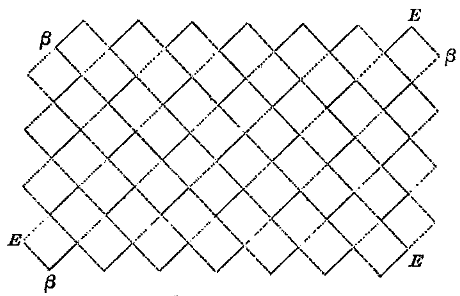
$G\ 48$

43a





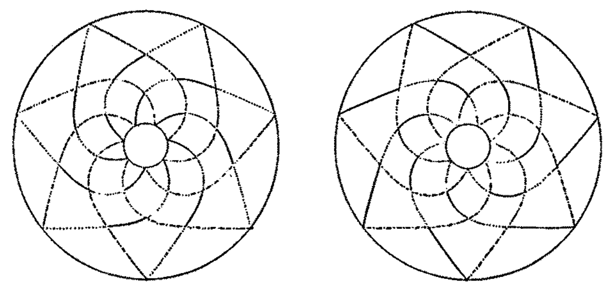
43a



$p = 1 \quad 2mn \quad m = 7 \quad n = 4$

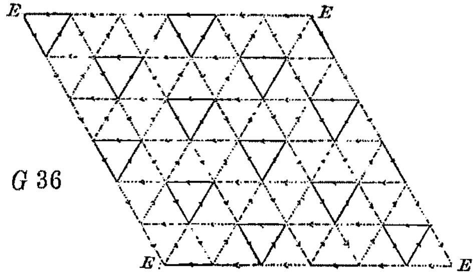
$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \alpha\beta\gamma\delta = E$

$(\alpha\beta)_m, \ (\beta\gamma)_n \qquad G\ 56$



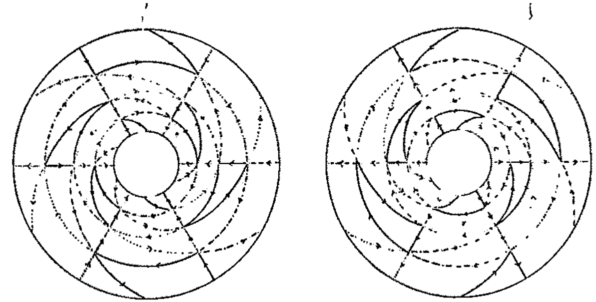
Ia

PLATE V.

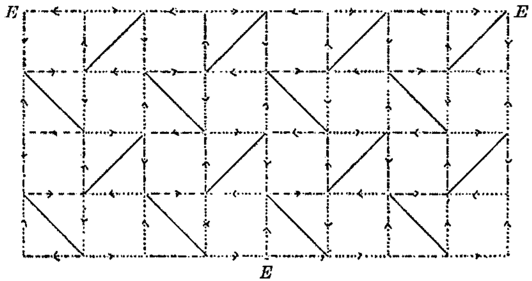


$= 1 \quad 9m^2 \quad m = 2$
$\alpha^2 = \beta^3 = \gamma^6 = \alpha\beta\gamma = E$
$(\alpha\beta^2)_{3m} \quad (\beta^2\alpha)_{3m}$
$(\alpha\beta^2)^m = (\beta^2\alpha)^m$
G 36

IIb

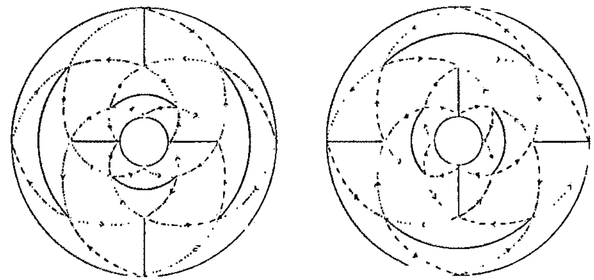


IIb

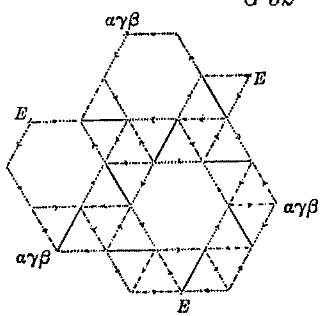


$p = 1 \quad 8m^2 \quad m = 2 \quad \alpha^2 = \beta^4 = \gamma^4 = \alpha\beta\gamma = E$
$(\beta^3\gamma)_{2m}, \ (\beta\gamma^3)_{2m} \quad (\beta^3\gamma)^m . (\beta\gamma^3)^m = E$
G 32

IIIb

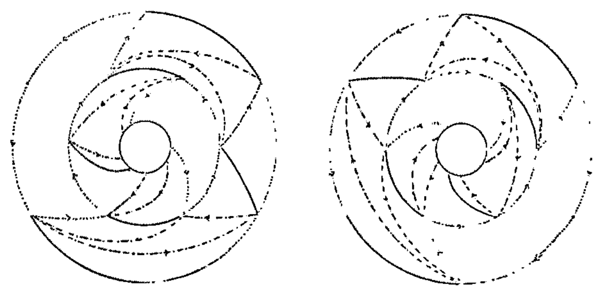


IIIb



$18m^2 \quad m = 1$
$\alpha^2 = \beta^3 = \gamma^6 = \alpha\beta\gamma = E$
$(\gamma^2\beta^2)_{3m}, \ (\gamma^4\beta)_{3m}$
$(\gamma^2\beta^2)^m . (\gamma^4\beta)^m = E$
G 18

IVb



IVb

CONCERNING CONTINUOUS IMAGES OF THE INTERVAL.

BY G. T. WHYBURN.

1. This paper is concerned with the following celebrated theorem of Hahn-Mazurkiewics: *

(A). *Every compact, metric and locally connected continuum is the image under a continuous transformation of the unit interval.*

At the Bologna congress † in 1928, Hahn gave a new proof for this theorem which (to quote him exactly), "nicht nur die ursprünglichen Beweise von Herrn Mazurkiewicz und mir, sondern auch die zeither von verscheidenen Seiten mitgeteilten Beweise an Einfachheit und Durchsichtigkeit übertrifft, indem er die Behauptung als unmittelbar Folge bekannter, mit elementaren Mitteln beweisbar Sätze aufweist. Es sind dies die folgenden Sätze:" . . . (There follows the statements of three theorems, of which the second is here quoted.)

II. *Every two points a and b of a self-compact, connected and locally connected set M can be joined by a subset M′ of M which is the continuous image of an interval.*

With the aid of the three theorems quoted, Hahn proceeds to prove (A) with very little difficulty.

The author has found that by a slight modification of Hahn's proof ‡ for II alone, one can obtain a complete proof for (A). This modification adds little if any to the complexity and very little to the length of the proof for II; and it will be noted that, aside from the separability of the space and the lemma given below in § 2, use is made of no concept or property which was not used by Hahn in proving II. We therfore obtain a proof for (A) which is in every way as simple and elementary as the proof required for II alone and which *completely avoids* the other two theorems used by Hahn in his simplified proof. Thus our proof, which will be given in detail in § 3 below, seems to be

* See Hahn, *Wiener Berichte*, Vol. 123 (1914), p. 2433; Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 1 (1920), p. 166, and note reference there given to an earlier paper by Mazurkiewicz.

† See the proceedings of this congress, Vol. 2, p. 217.

‡ See Hahn, *Wiener Berichte*, *loc. cit.*, p. 2436.

more direct and elementary than any which has yet been given for the very fundamental theorem (A).

2. *Definition.* If ϵ is any positive number, then by an ϵ-chain of points joining two given points a and b is meant a finite sequence of points $a = X_0$, $X_1, X_2, \ldots, X_n = b$ such that the distance between any two successive points in this sequence is $< \epsilon$.

It is well known that if a set is connected, it contains, for every $\epsilon > 0$, an ϵ-chain of points joining any two of its points. We shall make use of the following immediate consequence of this fact.

LEMMA. *If F is any finite subset of a connected set M and a and b are any two points of F, then for every $\epsilon > 0$, M contains an ϵ-chain of points joining a and b and containing all points of F.*

For let the points of F be $a = p_1, p_2, \cdots p_n = b$. Then to obtain an ϵ-chain of points of M from a to b we first take such a chain from $a = p_1$ to p_2, then such a chain from p_2 to p_3, then one from p_3 to p_4, and so on until we reach $p_n = b$. Clearly the sequence of points obtained in this order is an ϵ-chain of points from a to b containing all points of F.

3. *The Proof.* Let $P = \sum_1^\infty p_i$ be a countable set of points which is dense in M and, for each integer n, let $P_n = \sum_1^n p_i$. Let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a sequence of positive numbers such that $\sum_1^\infty \epsilon_i$ converges. Since M is uniformly locally connected,* there exists, for each of these numbers ϵ_k, a positive δ_k such that every two points x and y of M whose distance apart is $< \delta_k$ lie together in a connected subset of M of diameter $< \epsilon_k/2$.

Now let a and b be any two points of M. There exists an integer n_1 such that every point of M is at a distance $< \delta_1$ from some point of P_{n_1}. Since M is connected, it follows by the lemma that there exists in M a δ_1-chain C^1 of points joining a and b and containing P_{n_1} and which we may suppose consists of exactly $2^{v_1} + 1$ points

$$(1) \qquad a = X_0^1, X_1^1, \cdots X^1_{2^{v_1}} = b.$$

* See Hahn, *loc. cit.*, p. 2435. This property is proved as follows: If on the contrary, for some $\epsilon > 0$, there exists no such δ, it follows that there exist two infinite sequences of points [illegible] and $y_1, y_2, \cdots$ [illegible] same point p of M and such that for no n can x_n and y_n be joined by any connected subset of M of diameter $< \epsilon$. But clearly this contradicts the fact that M is locally connected at the point p.

There exists an integer $n_2 \geq n_1$ such that every point of M is at a distance $< \delta_2$ from some point of P_{n_2}. For each integer i, $0 \leq i \leq 2^{v_1}$, let F_i^1 be the set of all those points of P_{n_2} whose distances from X_i^1 are $< \delta_1$. Since the chain (1) contains P_{n_1}, it follows that $P_{n_2} = \sum_0^{2^{v_1}} F_i^1$. Now for each i, M contains a connected set M_i^1 of diameter $< \epsilon_1$ which contains $X_i^1 + F_i^1 + X^1_{i+1}$, because each point of $F_i^1 + X^1_{i+1}$ is at a distance $< \delta_1$ from the point X_i^1. Hence by the lemma, M_i^1 contains a δ_2-chain of points C_i^1 joining X_i^1 and X^1_{i+1} and containing all points of F_i^1. We may suppose that all of these chains C_i^1 contain the same number, say $2^{v_2} + 1$, of points which we shall denote by

$$X_i^1 = X^2_{i \cdot 2^{v_2}}, X^2_{i \cdot 2^{v_2}+1}, \cdots, X^2_{(i+1)2^{v_2}} = X^1_{i+1}.$$

Clearly the chains $[C_i^1]$ taken in the order $C_1^1, C_2^1, \cdots, C^1_{2^{v_1}}$ form a δ_2-chain C^2 from a to b which contains all points for P_{n_2}.

Let us continue in this way. In general, for each k, we choose an integer $n_k \geq n_{k-1}$ such that every point of M is at a distance $< \delta_k$ from some point of P_{n_k}. For each i, $0 \leq i \leq 2^{v_1+v_2+\cdots+v_{k-1}}$, let F_i^{k-1} be the set of all points of P_{n_k} whose distances from the point X_i^{k-1} are $< \delta_{k-1}$. Then M contains a connected set M_i^{k-1} of diameter $< \epsilon_{k-1}$ which contains $X_i^{k-1} + F_i^{k-1} + X^{k-1}_{i+1}$; and by the lemma, M_i^{k-1} contains a δ_k-chain of points C_i^{k-1} joining X_i^{k-1} and X^{k-1}_{i+1} and containing all points of F_i^{k-1}. We may suppose that all of these chains $[C_i^{k-1}]$ contain the same number, say $2^{v_k} + 1$, of points. Clearly then, these chains taken in the order $C_1^{k-1}, C_2^{k-1}, \cdots C_{2^{v_1+v_2+\cdots+v_{k-1}}}$ form a δ_k-chain C^k of points from a to b which contains P_{n_k} and consists of exactly $2^{v_1+v_2+\cdots+v_k} + 1$ points:

$$a = X_0^k, X_1^k, \cdots, X^k_{2^{v_1+v_2+\cdots+v_k}} = b,$$

in which notation we have always:

$$X^k_{i \cdot 2^{v_k}} = X_i^{k-1}. \tag{2}$$

Now to the values

$$t = i/2^{v_1+v_2+\cdots+v_k} \qquad (i = 0, 1, \cdots, 2^{v_1+v_2+\cdots+v_k})$$

of the parameter t we make correspond the points X_i^k, respectively, and set $T(t) = X_i^k$, so that we thus define our transformation T for the set D all values of $t (0 \leq t \leq 1)$ which are dyadically representable, i. e., which can be written as factions having powers of 2 for denominators. It is a consequence of (2) that T is single valued on D.

We shall now show that the transformation T thus defined on D is uniformly continuous on D. Let ϵ be any positive number and let us choose k so large that

$$(3) \qquad \sum_{k}^{\infty} \epsilon_i < \epsilon/2$$

and set

$$(4) \qquad \delta = 1/2^v, \text{ where } v = v_1 + v_2 + \cdots + v_k.$$

Then if t_1 and t_2 are any two values of t in D such that $|t_1 - t_2| < \delta$, they must lie between three successive values,

$$(j-1)/2^v, \quad j/2^v, \quad (j+1)/2^v$$

of t in D. Let us suppose t_1 lies between the last two of these values. We can write $t_1 = m/2^{v+u}$, where $u = v_{k+1} + \cdots + v_{k+w}$. Thus we have $T(t_1) = X_m^{k+w}$. Now the point X_m^{k+w} was arrived at in the following manner. We defined the chain C_j^k of points joining the two points X_j^k and $X^k{}_{j+1}$ and lying wholly in the ϵ_k-neighborhood of the point X_j^k; then joining some two points of C_j^k there was set up the chain $C_{j_1}^{k+1}$ so that it lay wholly in the ϵ_{k+1}-neighborhood of each of these points and hence lay in the $\epsilon_k + \epsilon_{k+1}$-neighborhood of X_j^k. Between some two points of $C_{j_1}^{k+1}$ there was set up the chain of points $C_{j_2}^{k+2}$ lying wholly in the ϵ_{k+2}-neighborhood of each of these points and hence in the $\epsilon_k + \epsilon_{k+1} + \epsilon_{k+2}$-neighborhood of X_j^k, and so on. After w steps of this sort we reach the point X_m^{k+w}, which therefore lies in the $\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_{k+w}$-neighborhood of X_j^k. By virtue of (3), we have

$$\rho[T(t_1), X_j^k] = \rho[X_m^{k+w}, X_j^k] < \epsilon/2.^*$$

In exactly the same manner we prove that

$$\rho[T(t_2), X_j^k] < \epsilon/2.$$

These two relations give at once that $\rho[T(t_1), T(t_2)] < \epsilon$, which proves that T is uniformly continuous on D.

Now by a well known theorem, it is possible † to extend the definition of

* We employ the usual notation $\rho(x, y)$ for the distance between the points x and y.

† This is done as follows. Let x be any limit point of D and let $x_1, x_2, x_3, \cdots$ be any sequence of points of D converging to x. By virtue of the uniform continuity of T on D it follows that the image points $x'_1, x'_2, x'_3, \cdots$ under T of the points $x_1, x_2, x_3, \cdots$, respectively, converge to some point x' of M. We then set $T(x) = x'$. Now if $y_1, y_2, y_3, \cdots$ is any sequence of points of D converging to x, it follows that [illegible] $y'_1, y'_2, y'_3, \cdots$ also converges to the point x'. Thus the extended transformation is single valued; and since we get the same conclusion if $y_1, y_2, y_3, \cdots$ is any sequence of points of $\overline{D}$, it follows that T is also continuous.

the transformation T to the limit points of D and thus to the entire interval $(0, 1) = I$ in such a way that the extended transformation T is single valued and continuous and has the same values on D as before. Now since $T(D) = \sum_1^\infty C^n$, and for each n, $C^n \supset P_{n_n} = \sum_1^{n_n} p_i$, it follows that $T(D) \supset \sum_1^\infty p_i = P$; and since P is dense in M, it follows that $T(I) = M$. Thus M is the image under the extended transformation T of I, and the proof is complete.

4. *Conclusion.* The author wishes to emphasize the small amount he has had to contribute in order to obtain the proof of (A) from the proof of II. Indeed, the essential change consists in altering the choice of a certain sequence of points in the proof of II in such a way as to insure that the subset M' of M which joins a and b and is the continuous image of the interval shall be identical with M. Thus our proof is really only a translation of Hahn's proof for II slightly modified in places so as to attain this more advantageous state of affairs and thus establish (A).

THE JOHNS HOPKINS UNIVERSITY.

ON QUASI-METRIC SPACES.*

By W. A. Wilson.

1. Let Z be a class of elements such that for each pair x and y there are two non-negative numbers called the distances from x to y and from y to x. These are designated by xy and yx and satisfy these axioms:

I. $xy = 0$ if and only if $x = y$.

II. $xz \leqq xy + yz$.

The same relations hold for yx, zy, and zx. Such a space will be called quasi-metric.†

If $xy = yx$, the common value is denoted by $\overline{xy}$; if this is true for every pair, Z is metric. Thus in one sense a quasi-metric space is merely the result of suppressing the axiom that $xy = yx$ from the definition of metric space. Usually the result of such a limitation on a set of axioms is to diminish the number of theorems easily deducible, but in this case there is an embarrassing richness of material. The first half of this paper (§§ 1-5) contains some of the properties of quasi-metric spaces, and relations between quasi-metric, metric, and topological spaces are discussed in the latter half (§§ 6-8).

As an example of a quasi-metric space consider any metric space M decomposed into mutually disjoint bounded closed sets. Let Z be the aggregate whose elements are these closed sets and for any pair let xy be the lower bound of the numbers $\{r\}$ such that every point of y has a (metric) distance from the set x less than r. Then Z is quasi-metric.

2. *Limiting points.* Let $A = \{x\}$ be a set in Z and a a point such that for every $r > 0$ there is at least one point x of A distinct from a which satisfies the relation $ax < r$. Then a is called a *u-limiting point* of A. If for at least one point x of $A - a$, $xa < r$, a is called an *l-limiting point* of A. If for at least one point x of $A - a$, both $ax < r$ and $xa < r$, a is called a *c-limiting point* of A. It is sometimes convenient to call u-limiting and l-limiting points collectively *quasi-limiting* points. In the example of § 1 let $A = \{x\}$ be a set of elements of Z and a some other element of Z; also

* Presented to the Society, September 9, 1930.

† Asymmetric definitions of distance have been used by various authors before (e.g., F. Hausdorff, *Mengenlehre*, pp. 145-146), but, as far as the writer knows, the properties of quasi-metric spaces have never been completely worked out.

let M be compact. If in M the set a contains the upper closed limiting set (in the Hausdorff sense) of a sequence $\{x_i\}$ chosen from A, then in Z the *point* a is a u-limiting point of A. If a is contained in the lower closed limiting set of $\{x_i\}$, a is an l-limiting point of A in Z. If a is the closed limiting set of $\{x_i\}$, a is a c-limiting point of A in Z.

It follows at once that for a to be a u-limiting, l-limiting, or c-limiting point of $A = \{x\}$, it is necessary and sufficient that for each $r > 0$, there is an infinity of points $\{x\}$ of A such that $ax < r$, $xa < r$, or $ax < r$ and $xa < r$, respectively.

A sequence $\{x_i\}$ is said to have the point a as a *u-limit, l-limit,* or *c-limit,* if for each $r > 0$ there is an i' such that for every $i > i'$, $ax_i < r$, $x_ia < r$, or $ax_i < r$ and $x_ia < r$, respectively. If from some i on the points $\{x_i\}$ are distinct, any limit of the sequence $\{x_i\}$ is of course the corresponding kind of a limiting point of the set $\{x_i\}$.

Simple examples show that a sequence may have any number of quasi-limits of either kind and that the Cauchy convergence criterion is not necessary for the existence of semi-limits. To see this, let $Z = A + B + C$, where A is the set of positive real numbers, B the set of negative real numbers, and C is any set whatever containing no real numbers. If $x + y \subset A$ or $x + y \subset B$, let $xy = yx = |x - y|$; if $x + y \subset C$, let $xy = yx = 1$; if x lies in A and y in B, let $xy = 1 + |x - y|$ and $yx = |x - y|$; if x lies in C and y in A, let $xy = |y|$ and $yx = 1 + |y|$; if x lies in C and y in B, let $xy = 1 + |y|$ and $yx = |y|$. Then Z is quasi-metric and, if $\{x_i\}$ is a sequence chosen from $A + B$ such that $|x_i| \to 0$, every point of C is a u-limit of $\{x_i\}$ when $\{x_i\} \subset A$ and an l-limit when $\{x_i\} \subset B$. Upper semi-continuous decompositions of compact continua usually furnish examples of the second phenomenon when the distances are defined as in § 1. However, we have the following theorems.

THEOREM I. *If a point a is both a u-limit and an l-limit of the sequence $\{x_i\}$, it is the only limit of any kind and the Cauchy criterion is satisfied.*

Proof. By hypothesis, for any $r > 0$, there is an i' such that $ax_i < r/2$ and $x_ia < r/2$ for every $i > i'$. If also $j > i'$, $ax_j < r/2$; hence $x_ix_j \leqq x_ia + ax_j < r$ for i and j both greater than i' and the Cauchy criterion is satisfied. If b is any u-limit of the sequence, there is an i'' such that $bx_i < r/2$ for $i > i''$. Taking i greater than both i' and i'', we have $ba \leqq bx_i + x_ia < r$, whence $b = a$. Thus the sequence has only one u-limit and in like manner a is the only l-limit.

COROLLARY. *No sequence has more than one c-limit.*

Note that in general a point a may be both a u-limiting and an l-limiting point of a set A, although not a c-limiting point.

THEOREM II. *If a is a u-limit or an l-limit of the sequence $\{x_i\}$, but is not the c-limit of the sequence, then $\{x_i\}$ has no l-limit or u-limit, respectively.*

Proof. Let a be a u-limit of $\{x_i\}$. If there were an l-limit b, we would have for any $r > 0$ an i', such that $ax_i < r/2$ and $x_i b < r/2$ for every $i > i'$. But then $ab < r$, or $b = a$.

COROLLARY. *If a sequence $\{x_i\}$ has more than one u-limit or l-limit, it has no l-limit or u-limit, respectively.*

THEOREM III. *Let a and c be fixed points and let x be a variable point. If $cx \to 0$, $\limsup ax \leqq ac$ and $\liminf xa \geqq ca$. If $xc \to 0$, $\liminf ax \geqq ac$ and $\limsup xa \leqq ca$. If both $cx \to 0$ and $xc \to 0$, then $ax \to ac$ and $xa \to ca$.*

The proofs of these statements come directly from Axiom II. The theorem shows that in general the distance functions are not continuous at semi-limiting points, but are semi-continuous. Both, however, are continuous at c-limiting points.

3. *Closed sets and regions.* A set is called *u-closed, l-closed,* or *c-closed* if it contains all of its u-limiting, l-limiting, or c-limiting points, respectively.

A *u-sphere, l-sphere,* or *c-sphere* of center a and radius r is the set of points $\{x\}$ such that $ax < r$, $xa < r$, or $ax < r$ and $xa < r$, respectively. These are denoted by $U_r(a)$, $L_r(a)$, and $S_r(a)$, respectively. Note that $S_r(a) = U_r(a) \cdot L_r(a)$.

A set A such that for each of its points $\{a\}$ some $U_r(a)$, $L_r(a)$, or $S_r(a)$ is wholly contained in A is called a *u-region, l-region,* or *c-region,* respectively.

For each type of closed set and region we have at once the usual theorems regarding divisors, unions, and complements. In particular, if A', A'', or $\bar{A}$ denotes the union of the set A and its u-limiting, l-limiting, or c-limiting points, respectively, A' is u-closed, A'' is l-closed, and $\bar{A}$ is c-closed. In addition there are various other properties due to the relations between the three types of limiting points.

THEOREM I. *Every quasi-closed set or quasi-region is a c-closed set or* [illegible]

Proof. To fix the ideas let A be u-closed. If a is a c-limiting point of A, it is a u-limiting point by the definitions and so belongs to A. Hence A is

c-closed. The converse is not true, because a u-limiting point is not necessarily a c-limiting point. The statement regarding regions follows from the complementary relations between regions and closed sets.

Consequently the quasi-closed sets are special types of c-closed sets. Constant vigilance is necessary to avoid confusing this with the fact that c-limiting points are special types of quasi-limiting points.

The u-spheres and l-spheres have certain features not analogous to those of metric spheres. Although a u-sphere or l-sphere of center a and radius r is a u-region or l-region, respectively, the set of points for which $ax > r$ or $xa > r$, respectively, is an l-region or u-region, respectively. The set for which $ax = r$ or $xa = r$ is the divisor of a u-closed and an l-closed set, and so merely c-closed. Finally, if $U = U_r(a)$ and $L = L_r(a)$, U' may contain points for which $ax > r$ and L'' points for which $xa > r$; in fact, a point b may be a u-limiting point of $U_r(a)$ for every r. Among the various relations between semi-closed sets and semi-regions suggested by these facts, the following generalization of the well-known separation theorem for metric spaces may be of interest.

THEOREM II. *Let A and B lie in a semi-metric space and $A' \cdot B + A \cdot B'' = 0$. Then there is an l-region R and a u-region S such that $A \subseteq R$, $B \subseteq S$, $R' \cdot S + R \cdot S'' = 0$, and $A' \cdot B'' = R' \cdot S''$.*

Proof. Since $A \cdot B'' = 0$, there is for each point x of A an $r > 0$ and less than one-third the lower bound of yx as y ranges over B''. Enclose x in an l-sphere L_x of center x and radius r. If R denotes the union of these spheres, it is an l-region and contains A.

Now $B'' \cdot R = 0$ by construction and, for each x, $B'' \cdot L_x' = 0$. For, if y lies in B'', $yx > 3r$, while if y lies in L_x', $yx \leqq r$. Hence if B contains a point y of R', y is a u-limit of a sequence $\{x_i\}$ of points, each in an l-sphere L_i of center a_i chosen from the set forming R and no two in the same L_i. Now $ya_i \leqq yx_i + x_ia_i$. If the radii $\{r_i\}$ have a lower bound, then $ya_i < 2r_i$ for i large enough, since $yx_i \to 0$. This is false, as $ya_i > 3r_i$. Hence for a partial sequence $r_i \to 0$ and so $ya_i \to 0$, which is another contradiction, as $A' \cdot B = 0$. Thus $R' \cdot B + R \cdot B'' = 0$.

Now enclose each point y of B in a u-sphere U_y of center y and radius r' less than one-third the lower bound of yx as x ranges over R'. If S is the union of these spheres, it is a u-region containing B and, as above, $R' \cdot S + R \cdot S'' = 0$.

Obviously $A' \cdot B'' \subseteq R' \cdot S''$. Let z lie in $R' \cdot S''$. Then z is not in R, but is a u-limit of a sequence $\{x_i\}$ of points of R, each lying in one of the l-spheres

forming R, say L_i with center a_i and radius r_i. Likewise z is an l-limit of a sequence of points $\{y_i\}$ of S, each lying in one of the u-spheres forming S, say U_i with center b_i and radius r_i'. As $zx_i \to 0$ and $y_iz \to 0$, $y_ix_i \to 0$. As in the earlier part of the proof there is a sub-sequence for which $r_i \to 0$ and $r_i' \to 0$. Then, since $za_i \leqq zx_i + x_ia_i \leqq zx_i + r_i$, $za_i \to 0$ and so z lies in A'. Likewise z lies in B''. Hence $R' \cdot S'' = A' \cdot B''$, which completes the proof.

COROLLARY. *If in the above A is u-closed, B is l-closed, and $A \cdot B = 0$, then also $R' \cdot S'' = 0$.*

4. The discussion of spheres in the previous section suggests the following sets of axioms:

Axiom III′. *For each pair of points a and b there is an $r > 0$ such that b does not lie in $U_r'(a)$.*

Axiom III″. *For each pair of points a and b there is an $r > 0$ such that b does not lie in $L_r''(a)$.*

Axiom IV′. *For each point a and each positive constant k there is an $r > 0$ such that, if $ab \geqq k$, b does not lie in $U_r'(a)$.*

Axiom IV″. *For each point a and each positive constant k there is an $r > 0$ such that, if $ba \geqq k$, b does not lie in $L_r''(a)$.*

Axiom III′ is clearly equivalent to the statement that for each pair of points a and b there is an $r > 0$ such that there is no point x for which both $ax < r$ and $bx < r$, and hence to the statement that no sequence has more than one u-limit. Similar equivalences are of course valid for Axiom III″. Axioms IV′ and IV″ are stronger forms of III′ and III″.

5. If the quasi-metric space Z contains an enumerable set $E = \{c_i\}$ such that every point of Z is the c-limit of some sub-sequence chosen from E, we say that E is *dense* in Z and that Z is *separable*. We then have the usual theorems on the cardinal number of the set of regions and closed sets, including the Lindelöf covering theorem. Also, if a proper part M of Z is considered as a space, it is quasi-metric and separable.

In addition to these theorems it should be noted that in a separable quasi-metric space Z every point x which is not a c-limiting point of $Z - x$ must belong to the fundamental set E. Hence the set of points which are u-limiting, but not l-limiting points of Z, and vice versa, is enumerable.

On the other hand the imposition of separability on Z does not remove the possibility of a sequence having two or more u-limits or l-limits. As

an example, let Z be the sum of the plane sets C and A, where C is the set defined by $x=0$, $0<y\leqq 1/2$, and A the set defined by $0<x\leqq 1$, $y=0$, and let E be the set of points of Z which have both coördinates rational. If $a+b\subset C$ or $a+b\subset A$, let $ab=ba$ be the ordinary Cartesian distance. If a lies in C and b in A, let ab be the abscissa of b and $ba=ab+1$. Then Z is quasi-metric and every point is the c-limit of some sequence chosen from E, but every point of C is a u-limit of the sequence of points of A whose abscissae are $\{1/n\}$.

If Z is separable and also satisfies Axiom IV′, Theorem II of § 3 may be replaced by the following: *If $A'\cdot B+A\cdot B'=0$, there are disjoint u-regions R and S containing A and B respectively.* An analogous theorem corresponds to Axiom IV″. The method of proof is indicated in § 8.

6. Relations between quasi-metric and metric spaces. We first note that there is no way of defining distance so that there is a unique correspondence between the u-limiting (or l-limiting) points and points which are limiting points by the new distance definition. For we may have two distinct points, both of which are u-limits of the same sequence. But in a metric space no sequence has more than one limit.

If we define the distance between two points x and y as the quantity $\rho(x,y)=(xy+yx)/2$, we obtain a metric space Z' which has the same points as Z. For points where $\overline{xy}$ exists, $\rho(x,y)=\overline{xy}$; for other pairs of points $\rho(x,y)$ has a value between xy and yx. It is a simple matter to show that, if a is the limit in Z' of a sequence $\{x_i\}$, it is the c-limit of $\{x_i\}$ in Z; and conversely. A point which is not a c-limiting point of Z is an isolated point of Z' and, if Z is separable, the number of such points is enumerable.

7. Relations between quasi-metric and topological spaces. In Hausdorff's *Mengenlehre* (pp. 228-229) the topological axioms are listed in three groups: axioms of vicinities (A, B, C); separation axioms $(4, 5, 6, 7, 8)$; and cardinal number axioms $(9, 10)$. This numbering will be used in the following theorems, which relate quasi-metric to topological spaces.

Theorem I. *Let Z be quasi-metric and let the vicinities of each point x be the u-spheres or l-spheres of center x and rational radii. Then Z is a topological space satisfying Axioms A, B, C, 4, and 9, but not necessarily satisfying Axiom 5. A u-limit, or l-limit, respectively, of a sequence is the topological limit of the sequence, and vice versa.*

Proof. The proof of the positive statements is immediate. That Z need not satisfy Axiom 5 follows from the fact that u-limits and l-limits may not be unique. In this and the following theorems analogous results are obtained

when the vicinities are c-spheres, but this case is not worth considering, as it was seen in § 6 that the space can be made metric with a preservation of c-limiting points.

THEOREM II. *Let Z be quasi-metric and separable, E being the enumerable set dense in Z. Let the vicinities be the u-spheres or l-spheres whose radii are rational and whose centers are points of E, and let the vicinity of a point x be any such sphere containing x. Then Z is a topological space satisfying Axioms A, B, C, 4, and* 10, *but not necessarily satisfying Axiom* 5. *A u-limit or l-limit, respectively, of a sequence is the topological limit of the sequence, and conversely.*

Proof. Consider the u-case; the other is similar. If a is any point of Z, there is a sub-sequence $\{c_n\}$ of E such that a is the c-limit of $\{c_n\}$; i. e., for any rational $r > 0$, $ac_n < r$ and $c_n a < r$ for every n greater than some n'. Hence a lies in the u-sphere of center c_n and radius r. Thus Axiom A is satisfied; the validity of the other axioms readily follows.

Let a be any u-limit of a sequence $\{x_n\}$ and V be any vicinity of a. Now V is a u-sphere having some point c of E as its center and a rational radius r, and $ca < r$. Taking $r' < r - ca$, we have $ax_n < r'$ for every n larger than some n'. Hence $cx_n < ca + r' < r$, or V contains every x_n for $n > n'$.

Conversely, let a be the topological limit of the sequence $\{x_n\}$. Since E is dense in Z, there is for each rational $r > 0$ a point c of E such that $ac < r$ and $ca < r$. Then the u-sphere of center c and radius r is a vicinity of a and consequently contains every x_n for n larger than some n'. This gives $ax_n \leqq ac + cx_n < r + r = 2r$ for $n > n'$, which was to be proved.

THEOREM III. *Let Z be a topological space satisfying Axioms A, B, C, 4, and* 10. *Then distances can be defined so that Z is quasi-metric and separable and, if a is any topological limit of the sequence $\{x_i\}$, then a is a u-limit of the sequence, and conversely.*

Proof. Let the enumerable set of vicinities be $\{V_i\}$. For any two points x and y and for each i, set $f_i(x, y) = 1$ if x lies in V_i and y in $Z - V_i$; otherwise $f_i(x, y) = 0$. Let $xy = \sum_1^\infty f_i(x, y)/2^i$. If $x = y$, every $f_i(x, y) = 0$ and $xy = 0$. If $x \neq y$, there is some V_i such that x lies in V_i and y in $Z - V_i$ by Axiom 4. Hence this $f_i(x, y) = 1$ and consequently $xy > 0$. [illegible]
If x and y are both in V_i, $f_i(x, y) = 0$, and $f_i(y, z)$ and $f_i(x, z)$ are both 0 or both 1, according as z lies in V_i or in $Z - V_i$. If x lies in V_i and y in

$Z - V_i$, $f_i(x,y) = 1$. Thus in every case $f_i(x,y) + f_i(y,z) \geqq f_i(x,z)$ for each i, and so $xy + yz \geqq xz$. Hence Z is quasi-metric.

Now let a be a topological limit of the sequence $\{x_n\}$. Take any $r > 0$ and k so large that $1/2^k < r$. If a is not in V_i, $f_i(a, x_n) = 0$ for every n. If a lies in V_i, $f_i(a, x_n) = 0$ for every n larger than some n_i by the definition of topological limit. It we now take m as the largest of the integers n_i, for $i \leqq k$, we have $ax_n = \sum_1^\infty f_i(a, x_n)/2^i \leqq \sum_{k+1}^\infty 1/2^i = 1/2^k < r$ for every $n > m$. Hence a is a u-limit of $\{x_n\}$. Conversely, let $ax_n \to 0$. Let V_k be any vicinity containing a. Since $ax_n \to 0$, there is an integer m such that $ax_n < 1/2^k$ for every $n > m$. This means that x_n lies in V_k for every $n > m$, as otherwise $f_k(a, x_n) = 1$ and $ax_n \geqq 1/2^k$. But this is the definition of topological limit.

To show that Z is separable, consider the sequence of vicinities $\{V_i\}$ and let $W_i = Z - V_i$. In each of the sets V_1 and W_1, if not void, choose a point. In each of the sets $V_1 \cdot V_2$, $V_1 \cdot W_2$, $W_1 \cdot V_2$, and $W_1 \cdot W_2$ which is not void, choose a point. Do the same for $V_1 \cdot V_2 \cdot V_3$, $V_1 \cdot V_2 \cdot W_3$, $V_1 \cdot W_2 \cdot V_3$, $W_1 \cdot V_2 \cdot V_3$, $V_1 \cdot W_2 \cdot W_3$, $W_1 \cdot V_2 \cdot W_3$, $W_1 \cdot W_2 \cdot V_3$, and $W_1 \cdot W_2 \cdot W_3$; etc. At the k-th stage we add at most 2^k new points; hence the set E of these point is an enumerable set.

Now let a be any point of Z. If a lies in V_1, let x_1 be a point of $E \cdot V_1$; if a lies in W_1, let x_1 be a point of $E \cdot W_1$. Also a lies in one of the four sets $V_1 \cdot V_2$, etc.; let x_2 be a point of E in the same set. Let x_3 be a point of E in that one of the eight sets $V_1 \cdot V_2 \cdot V_3$, etc., which contains a; etc. This method of choice insures that a and all the points $\{x_n\}$ lie either in V_1 or in W_1. Hence for every n, $f_1(a, x_n) = f_1(x_n, a) = 0$; and so for every n both ax_n and x_na are less than 1. Continuing, we see that for any integer m and $n \geqq m$, $f_i(a, x_n) = f_i(x_n, a) = 0$ for $i \leqq m$ and so both ax_n and x_na are less than $1/2^{m-1}$. Therefore a is the c-limit of the sequence $\{x_n\}$ and Z is separable.

THEOREM IV. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 10. Then distances can be defined so that Z is quasi-metric and separable and, if a is a topological limit of the sequence $\{x_i\}$, then a is an l-limit of the sequence, and conversely.*

To prove this set $f_i(x,y) = 1$ if y lies in V_i and x in $Z - V_i$; otherwise set $f_i(x,y) = 0$. Then proceed much as in the proof of Theorem III.

THEOREM V. *In Theorem III or IV let Z satisfy Axiom 5 as well as 4. Then the same conclusions are valid and also Z as a quasi-metric space satisfies Axiom III′ or III″, respectively.*

Proof. The first statement is true, since Axiom 5 is stronger than Axiom 4. If a and b were both u-limits or l-limits, respectively, of the same sequence $\{x_i\}$, they would both be the topological limits of this sequence by Theorem III or IV, respectively. This, however, is impossible by virtue of Axiom 5. Hence Axiom III′ or III″, respectively, is valid.

THEOREM VI. *Let Z be quasi-metric and separable and satisfy Axiom III′ or III″. Let u-spheres or l-spheres, respectively, be taken for vicinities as in Theorem II. Then Z is a topological space satisfying Axioms A, B, C,* 5, *and* 10. *The u-limit or l-limit, respectively, of any sequence is the topological limit of the sequence, and vice versa.*

Proof. By Theorem II we need only to prove that Axiom 5 is satisfied. Let a and b be any two points and Axiom III′ be satisfied. Then there is an $r > 0$ such that $U_r(a)$ and $U_r(b)$ have no common points. But by Theorem II these are topological regions and hence Axiom 5 is satisfied. The proof for Axiom III″ is similar.

THEOREM VII. *Theorems V and VI are valid if III′ and III″ are replaced by IV′ and IV″, respectively, and* 5 *by* 6.

Proof. Let us take the u-case and assume Axiom IV′. Let C be u-closed and a be a point not in C. Then there is a $k > 0$ such that $ax > k$ if x lies in C. By Axiom IV′ there is an $r > 0$ and for each point x of C, an $r' > 0$ such that $U_r(a) \cdot U_{r'}(x) = 0$. If $R = U_r(a)$ and S is the union of the sets $U_{r'}(x)$ as x runs over C, R and S are u-regions containing a and C, respectively, and $R \cdot S = 0$. By Theorem II, R and S are topological regions, and hence Axiom 6 is valid.

Conversely, let B be the set of points $\{x\}$ for which $ax \geqq k$. Then B is closed and by Axiom 6 there are disjoint topological regions R and S containing a and B, respectively. By Theorem III there are u-regions; hence some $U_r(a) \subset R$ and, for each x in R, some $U_{r''}(x) \subset S$. Thus Axiom IV′ is satisfied.

8. It has been shown by the researches of Urysohn and Tychonoff* that topological spaces satisfying Axioms A, B, C, 6, and 10 are identical with separable metric spaces. The theorems of the previous section show that, if Axiom 6 is replaced by the weaker Axiom 4 or 5, the resulting topological

* P. Urysohn, "Zum Metrisationsproblem," *Mathematische Annalen*, Vol. 94, pp. 309-315, and A. Tychonoff, "Über einen Metrisationssatz von P. Urysohn," *ibid.*, Vol. 95, pp. 139-142.

spaces can be identified with separable quasi-metric spaces, which in the latter case satisfy Axiom III′ or III″. Theorem VII has been added for the sake of completeness and to show the correspondence between the topological axioms of separation 4, 5, and 6 and the quasi-metrical axioms I, III′ or III″, and IV′ or IV″. Whether this correspondence can be pushed further depends upon the possibility of a converse to Theorem I.

In connection with the above references it should be noted that Theorem II of § 3 as extended at the close of § 5 is Tychonoff's separation theorem (*loc. cit.*, p. 140) and can be proved in a similar manner by the aid of Axiom IV′. Urysohn's method can then be applied to transform a separable quasi-metric space satisfying Axiom IV′ into a separable metric space. For, let $Z = \{x\}$ be the space in question and $E = \{c_i\}$ be the enumerable set dense in Z. For each rational number k and each c_i there is an $r_i > 0$ such that $U_{r_i}(c_i) \subset U_k(c_i)$ and the u-closed sets $U_{r_i}'(c_i)$ and $Z - U_k(c_i)$ have no common points. We can then set up Urysohn's continuous function $f(x)$ and his distance formula just as he does (*loc. cit.*, pp. 311-312).

The following example indicates the possible utility of quasi-metric notions in the study of decompositions of spaces into disjoint sub-sets. Let M be a compact metric space, $M = \sum [X]$ be any decomposition of M into disjoint closed sets, Z be the aggregate whose elements are the sets $\{X\}$, and distances be defined as in the example in § 1. Then Z is quasi-metric. Since M is separable, there is an enumerable set E dense in M. If F denotes any finite sub-set of E, the system of possible sets $\{F\}$ is enumerable. For each F and each rational number $r > 0$ select, if possible, an element Y of Z such that $FY < r$ and $YF < r$; this gives a finite or enumerable aggregate $G = \{Y_i\}$. But by the Borel theorem we see at once that for any $r > 0$ and each X there is some F such that $FX < r$ and $XF < r$. Hence G is dense in Z and Z is separable. This shows incidentally that for any decomposition of a compact metric space into closed sets $\{X\}$ there is an enumerable system G of these sets such that every X is the closed limiting set of some sequence chosen from G. The space Z also satisfies Axiom III′. For, in consequence of our distance definitions, $AX_i \to 0$ if and only if the upper closed limiting set of $\{X_i\}$ is a sub-set of A. As the elements of Z, considered as sub-sets of M, are mutually disjoint, no sequence of elements of Z has more than one u-limit. As a topological space Z satisfies Axioms A, B, C, 5, and 10.

YALE UNIVERSITY,
NEW HAVEN, CONN.

DOUBLE IMPLICATION AND BEYOND.

By H. B. Smith.

The general problem which we have here in mind and which will only be begun in the present paper but continued in a later one, is to determine (to construct) all the true and all the untrue propositions into which any number of symbols of implication may enter in any way whatsoever. The word "true" is to be understood as "true in all senses of the word true," whereas the word "untrue" is to be taken to mean "untrue in at least one sense of the word," that is to say, not generally true. The construction of an infinite series of meanings of the true and the untrue and their definitions as well as a statement of the elementary properties of the existential function is contained in an article in the *Bulletin of the American Mathematicol Society*, Jan.-Feb. 1929. This function is defined by

$$|XY|' = X \angle Y'$$

which may be read: If X (is true) then Y (is untrue). Our fundamental assumption is,

$$X \angle |X|$$

that is: If X (is true) then X (is true for some meanings of the terms that enter into X). The converse is in general untrue. The *degree* of the function is the number of its elements or variables. Its *order* is the largest number of operations that occur among its terms when the function is expanded. Thus $|QR|$ is first order, $|P|QR||$ is second order and so on.

Since every proposition into which any number of symbols of implication enters in any way undetermined, may be expressed as a function of existentials of varying orders and degrees and of the free variables, the problem may be otherwise stated: To determine all the cases in which ϕ becomes unity, where ϕ is a function of any form whatever of existentials of undetermined order and degree and of the free variables.

Without loss of generality we may suppose the symbol of implication to appear nowhere explicitly in ϕ, being implicit in the existentials, and that the variables in turn, unless otherwise stated, are free variables, that is free of the symbol of implication. If we say that ϕ and the variables entering into ϕ are general, we mean that they may represent either sums of products or products of sums. This latter provision may [illegible] a product of sums may always be represented as a sum of products by direct

multiplication. It may be supposed, then, that ϕ is of the form of a sum of products and the question then is, under what conditions does ϕ become unity.

The problem can again be stated otherwise and in a way that will lead more directly to a solution, for if ϕ is primarily a sum of products, ϕ' is in turn reducible to a sum of products. The condition for ϕ becoming unity will be the same as for the vanishing of ϕ', and since for a sum to vanish each term must vanish separately, our problem may now be stated in its simplest and most general form: To determine all the cases in which ϕ vanishes (replacing ϕ' by ϕ) where ϕ is a product of existentials of any order and degree and the free variables, the latter being in general sums of products of the categorical forms.

Let us begin with the consideration of some simple cases. The double implication,

$$P \angle (Q \angle R') = | P | QR | |'$$

is true when and only when $P | QR |$ vanishes. This will happen if P or Q or R or QR implies zero but we shall find cases in which

$$P | QR | \angle o$$

holds without any of these conditions being satisfied. Again the triple implication,

$$P \angle [Q \angle (R \angle S')] = | P | Q | RS | | |'$$

will only be true when $P | Q | RS | |$ vanishes, which will occur if any of the variables imply zero or again if

$$Q | RS | \angle o.$$

These last cases having been determined we should still have to consider if there are instances in which

$$P | Q | RS | | \angle o$$

without the aforementioned conditions being fulfilled. Here our task is somewhat simplified by the fact that

$$Q | RS | \angle o$$

is not true unless

$$QRS \angle o$$

that the second order expression

$$P | Q | RS | | \angle o$$

does not hold except for the cases in which

$$PQ \mid RS \mid \angle o$$

holds true, and so on. This is because QRS is a term in the expansion of $Q \mid RS \mid$, $PQ \mid RS \mid$ is a term in the expansion of $P \mid Q \mid RS \mid \mid$ etc.

The other double implication,

$$(P \angle Q') \angle R' = \mid \mid PQ \mid' R \mid'$$

requires the vanishing of $\mid PQ \mid' R$ and this will come about if $R = o$ or if $P' = Q' = o$, or if $R = PQ$, but we shall find cases in which

$$\mid PQ \mid' R \angle o$$

holds independently of these conditions. Again the triple implication,

$$[(P \angle Q') \angle R'] \angle S' \quad \mid \mid PQ \mid' R \mid' S \mid'$$

is satisfied if $S = \mid PQ \mid' R$ or if

$$(1) \; S = o \quad \text{or} \quad (2) \; R' = o \quad \text{and} \quad PQ = o$$

and, in the general case if,

$$(1) \; T = o \quad \text{or} \quad (2) \; S' = o \quad \text{and} \quad \cdots R' = o \quad \text{and} \quad PQ = o.$$

The other possibilities, however, remain and the solution of the general case will have to be built up inductively from the simple to the more complex.

The traditional categorical forms we shall represent by the notation:

$$\begin{aligned} A(ab) &= \text{All } a \text{ is } b \\ E(ab) &= \text{No } a \text{ is } b \\ I(ab) &= \text{Some } a \text{ is } b \\ O(ab) &= \text{Not all } a \text{ is } b \end{aligned}$$

the term-order being the order subject-predicate. Whenever it is desired to indicate that the term-order is unsettled, a comma will appear between the terms. The set of propositions of a given type will be called the *array* of propositions of that type. Let us consider at the outset the array,

$$\mid X(a, b) \mid' \mid Y(a, b) \mid' \angle o$$

wherein X and Y are conceived as capable of taking on any of the values A, E, I, O and for which

$$\mid X(a, b) \mid' = X(a, b) \angle o.$$

There will evidently be sixteen propositions in the array obtained by taking the permutations of the four letters two at a time and by taking each letter once with itself. We shall refer to the case in which the term-order is the same in X and Y as the first *figure* of the array, to the case in which the term-order in X is the reverse of the term-order in Y as the second figure. There will then be thirty-two instances to consider. The true members of the set will be called valid *moods*, the others invalid. The valid moods, AO, EI, IE, IO, OA, OI in the first figure follow at once from $A'O' \angle o$, $E'I' \angle o$, $I'O' \angle o$, results of single implication (see the writer's *Symbolic Logic*, Chap. III, Crofts, N. Y.) by "strengthening," that is, by the principle,*

$$(XY \angle Z)(W \angle X) \angle (WY \angle Z)$$

for since $X(ab) \angle \mid X(ab) \mid$ and therefore $\mid X(ab) \mid' \angle X'(ab)$, we have

$$\{A'(ab)O'(ab) \angle o\}\{\mid A(ab)\mid' \angle A'(ab)\} \angle \{\mid A(ab)\mid' O'(ab) \angle o\}$$
$$\{\mid A(ab)\mid' O'(ab) \angle o\}\{\mid O(ab)\mid' \angle O'(ab)\} \angle \{\mid A(ab)\mid' \mid O(ab)\mid' \angle o\}$$

and the valid moods of the second figure, EI, IE, IO, OI, follow from those of the first by converting simply, that is by interchanging terms in an E or I form. This process could of course be represented symbolically. The invalid moods composed of two affirmative forms, A and I, can be shown to be invalid by the substitution $a = b'$, those composed of two negative forms, E and O, by the substitution $a = b$.† The rest are theorems from the

Postulate: $\mid A(ab) \mid' \mid O(ba) \mid' \angle o$ is untrue.

We have

$$\begin{aligned}\{X \angle Y\} &\angle \{\mid X \mid \angle \mid Y \mid\}\\ &\angle \{\mid \mid Y \mid' \mid \angle \mid \mid X \mid' \mid\}\\ &\angle \{\mid \mid X \mid' \mid' \angle \mid \mid Y \mid' \mid'\}\\ \{X \angle Y\} &\angle \{[U \mid X \mid' \angle o] \angle [U \mid Y \mid' \angle o]\}\end{aligned}$$

and accordingly,

$$\{\mid A(ab) \mid' \mid O(ba) \mid' \angle o\}' \angle \{\mid A(ab) \mid' \mid E(ba) \mid' \angle o\}'.$$

The rest follow by converting simply in the E-form. We may note in passing

* The principles which we take from the calculus of propositions, may, however, be derived by assuming certain very simple properties of the existential function together with the formulas for its expansion.

† For a proof that the affirmative forms become true, the negative forms false, when subject and predicate are identified, that the negative forms become true, the affirmative forms false, when the terms are made contradictory, see *Symbolic Logic*, p. 77.

that this assumption enables us to save the postulate we were forced to introduce in *Symbolic Logic,* page 80.

There are no valid moods in the array,

$$| X(a, b) Y(b, c) |' \, | Z(c, a) |' \angle o$$

for if X or Y is negative we have only to identify terms in the negative, if X and Y are affirmative and Z negative to identify terms in Z, if X, Y and Z are affirmative to identify terms in X or Y, in order to make the invalidity of the mood depend on the invalidity of a simpler case.

The array $X(a, b) \, | Y(a, b) |' \angle o$.

The valid moods follow at once from the valid moods of immediate inference by "strengthening," thus,

$$\{O(ab) O'(ab) \angle o\}\{ \, | O(ab) |' \angle O'(ab)\} \angle \{O(ab) \, | O(ab) |' \angle o\}$$
$$\{A(ab) \, I'(ba) \angle o\}\{ \, | I(ba) |' \angle I'(ba)\} \angle \{A(ab) \, | I(ba) |' \angle o\} \, *$$

The invalid moods are deduced by the methods illustrated below.

$$\{ \, | A(ab) |' \angle A'(ab)\}\{ \, | A(ab) |' \, | O(ba) |' \angle o\}' \angle \{A'(ab) \, | O(ba) |' \angle o\}'$$
by $(XY \angle Z)'(X \angle W) \angle (WY \angle Z)'$

$$\{A'(ab) \angle O(ab)\}\{A'(ab) \, | O(ba) |' \angle o\}' \angle \{O(ab) \, | O(ba) |' \angle o\}'$$
by the same principle.

$$\{E(ab) \angle O(ba)\} \angle \{[O(ab) \, | O(ba) |' \angle o]' \angle [O(ab) \, | E(ab) |' \angle o]'\}$$
by $\{X \angle Y\} \angle \{[U \, | Y |' \angle o]' \angle [U \, | X |' \angle o]'\}$.

If $A(ab) \, | E(ab) |' \angle o$ were valid it would have to remain valid for all special meanings of the terms. Substitute $a = b$. The antecedent then becomes true and the consequent false.

It might be well to indicate at this point a method alternative to the one we have employed so far for the determination of the invalid moods.

Postulate: There exists a meaning of a and b, viz. $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $A(\boldsymbol{ab}) \angle o$ and $O(\boldsymbol{ba}) \angle o$.

These meanings might be regarded as empirical or definitional (from some other science), for example *plane figures* and *triangles.* Our derivations would then proceed as follows:

THEOREM. $| A(\boldsymbol{ab}) |' \, | O(\boldsymbol{ba}) |' \angle o$ *is untrue,*

[illegible]

* For a proof that the current view that subalternation fails rests on a misunderstanding see *Symbolic Logic*, Chap. III.

THEOREM. $|\, A(ab)\, |' \,|\, O(ba)\, |' \angle o$ *is untrue,*

since it is true for *not all* meanings of the terms, and accordingly,

$$\{\, |\, A(\boldsymbol{ab})\, |' \,|\, O(\boldsymbol{ba})\, |' \angle o\}' \angle \{\, |\, A(\mathbf{ab})\, |' \,|\, E(\mathbf{ba})\, |' \angle o\}'.$$

The antecedent *being true may be suppressed* and we have,

THEOREM. $|\, A(\boldsymbol{ab})\, |' \,|\, E(\boldsymbol{ba})\, |' \angle o$ *is untrue,*

THEOREM. $|\, A(ab)\, |' \,|\, E(ba)\, |' \angle o$ *is untrue,*

since it is true for *not all* meanings of the terms.

Similarly, in the case of the assumption we introduce below, we might write

Postulate: There exists a meaning of a, b and c, viz. $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, such that $O(\boldsymbol{ab}) \angle o$, $O(\boldsymbol{cb}) \angle o$ and $I(\boldsymbol{ca}) \angle o$.

THEOREM. $|\, O(\boldsymbol{ab})\, |' \,|\, O(\boldsymbol{cb})\, |' \,|\, I(\boldsymbol{ca})\, |' \angle o$ *is untrue,*

THEOREM. $|\, O(ab)\, |' \,|\, O(cb)\, |' \,|\, I(ca)\, |' \angle o$ *is untrue,*

$$\{\, |\, O(\boldsymbol{ab})\, |' \,|\, O(\boldsymbol{cb})\, |' \,|\, I(\boldsymbol{ca})\, |' \angle o\}' \angle \{\, |\, E(\boldsymbol{ab})\, |' \,|\, O(\boldsymbol{cb})\, |' \,|\, I(\boldsymbol{ca})\, |' \angle o\}'$$
$$\text{by } \{X \angle Y\} \angle \{[U \,|\, X\, |' \angle o] \angle [U \,|\, Y\, |' \angle o]\}.$$

THEOREM. $|\, E(\boldsymbol{ab})\, |' \,|\, O(\boldsymbol{cb})\, |' \,|\, I(\boldsymbol{ca})\, |' \angle o$ *is untrue,*

THEOREM. $|\, E(ab)\, |' \,|\, O(cb)\, |' \,|\, I(ca)\, |' \angle o$ *is untrue*

and so on. This method does not apply, however, to the later postulates we shall set down.

The array $|\, X(a, b)\, |' \,|\, Y(b, c)\, |' \,|\, Z(c, a)\, |' \angle o$.

We should have here sixty-four cases to consider, each one in each one of the ordinary four figures, though because logical multiplication is commutative not all of these will be distinct. The valid moods are obtained by "strengthening" the premises in $X'(a, b)\ Y'(b, c)\ Z'(c, a) \angle o$. Thus from BARBARA by three steps:

$$\{O'(ba)\ O'(cb)\ A'(ca) \angle o\}\{|\, O(ba)\, |' \angle O'(ba)\} \angle \{|\, O(ba)\, |'\ O'(cb)\ A'(ca) \angle o\}$$
$$\{|\, O(ba)\, |'\ O'(cb)\ A'(ca) \angle o\}\{|\, O(cb)\, |' \angle O'(cb)\} \angle \{|\, O(ba)\, |' \,|\, O(cb)\, |'\ A'(ca) \angle o\}$$
$$\{|\, O(ba)\, |' \,|\, O(cb)\, |'\ A'(ca) \angle o\}\{|\, A(ca)\, |' \angle A'(ca)\} \angle \{|\, O(ba)\, |' \,|\, O(cb)\, |' \,|\, A(ca)\, |' \angle o\}$$

For the deduction of the invalid moods we shall introduce a

Postulate: $|\, O(ab)\, |' \,|\, O(cb)\, |' \,|\, I(ca)\, |' \angle o$ is untrue.

From this assumption and by the methods exemplified below all the invalid moods can be established. We may note in passing that we are now able to dispense with the postulate introduced in *Symbolic Logic,* page 98.

(1) Suppose $|A(ba)|'\,|A(cb)|'\,|A(ca)|' \angle o$ were valid and make $b=a'$. The mood then reduces to $|E(ca)|'\,|A(ca)|' \angle o$ a form already established as invalid.*

(2) Suppose $|A(ab)|'\,|O(bc)|'\,|O(ca)|' \angle o$ were valid and make $c=a$. The mood then reduces to $|A(ab)|'\,|O(ba)|' \angle o$.

(3) $\{\,|O(ab)|'\,|O(cb)|'\,|I(ca)|' \angle o\}' \angle \{\,|E(ab)|'\,|O(cb)|'\,|I(ca)|' \angle o\}'$ by $\{X \angle Y\} \angle \{[U\,|X|' \angle o] \angle [U\,|Y|' \angle o]\}$.

There are no valid moods in the array,

$$|X(a,b)\;Y(b,c)|'\,|Z(c,d)|'\,|W(d,a)|' \angle o.$$

In order to show this identify terms in a negative form if a negative form occurs. If all the forms are affirmative, we have only to identify terms in one of the forms of the first bracket.

$$\text{The array}\quad X(a,b)\;Y(b,c)\;|Z(c,a)|' \angle o.$$

The valid moods are derivable from and correspond exactly to the valid moods of the syllogism. The invalid moods are gotten from the invalid moods of the last array. The following examples will be enough to illustrate the method.

(1) $$\{A(ba)A(cb)A'(ca) \angle o\}\{\,|A(ca)|' \angle A'(ca)\}$$
$$\angle\ \{A(ba)A(cb)\;|A(ca)|' \angle o\}$$

(2) $$\{E(ca)\;|O(ca)|' \angle o\}\{A(ab)E(cb) \angle E(ca)\}$$
$$\angle\ \{A(ab)E(cb)\;|O(ca)|' \angle o\}$$
by $(XY \angle Z)(W \angle X) \angle (WY \angle Z)$

(3) $$\{\,|O(ab)|'\,|O(cb)|'\,|I(ca)|' \angle o\}'\{\,|O(ab)|' \angle O'(ab)\}$$
$$\angle\ \{O'(ab)\,|O(cb)|'\,|I(ca)|' \angle o\}'$$
$$\{O'(ab)\;|O(cb)|'\,|I(ca)|' \angle o\}'\{\,|O(cb)|' \angle O'(cb)\}$$
$$\angle\ \{O'(ab)O'(cb)\;|I(ca)|' \angle o\}'$$
$$\{O'(ab)O'(cb)\;|I(ca)|' \angle o\}'\{O'(ab) \angle A(ab)\}$$
$$\angle\ \{A(ab)O'(cb)\;|I(ca)|' \angle o\}'$$
$$\{A(ab)O'(cb)\;|I(ca)|' \angle o\}'\{O'(cb) \angle A(cb)\}$$
$$\angle\ \{A(ab)A(cb)\;|I(ca)|' \angle o\}'$$
by $(XY \angle Z)'(X \angle W) \angle (WY \angle Z)'$

* For a proof of obversion see *Symbolic Logic,* page 76.

$$\text{The array } X(a,b) \mid Y(b,c)\mid' \mid Z(c,a)\mid' \angle\, o.$$

There is no novel principle involved in this case. The valid moods come from the array of the syllogism $X(a,b)\ Y'(b,c)\ Z'(c,a) \angle\, o$ by "strengthening," the invalid moods from the array $\mid X(a,b)\mid' \mid Y)b,c)\mid' \mid Z(c,a)\mid' \angle\, o$ by "weakening" as before.

$$\text{The array } \mid X(1,2)\mid' \mid Y(2,3)\mid' \cdots \mid Z(n,1)\mid' \angle\, o$$

wherein the n terms are arranged in a cycle and the number of premises is the same as the number of terms. All valid moods of this type are evidently gotten from valid moods of the sorites as those of the cycle of three terms are gotten from valid moods of the cycle of three terms or the syllogism. These types already established (*Symbolic Logic,* Chap. V) are:

$$O'(21)\ O'(32) \cdots O'(n\, n-1) \angle\, O'(n\, 1)$$
$$O'(21)\ O'(32) \cdots O'(r\, r-1)\ O'(r\, r+1) \cdots O'(n-2\, n-1)\ O'(n-1\, n) \angle\, E'(n\, 1)$$
$$O'(21)\ O'(32) \cdots O'(t\, t-1)\ E'(t, t+1)\ O'(t+1\, t+2) \cdots O'(n-2\, n-1)\ O'(n-1\, n) \angle\, E'(n\, 1).$$

The invalidity of all other moods can be established by the same methods (*Symbolic Logic,* Chap. V) as before and it will be easy to show that no bracket can contain more than one form of the cycle. More generally, valid moods in which only some existentials occur correspond exactly to valid moods of the sorites wherein any $Y'(r, r-1)$ is strengthened to $\mid Y(r, r-1)\mid'$.

It may be useful to point out one method of establishing invalidity by reduction to a special case, a method not needed and therefore not employed in *Symbolic Logic,* Chap. V. Suppose we are dealing with an implication of the form,

$$X(1,2)\ Y(2,3) \cdots Z(n-1,n) \mid U(n, n+1) \cdots V(r\, 1) \mid' \angle\, o.$$

If there is more than one negative form in the second part, eliminate the affirmative forms through the identification of terms. If Z and U be adjacent, that is, if they have a term in common, make $n + 1 = n'$ and get rid of the $(n+1)'$ in Z by obversion, and continue thus to eliminate the forms in the second bracket.

Zero Conjunction of Cycles, Chains and Zero Conjunction of Chains.

Two categorical forms which have a common term are called *adjacent.* A product of categorical forms and of existentials of these products wherein

each form is adjacent to at least one other form is called a *chain*. Thus a *cycle,* a conjunction of cycles or a series of cycles one-directionally joined is a special case of a chain closed at some of its terms. The terms common to two chains are called their *intersections*. The cycles formed at the intersections in a product of two or more chains are called their *cross-products*. A chain of forms represented as implying zero is called a *zero chain*. A product of chains represented as implying zero is called a *zero conjunction of chains*.

If $P \mid Q \mid' \angle o$ be of such a form that PQ represents a zero conjunction of cycles, it being understood that existentials may appear as factors, then Q contains among its factors no zero cycle and no cross-product that vanishes unless P also vanishes, for in that case $\mid Q \mid'$ reduces to unity. We may suppose, then, that if PQ contains any vanishing cycle, that its factors lie partly in P and partly in Q. Moreover there are no terms among the categorical forms that appear in Q that do not also appear in P. For if any term in Q did not appear in P we could identify it with an adjacent term in the case of a negative, make it contradictory with its adjacent term in the case of an affirmative, and so cause Q to vanish. The invalidity of the mood would then be establshed since P is assumed not to vanish independently.

Consider now the implication,

$$C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle o$$

in which each C is in the form of a cycle of any number of terms. The ordinary zero conjunction of cycles of single implication if valid contains a zero factor, either one of the C's or some cross product. Let this factor be C and its form,

$$X'(1, 2)\; Y'(2, 3) \cdot \cdot \cdot Z'(n\, 1) \angle o.$$

By strengthening each $W'(r, r-1)$ to $\mid W(r, r-1) \mid'$ in succession there would arise corresponding valid moods of

$$C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle o$$

Principle: If none of the cycles of a valid mood of the zero conjunction of cycles vanishes, then one of the cross-products vanishes.

From this principle all valid moods are evidently constructible, for if K be the cross-product in question,

$$(K \angle o) \angle (C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle o)$$

Principle: A valid mood of the zero chain contains at least one vanishing cycle.

Principle: A valid mood of the zero conjunction of chains contains at least one vanishing cycle.

These principles, of which the first two are special cases of the last, evidently contain the general solution of $P \mid Q \mid' \angle o$ for we have now constructed all valid implications of this type, P and Q being in general sums of products of the categorical forms. If

$$P = lm \cdots + pq \cdots + uv \cdots + \cdots$$
$$\mid Q \mid' = \mid n + r + w + \cdots + S \mid' = \mid n \mid' \mid r \mid' \mid w \mid' \cdots \mid S \mid'$$

the valid moods would be generally those in which each one of the factors, say,

$$l \mid n \mid', \qquad p \mid r \mid', \qquad u \mid w \mid',$$

vanishes.

The array $X(a, b) \mid Y(a, b) \mid \angle o$.

The valid moods are gotten from a postulate we shall introduce later on, viz. $\mid A(ab) \mid \mid E(ab) \mid \angle o$, as follows:

$$\{ \mid A(ab) \mid \mid E(ab) \mid \angle o\}\{A(ab) \angle \mid A(ab) \mid \} \angle \{A(ab) \mid E(ab) \mid \angle o\}$$
$$\{ \mid E(ab) \mid \mid A(ab) \mid \angle o\}\{E(ab) \angle \mid E(ab) \mid \} \angle \{E(ab) \mid A(ab) \mid \angle o\}$$
$$\text{by} \quad (XY \angle Z)(W \angle X) \angle (WY \angle Z).$$

We derive the invalid moods in part from two later postulates, viz.

$$\mid E(ba)\, A(bc) \mid A(ca) \angle o \quad \text{and} \quad E(ba)\, A(bc) \mid A(ca) \mid \angle o$$

and a postulate already introduced,

$$\mid A(ab) \mid' \mid O(ba) \mid' \angle o$$

all of these being assumptions of invalidity, as follows:

$$\{ \mid E(ba) A(bc) \mid A(ca) \angle o\}'\{ \mid E(ba) A(bc) \mid \angle \mid O(ca) \mid \} \angle \{ \mid O(ca) \mid A(ca) \angle o\}'$$
$$\{E(ba) A(bc) \mid A(ca) \mid \angle o\}'\{E(ba) A(bc) \angle O(ca)\} \angle \{O(ca) \mid A(ca) \mid \angle o\}'$$
$$\text{by} \quad (XY \angle Z)' (X \angle W) \angle (WY \angle Z)'$$
$$\{ \mid A(ab) \mid' \mid O(ba) \mid' \angle o\}' \{ \mid A(ab) \mid' \angle \mid O(ab) \mid \} \angle \{ \mid O(ab) \mid \mid O(ba) \mid' \angle o\}'$$
$$\{ \mid O(ab) \mid \mid O(ba) \mid' \angle o\}' \{ \mid O(ba) \mid' \angle O'(ba)\} \angle \{ \mid O(ab) \mid O'(ba) \angle o\}'.$$

This is $A(ba) \mid O(ab) \mid \angle o$ is untrue, and in the same way we obtain $O(ba) \mid A(ab) \mid \angle o$ is untrue, while $E(ab) \mid I(ab) \mid \angle o$ is untrue and $\mid E(ab) \mid I(ab) \angle o$ is untrue follow from these by obversion. Again the invalidity of $I(ab) \mid O(ab) \mid \angle o$ and $\mid I(ab) \mid O(ab) \angle o$ are derived from the same theorems by weakening and the remainder can be established by one the substitutions $a = b$ or $a = b'$.

The array $| X(a, b) | \, | Y(a, b) | \angle o$.

There is only one valid mood in this set and this we shall introduce provisionally as a

Postulate: $| A(ab) | \, | E(ab) | \angle o$.

It is a matter of indifference whether we assume this or the two that follow from it in the array that has just gone before, for these give together

$$| A(ab) | \, | E(ab) | \angle A'(ab)E'(ab)$$

or

$$| A(ab) | \, | E(ab) | \angle A'(ab)E'(ab) \;\; | A(ab) | \, | E(ab) |$$

and this last vanishes, since it is a term in the expansion of

$$| A(ab)E(ab) | \angle o$$

by the fundamental formula introduced elsewhere (*Symbolic Logic,* p. 127)

$$| XY | = XY + X'Y' \, | X | \, | Y | + | X | \, | Y' |' + | Y | \, | X' |'.$$

All the invalid moods follow from those that have been established in the array that has gone before by weakening.

It might be well before we leave these arrays of two-term cycles to give at least one illustration of the fallacy of assuming

$$X \angle Y = X' + Y.$$

Thus,

$$X \angle (Y \angle Z) = X' + (Y \angle Z) = X' + Y' + Z = XY \angle Z$$
$$\{A(ba)A(cb) \angle A(ca)\} \angle \{A(ba) \angle [A(cb) \angle A(ca)]\}$$

gives for $b = c$,

$$A(ba) \angle | A'(ba) |' = A(ba) \angle | O(ba) |' = | O(ba) | \angle O(ba)$$

which is fallacious.

The array $| X(a, b) Y(b, c) | \, Z(c, a) \angle o$.

The valid moods, six in number, are derived from those of the last array but one by strengthening as follows:

$$\{ | A(ca) | \, E(ca) \angle o \} \{ | A(ba)A(cb) | \angle | A(ca) | \} \angle \{ | A(ba)A(cb) | \, E(ca) \angle o \}$$
$$\{ | E(ca) | \, A(ca) \angle o \} \{ | E(ab)A(cb) | \angle | E(ca) | \} \angle \{ | E(ab)A(cb) | \, A(ca) \angle o \}.$$

For the derivation of the invalid moods we shall introduce

Postulate: $| A(ba) A(bc) | \, E(ca) \angle o$ is *untrue*.

Theorem: $| E(ba) A(bc) | \, A(ca) \angle o$ is *untrue*,

by making $a = a'$.

One illustration will serve to indicate the method by which many theorems can be derived:

$$\{A(ab)A(cb)E(ca) \angle o\}' \{A(ab)A(cb) \angle \mid A(ab)A(cb) \mid \}$$
$$\angle \{ \mid A(ab)A(cb) \mid E(ca) \angle o\}'.$$

When this method is not applicable the invalidity of the mood can be established by one of the substitutions, $a = b, b'$, $c = b, b'$.

The array $X(a, b)Y(b, c) \mid Z(c, a) \mid \angle o$.

Here the derivations are entirely analogous to the last case. We have:

Postulate: $A(ba)A(bc) \mid E(ca) \mid \angle o$ is untrue,

THEOREM: $E(ba)A(bc) \mid A(ca) \mid \angle o$ is untrue,

and the procedure is the same as before. In passing, however, we may note again one case of independent interest:

$$\{ \mid A(ba)A(bc) \mid E(ca) \angle o\}' \{ \mid A(ba)A(bc) \mid \angle \mid I(ca) \mid \}$$
$$\angle \{I(ca) \mid E(ca) \mid \angle o\}'$$
$$\{A(ba)A(bc) \mid E(ca) \mid \angle o\}' \{A(ba)A(bc) \angle I(ca)\}$$
$$\angle \{E(ca) \mid I(ca) \mid \angle o\}'$$

by the principle that we have repeatedly used before.

The valid moods of the zero cycle are gotten at once by strengthening the factors in

$$A(n\,1) \mid E(n\,1) \mid \angle o, \qquad E(n\,1) \mid A(n\,1) \mid \angle o,$$

by means of the implications,

$$A(12) \cdots A(t-1\,t)E(t, t+1)A(t+2\,t+1) \cdots A(n\,n-1) \angle E(n\,1)$$
$$A(21)A(32) \cdots A(n-1\,n-2)A(n\,n-1) \angle A(n\,1)$$

giving rise to four types. All the others can be shown to be invalid by the methods developed already (*Symbolic Logic*, Chap. V). If now we introduce again the principles laid down before, the valid moods of the zero chain, the zero conjunction of cycles and the zero conjunction of chains will be determined. We have then arrived at the general solution of $P \mid Q \mid \angle o$, P and Q being perfectly general, that is, variables of any form free of the symbol of implication.

UNIVERSITY OF PENNSYLVANIA.

REGULAR BILINEAR TRANSFORMATIONS OF SEQUENCES.

By P. A. Fraleigh.

Introduction. Linear transformations of sequences of the form

$$S: \qquad y_n = \sum_{k=1}^{n} a_{nk} x_k,$$

where (x_k) is a given sequence and a_{nk} is constant, have been used * to evaluate divergent series. Such a transformation is said to be *regular* if $\lim_{n\to\infty} y_n$ exists and equals $\lim_{n\to\infty} x_n$, whenever the latter exists. The following theorem is due to Silverman and Toeplitz.

Theorem A. *A necessary and sufficient condition that transformation S be regular is that $\lim_{n\to\infty} a_{nk} = 0$, for each k; $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$; $\sum_{k=1}^{n} | a_{nk} | < C$, for all n, where C is a constant.*

Another theorem due to Kojima † is as follows:

Theorem B. *A necessary and sufficient condition that $\lim_{n\to\infty} y_n$ exist, whenever $\lim_{n\to\infty} x_n$ exists, is that $\lim_{n\to\infty} a_{nk}$ exist for each k; $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}$ exist; $\sum_{k=1}^{n} | a_{nk} | < C$, for all n. If these conditions hold then*

$$\lim_{n\to\infty} y_n = \alpha l + \sum_{k=1}^{\infty} \alpha_k (x_k - l),$$

where

$$\alpha = \lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}, \qquad \alpha_k = \lim_{n\to\infty} a_{nk}, \qquad \textit{and} \qquad \lim_{n\to\infty} x_n = l.$$

In this paper we shall be concerned with bilinear transformations of sequences of the form

$$T: \qquad y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} u_k v_l,$$

where (u_k) and (v_l) are sequences and a_{nkl} is constant. Such a transformation is *regular* if $y_n \to uv$ whenever $u_n \to u$ and $v_n \to v$.

* First studied by L. L. Silverman, *Missouri Dissertation* (1910), and Toeplitz, *Prace matematyczno-fizyczne*, Vol. 28 (1911), p. 113.

† Tetsuzô Kojima, *Tôhoku Mathematical Journal*, Vol. 12 (1917).

In § 1 we shall give a set of necessary and sufficient conditions for regularity of T. A further necessary condition will be found in § 2. An application will be made, in § 3, to the Cauchy product of two series each of which is Cèsaro summable. A set of necessary and sufficient conditions will be given under which the Cauchy product of two Cèsaro summable series is evaluated correctly by the transformation S. Finally in § 4 there are some remarks concerning the possibility of further simplifying the conditions for regularity.

1. *Necessary and sufficient conditions for regularity.* Given two sequences (u_n) and (v_n), consider the bilinear transformation defined by

$$T: \qquad y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} u_k v_l, \qquad a_{nkl} \text{ constant.}$$

If $y_n \to uv$, whenever $u_n \to u$ and $v_n \to v$, we say that T is *regular*.

Let
$$V_{nk} = \sum_{l=1}^{n} a_{nkl} v_l.$$
Then
$$y_n = \sum_{k=1}^{n} V_{nk} u_k,$$

and, by Theorem B, a necessary and sufficient condition that $y_n \to uv$ whenever $u_n \to u$ is that $\lim\limits_{n\to\infty} V_{nk} = 0$, for each k; $\lim\limits_{n\to\infty} \sum_{k=1}^{n} V_{nk} = v$; $\sum_{k=1}^{n} | V_{nk} | < C$, for all n, where C is a constant.

A necessary and sufficient condition that $\lim\limits_{n\to\infty} V_{nk} = 0$ for each k, whenever $v_n \to v$, is that $\lim\limits_{n\to\infty} a_{nkl} = 0$, for each k and l; $\lim\limits_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ for each k; $\sum_{l=1}^{n} | a_{nkl} | < K(k)$ for each k and all n, where $K(k)$ is a constant depending on k.

We may write

$$\sum_{k=1}^{n} V_{nk} = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl}\right) v_l,$$

so that a necessary and sufficient condition that $\sum_{k=1}^{n} V_{nk} \to v$, whenever $v_n \to v$, is that $\lim\limits_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$, for each l; $\lim\limits_{n\to\infty} \sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl} = 1$; $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \right| < C$, for all n, where C is a constant.

We may write $\sum_{k=1}^{n} | V_{nk} | < C$ in the form

$$\sum_{k=1}^{n} \Big| \sum_{l=1}^{n} a_{nkl} v_l \Big| < B(v_1, v_2, \cdots),$$

where $B(v_1, v_2, \cdots)$ is a constant for each convergent sequence (v_n). We have, therefore, the following theorem:

THEOREM I. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$, *for each* k *and* l; $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$, *for each* l; $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$, *for each* k; $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$; $\sum_{l=1}^{n} \big| \sum_{k=1}^{n} a_{nkl} \big| < C$, *for all* n; $\sum_{l=1}^{n} | a_{nkl} | < K(k)$ *for each* k *and all* n; $\sum_{k=1}^{n} \big| \sum_{l=1}^{n} a_{nkl} v_l \big| < (B(v_1, v_2, \cdots)$ *for all* n; *where* C *is a constant,* $K(k)$ *is a constant for each* k, *and* $B(v_1, v_2, \cdots)$ *is a constant for each convergent sequence* (v_n).

By interchanging the rôles of k and l, we obtain the following theorem.

THEOREM I′. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each* k *and* l, $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each* l, $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each* k, $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \big| \sum_{l=1}^{n} a_{nkl} \big| < C$ *for all* n, $\sum_{k=1}^{n} | a_{nkl} | < L(l)$ *for each* l *and all* n, $\sum_{l=1}^{n} \big| \sum_{k=1}^{n} a_{nkl} u_k \big| < A(u_1, u_2, \cdots)$ *for all* n, *where* C *is a constant,* $L(l)$ *is a constant for each* l, *and* $A(u_1, u_2, \cdots)$ *is a constant for each convergent sequence* (u_n).

We now prove certain lemmas which simplify the conditions of these theorems.

LEMMA 1. *From* $\sum_{l=1}^{n} \big| \sum_{k=1}^{n} a_{nkl} u_k \big| < A(u_1, u_2, \cdots)$ *and* $\sum_{l=1}^{n} | a_{nkl} | < K(k)$, *follows* $\sum_{l=1}^{n} | a_{nkl} | < K$ *for all* k *and* n, *where* K *is a constant.*

We will assume that

$$\sum_{l=1}^{n} | a_{nkl} | \geqq K$$

is true for a sequence of pairs of values of n and k and obtain a contradiction.

Let M_k be the greatest of $K(1), K(2), \cdots, K(k)$. Choose n_1 and k_1 such that

$$\sum_{l=1}^{n_1} \big|\ \text{[illegible]}\ \big| \geqq \text{[illegible]}\,1.$$

Define $\qquad u_k = 0, \qquad 1 \leqq k \leqq n_1, \qquad k \neq k_1,$

$$u_{k_1} = 1.$$

Then
$$\sum_{l=1}^{n_1} \left| \sum_{k=1}^{n_1} a_{n_1kl} u_k \right| = \sum_{l=1}^{n_1} | a_{n_1k_1l} | \geqq 1.$$

In general, choose $n_p > n_{p-1}$ and $k_p > n_{p-1}$ such that
$$\sum_{l=1}^{n_p} | a_{n_pk_pl} | \geqq 2^{p-1}(p-1)M_{n_{p-1}} + 2^{2p-1}.$$

Define
$$u_k = 0, \qquad n_{p-1} < k \leqq n_p, \qquad k \neq k_p,$$
$$u_{k_p} = \frac{1}{2^{p-1}}.$$

Then
$$\left| \sum_{k=1}^{n_p} a_{n_pkl} u_k \right| = \left| a_{n_pk_1l} + \tfrac{1}{2} a_{n_pk_2l} + \cdots + \frac{1}{2^{p-1}} a_{n_pk_pl} \right|,$$
and
$$\sum_{l=1}^{n_p} \left| \sum_{k=1}^{n_p} a_{n_pkl} u_k \right| \geqq \frac{1}{2^{p-1}} \sum_{l=1}^{n_p} | a_{n_pk_pl} | - \frac{1}{2^{p-2}} \sum_{l=1}^{n_p} | a_{n_pk_{p-1}l} | - \cdots - \sum_{l=1}^{n_p} | a_{n_pk_1l} |$$
$$\geqq \frac{1}{2^{p-1}} [2^{p-1}(p-1)M_{n_{p-1}} + 2^{2p-1}] - \frac{K(k_{p-1})}{2^{p-1}} - \cdots - K(k_1)$$
$$\geqq 2^p.$$

For this particular sequence $u_n \to 0$, while the expression
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right|$$
is not bounded. The condition of lemma 1 is therefore necessary.

LEMMA 2. *When the first six conditions of Theorem I are satisfied, a necessary and sufficient condition for*
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A(u_1, u_2, \cdots)$$
is
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C,$$
for all n, and in the case of each n, for every possible choice of σ_k, *where* σ_k *is any number such that* $|\sigma_k| = 1$, *and C is a constant.*

We will assume
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C$$
violated and show that a convergent sequence (u_n) exists such that
$$z_n = \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A$$
is not true for any value of A.

Choose n_1 and $\sigma_k{}^{(1)}$, $1 \leqq k \leqq n_1$, such that $|\sigma_k{}^{(1)}| = 1$ and

$$\sum_{l=1}^{n_1} \Big| \sum_{k=1}^{n_1} a_{n_1kl}\sigma_k{}^{(1)} \Big| > 1.$$

Define
$$u_k = \sigma_k{}^{(1)}, \qquad 1 \leqq k \leqq n_1.$$

Then
$$z_{n_1} = \sum_{l=1}^{n_1} \Big| \sum_{k=1}^{n_1} a_{n_1kl}u_k \Big| > 1.$$

By lemma 1

$$\sum_{l=1}^{n} | a_{nkl} | < K.$$

In general, choose

$$n_p > n_{p-1} \quad \text{and} \quad \sigma_k{}^{(p)}, \qquad 1 \leqq k \leqq n_p,$$

such that $|\sigma_k{}^{(p)}| = 1$ and

$$\sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_p} a_{n_pkl}\sigma_k{}^{(p)} \Big| > p^2 + 2pn_{p-1}K.$$

Define

$$u_k = \sigma_k{}^{(p)}/p, \qquad n_{p-1} < k \leqq n_p.$$

Then

$$\begin{aligned} z_{n_p} &= \sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_{p-1}} a_{n_pkl}(u_k - \sigma_k{}^{(p)}/p) + \sum_{k=1}^{n_p} a_{n_pkl}(\sigma_k{}^{(p)}/p) \Big| \\ &\geqq (1/p) \sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_p} a_{n_pkl}\sigma_k{}^{(p)} \Big| - \sum_{k=1}^{n_{p-1}} \sum_{l=1}^{n_p} | a_{n_pkl} | \cdot 2 \\ &> (1/p)\,[p^2 + 2pn_{p-1}K] - 2n_{p-1}K \\ &> p. \end{aligned}$$

Since z_{n_p} can be made as great as we please while $u_n \to 0$, the condition is necessary.

To prove the sufficiency of the condition assume *

$$\sum_{l=1}^{n} \Big| \sum_{k=1}^{n} a_{nkl}\sigma_k \Big| < C.$$

Let (u_n) be a convergent sequence; then $|u_n| < U$. Define

$$\sigma_l = \operatorname{sgn} \sum_{k=1}^{n} a_{nkl}u_k, \quad \text{when} \quad \sum_{k=1}^{n} a_{nkl}u_k \neq 0,$$

* The assumption insures that

$$\sum_{k=1}^{n} \Big| \sum_{l=1}^{n} a_{nkl}\sigma_l \Big| < C,$$

as may be seen by interchanging the rôles of ρ and σ in the latter part of the proof of lemma 3.

$$\sigma_l = 1, \qquad \text{when} \quad \sum_{k=1}^{n} a_{nkl}u_k = 0.$$

Then

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl}u_k \right| = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl}u_k\right)\sigma_l \leqq \sum_{k=1}^{r} | u_k | \; \left| \sum_{l=1}^{n} a_{nkl}\sigma_l \right|$$

$$< U \sum_{k=1}^{n} \left| \sum_{l=1}^{n} a_{nkl}\sigma_l \right| < UC.$$

LEMMA 3. *Condition* $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl}\sigma_k \right| < C$ *of lemma 2 may be replaced by*

$$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl}\sigma_k\rho_l < C,$$

for all n, and in the case of each n, for every possible choice of σ_k and ρ_l, where $| \sigma_k | = 1$, $| \rho_l | = 1$, and C is a constant.

Assuming

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl}\sigma_k \right| < C,$$

we have

$$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl}\sigma_k\rho_l \leqq \sum_{l=1}^{n} \left| \rho_l \sum_{k=1}^{n} a_{nkl}\sigma_k \right| \leqq \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl}\sigma_k \right| < C.$$

Assuming $$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl}\sigma_k\rho_l < C,$$

define $$\rho_l = \operatorname{sgn} \sum_{k=1}^{n} a_{nkl}\sigma_k, \quad \text{when} \quad \sum_{k=1}^{n} a_{nkl}\sigma_k \neq 0,$$

$$\rho_l = 1, \qquad \text{when} \quad \sum_{k=1}^{n} a_{nkl}\sigma_k = 0.$$

Then $$\sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl}\sigma_k\right)\rho_l = \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl}\sigma_k \right| < C.$$

The fifth condition in Theorem I′ follows from the seventh condition in Theorem I, and conversely. We may therefore omit the fifth condition in both theorems. Referring then to the results in the preceding lemmas, we may state Theorems I and I′ together in the following form.

THEOREM II. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each k and l,* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each l,* $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each k,* $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl}\sigma_k\rho_l < C$,

for all n, and in the case of each n, for every possible choice of σ_k *and* ρ_l, *where* $|\sigma_k| = 1$, $|\rho_l| = 1$, *and C is a constant; and one or the other of*

$$\begin{cases} \sum_{k=1}^{n} |a_{nkl}| < L & \text{for all } l \text{ and } n, \\ \sum_{l=1}^{n} |a_{nkl}| < K & \text{for all } k \text{ and } n, \end{cases}$$

where K and L are constants.

2. *A necessary condition for regularity.* We notice that the last three conditions of Theorems I and I′ will follow from $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C$ for all n, where C is a constant. We can therefore state the following theorem.

THEOREM III. *A sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each k and l,* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each l,* $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each k,* $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C$ for all n, *where C is a constant.*

We have not been able to show that the last condition of this theorem is necessary for regularity of T. The following theorem indicates to what extent we have been able to state a necessary condition in terms of $|a_{nkl}|$.

THEOREM IV. *A necessary condition that the transformation T be regular is that* $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C(n)^{1/2}$ *for all n, where C is a constant.*

The proof of this theorem will be given at the end of this section after we have first stated certain lemmas. It is to be noted here that the last condition of Theorem II shows that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < Cn,$$

where C is a constant, is certainly necessary.

LEMMA 4. *Let* $G(x_1, \cdots, x_n)$ *denote the greatest of* $x_1, x_2, \cdots, x_n$, $x_1, x_2, \cdots, x_n$ *being real and non-negative. Then* $G(x_1, \cdots, x_n)$ *is continuous.*

A similar statement holds for $L(x_1, \cdots, x_n)$, the least of the numbers $x_1, x_2, \cdots, x_n$.

In the following lemmas we shall introduce the notation

$$\sum{}^{*} \mid \pm a_1 \pm a_2 \pm \cdots \pm a_n \mid$$

to denote the sum of absolute values for every possible combination of signs. For example

$$\sum{}^{*} \mid \pm a_1 \pm a_2 \mid = \mid a_1 + a_2 \mid + \mid a_1 - a_2 \mid + \mid -a_1 + a_2 \mid + \mid -a_1 - a_2 \mid .$$

LEMMA 5. *If any two a's in $\sum^{*} \mid \pm a_1 \pm \cdots \pm a_n \mid$ are unequal, the new summation in which each of these a's is replaced by their arithmetic mean, has a value not greater than the original sum.*

LEMMA 6. *The function,*

$$\frac{\sum^{*} \mid \pm a_1 \pm \cdots \pm a_n \mid}{\sum_{k=1}^{n} \mid a_k \mid},$$

of n real variables, $a_1, a_2, \cdots, a_n$, not all zero, takes on its minimum value when all of the a's are equal.

Let us investigate this minimum value by putting

$$a_k = 1, \qquad (k = 1, 2, \cdots, n).$$

Call

$$A_n = \sum{}^{*} \mid \pm 1 \pm 1 \pm \cdots \pm 1 \mid,$$
$$S_n = \mid 1 \mid + \mid 1 \mid + \cdots + \mid 1 \mid = n,$$
$$Q_n = A_n / S_n.$$

Then by lemma 6

$$\frac{\sum^{*} \mid \pm a_1 \pm \cdots \pm a_n \mid}{\sum_{k=1}^{n} \mid a_k \mid} \geqq Q_n$$

whatever the values of the a's may be. It is not difficult to show that

$$Q_n = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \qquad n \text{ even};$$

$$Q_n = \frac{2(n-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}, \qquad n \text{ odd};$$

and as a result †

$$Q_n \sim \frac{2^{n+1}}{(2n\pi)^{1/2}}.$$

† $\alpha_n \sim \beta_n$ means $\lim_{n\to\infty} (\alpha_n/\beta_n) = 1$.

LEMMA 7. *When the transformation T is regular and a_{nkl} is complex, $a_{nkl} = \alpha_{nkl} + i\beta_{nkl}$, then it is necessary that*

$$\sum_{l=1}^{n} \Big| \sum_{k=1}^{n} \alpha_{nkl} x_k \Big| < C,$$

and

$$\sum_{l=1}^{n} \Big| \sum_{k=1}^{n} \beta_{nkl} x_k \Big| < C,$$

for all n, and in the case of each n, for every possible choice of $x_k = \pm 1$ where C is a constant.

We are now ready to prove Theorem IV.

(a) Let a_{nkl} be real. By lemma 2, calling $\sigma_k = \pm 1$, we have

$$\sum_{l=1}^{n} \Big| \sum_{k=1}^{n} a_{nkl} x_k \Big| < C,$$

for all n, and in the case of each n, for every possible choice of signs, $x_k = \pm 1$, where C is a constant. It follows immediately that

$$\Sigma^{*}[\,|\pm a_{n11} \pm \cdots \pm a_{nn1}| + \cdots + |\pm a_{n1n} \pm \cdots \pm a_{nnn}|\,] < 2^n C,$$

for there are evidently 2^n choices of signs.

But $$\sum_{k=1}^{n} | a_{nkl} | \leqq (1/Q_n) \Sigma^{*} | \pm a_{n1l} \pm \cdots \pm a_{nnl} |$$

for each l; therefore

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | \leqq (1/Q_n) \sum_{l=1}^{n} \Sigma^{*} | \pm a_{n1l} \pm \cdots \pm a_{nnl} | < 2^n C/Q_n.$$

Referring to the formula for Q_n, we have immediately

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | \sim \frac{2^n C}{2^{n+1}/(2n\pi)^{1/2}} = C'(n)^{1/2},$$

where C' is a constant independent of n.

(b) By lemma 7, and as a result of part (a) just proved, it follows immediately that when $a_{nkl} = \alpha_{nkl} + i\beta_{nkl}$, then

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | < C''(n)^{1/2}.$$

[illegible]

$$S: \qquad y_n = \sum_{k=1}^{n} a_{nk} x_k, \qquad\qquad a_{nk} \text{ constant,}$$

will correctly evaluate the Cauchy product of two series each of which is Cèsaro summable. It will at times be convenient to write

$$a_{nk} = 0, \qquad\qquad (k > n).$$

Let $\sum u_n$ and $\sum v_n$ be two series, and let us call

$$U_n = \sum_{k=1}^{n} u_k, \qquad V_n = \sum_{k=1}^{n} v_k.$$

We shall write

$$U_0 = 0, \qquad V_0 = 0.$$

The Cauchy product of the two series is $\sum_{k=1}^{\infty} w_k$, where

$$w_k = \sum_{l=1}^{k} u_l v_{k-l+1}.$$

Let us write

$$W_n = \sum_{l=1}^{n} w_k = \sum_{l=1}^{n} u_l V_{n-l+1}.$$

The transformation S applied to W_n gives

$$y_n = \sum_{l=1}^{n} \sum_{k=1}^{n} (a_{n,k+l-1} - a_{n,k+l}) U_l V_k.$$

This is of the form T where $a_{nkl} = a_{n,k+l-1} - a_{n,k+l}$.

In defining Cèsaro summability of non-integral orders for the series $\sum u_n$, Chapman * considers

$$C_r: \qquad \phi_k = U_k^{(r)} / A_k^{(r)},$$

where

$$U_k^{(r)} = \sum_{l=1}^{k} \binom{r}{k-l} U_l,$$

and

$$A_k^{(r)} = \sum_{l=1}^{k} \binom{r}{k-l} = \binom{r+k-1}{k-1}.$$

Whenever

$$\lim_{k\to\infty} \phi_k = \lambda$$

the series $\sum u_n$ is said to be *summable* C_r to the value λ.

It is easy to obtain the inverse transformation

$$C_r^{-1}: \qquad U_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{r}{l-1} \binom{r+k-l}{k-l} \phi_{k-l+1}.$$

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* *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 9 (1910), p. 369.

LEMMA 8. *If transformation S evaluates to uv the Cauchy product of every two series summable to values u and v by C_r and C_s respectively, where $r > 0$, $s > 0$, then S is regular.*

Let $\sum u_n$ be any convergent series; it is summable C_r, $r > 0$. Let

$$\sum v_n = 1 + 0 + 0 + \cdots,$$

which is summable C_s. The Cauchy product of these series is $\sum u_n$. Hence S must be regular since it evaluates $\sum u_n$ correctly.

Let $\sum u_n$ be summable C_r, $r > 0$, to the value u, and let $\sum v_n$ be summable C_s, $s > 0$, to the value v. In the notation previously explained write

$$C_r: \qquad \phi_k = U_k^{(r)}/A_k^{(r)},$$

$$C_s: \qquad \psi_k = V_k^{(s)}/A_k^{(s)},$$

$$C_r^{-1}: \qquad U_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{r}{l-1} \binom{r+k-l}{k-l} \phi_{k-l+1},$$

$$C_s^{-1}: \qquad V_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{s}{l-1} \binom{s+k-l}{k-l} \psi_{k-l+1}.$$

Applying transformation S to the Cauchy product we have *

$$y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} (a_{n,k+l-1} - a_{n,k+l}) U_l V_k$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \phi_l \psi_k.$$

This latter form of y_n is obtained by substituting the values of U_l and V_k as above and carrying out the necessary reductions. We have now a case of transformation T wherein

$$a_{nkl} = \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1}.$$

The necessary and sufficient conditions that the transformation S evaluate correctly the Cauchy product of $\sum u_n$ and $\sum v_n$ are obtained by using this value of a_{nkl} in the conditions of Theorem II. The results thus obtained may be simplified † to the form given in the following theorem.

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* $\Delta^q x_p = \sum_{l=0}^{\infty} (-1)^l \binom{q}{l} x_{p+l}$, where $x_k = 0$ for sufficiently great values of k.

† For example

$$\sum_{l=1} \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} \text{[illegible]} \binom{r+l-1}{l-1} \Delta^{r} \text{[illegible]}.$$

THEOREM V. *A necessary and sufficient condition that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r, $r > 0$, and C_s, $s > 0$, to values u and v respectively, is that S be regular,*

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1}\sigma_k\rho_l < C,$$

for all n, and in the case of each n, for every possible choice of σ_k and ρ_l, where $|\sigma_k| = 1$, $|\rho_l| = 1$, and C is a constant; and one or the other of

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \left| \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C, \text{ for all } n \text{ and } l, \\ \displaystyle\sum_{l=1}^{n} \left| \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C, \text{ for all } n \text{ and } k. \end{cases}$$

By Theorem III we may evidently state a sufficient condition thus:

THEOREM VI. *A sufficient condition that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r, $r > 0$, and C_s, $s > 0$, to values u and v respectively, is that S be regular,*

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C,$$

for all n, where C is a constant.

Also, by Theorem IV, we may state a further necessary condition as follows:

THEOREM VII. *A necessary condition, (in addition to those of Theorem V), that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r and C_s to values u and v respectively is that*

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C(n)^{1/2},$$

for all values of n, where C is a constant.

4. *Conclusion.* There is still the possibility that the last condition of Theorem III may be necessary for regularity of T. If this is not the case it may happen that when this condition is replaced by that of Theorem IV

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whence

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} = \sum_{k=1}^{\infty} a_{nk}.$$

Also $\lim_{n\to\infty} \Delta^r a_{nl} = 0$, $r > 0$, for each l. Lemma 8 reduces the first four conditions obtained to the condition of regularity of S.

the set of conditions will then be necessary and sufficient. Both of these questions remain open.

In this connection it is interesting to note that regularity of S together with

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \binom{r+l-1}{l-1}\binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C$$

is a necessary and sufficient condition that S include C_{r+s+1}, when $r > 0$, $s > 0$.

If it should happen that the last condition of Theorem III be necessary, it would follow that a transformation S, evaluating correctly the Cauchy product of every two series summable C_r and C_s respectively, must include C_{r+s+1}. This would show the connection between our theory and Chapman's result that the Cauchy product of two series summable C_r and C_s respectively, is always summable C_{r+s+1}.

UNIVERSITY OF VERMONT.

ON SOME GENERAL COMMUTATION FORMULAS.*

By NEAL H. McCOY.†

Let x_i $(i = 1, 2, \cdots, n)$ be a class of elements satisfying the ordinary laws of algebra with the exception that multiplication is not necessarily commutative.‡ In the first part of this paper we give two identities for the commutator, $fg - gf$, where f is an arbitrary polynomial in the x's and g is a polynomial of the form $\sum a_{m_1 m_2 \ldots m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, each with real or complex coefficients. These identities are obtained in terms of expressions of the form $\phi x_i - x_i \phi$ where ϕ is a function of the x's, and are generalizations of some formulas given by Wentzel.§

As the first application of these identities we consider a special non-commutative algebra which arises in quantum mechanics.¶ For a single pair of quantum "variables," p and q, the properties of the algebra are determined by the fundamental rule,

$$pq - qp = cI, \tag{1}$$

where c is a real or complex number. As is well known these variables may be interpreted either as infinite matrices, in which case I in relation (1) indicates the unit matrix, or they may be certain operators and in this case I represents the unit operator. The results which we obtain are independent of the particular interpretation to be placed on the variables. We shall omit the symbol "I" in what follows as there can be no confusion.

If ϕ is a polynomial in p and q we define,

$$\phi p - p\phi = -c\partial\phi/\partial q, \qquad \phi q - q\phi = c\partial\phi/\partial p, \tag{2}$$

from which it follows that the usual formulas for differentiating polynomials hold.‖ By means of the general identities discussed above we can obtain two

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† National Research Fellow.

‡ By this statement we shall understand that the class of polynomials in the x's with real or complex coefficients constitute a non-commutative domain of integrity.

§ *Zeitschrift für Physik*, Vol. 37 (1926), p. 85.

¶ For references to this algebra see a previous paper, "On Commutation Formulas in the Algebra of Quantum Mechanics," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 793-806.

‖ Care must be taken to preserve the order of factors. That is,

$$\partial f\phi/\partial p = f\partial\phi/\partial p + (\partial f/\partial p)\phi.$$

formulas for the commutator of any two polynomials in p and q in terms of the various derivatives of these polynomials. These formulas are in a different form from those given previously. Corresponding results are obtained for the case of polynomials in n pairs of quantum variables, these being subject to the set of relations

$$(3) \qquad p_r q_s - q_s p_r = c\delta_{rs}, \qquad p_r p_s - p_s p_r = 0, \qquad q_r q_s - q_s q_r = 0.$$

We shall also give some commutation relations for functions of three variables, α, β, γ, which satisfy the conditions,

$$(4) \qquad \alpha\beta - \beta\alpha = c\gamma, \qquad \gamma\alpha - \alpha\gamma = c\beta, \qquad \beta\gamma - \gamma\beta = c\alpha.$$

These are the relations which hold for the components of angular momentum in quantum mechanics. As a special case it may be verified that the relations (4) are satisfied if we take

$$\alpha = q_2 p_3 - q_3 p_2, \qquad \beta = q_1 p_3 - q_3 p_1, \qquad \gamma = q_1 p_2 - q_2 p_1,$$

where p_1, p_2, p_3, q_1, q_2, q_3 are subject to the relations (3). As before, the results obtained depend only upon the fact that the variables satisfy relation (4) and not upon any special interpretation of the variables.

1. *Two general identities.* Let x_i $(i = 1, 2, \cdots, n)$ be the elements considered. These elements are assumed to satisfy the ordinary laws of algebra with the exception that multiplication is non-commutative. As a special case we may at any time let certain of these x's be identical.

Let f be any polynomial in the x's with real or complex coefficients. We define operators D_{x_i} as follows:

$$(5) \qquad f x_i - x_i f = c D_{x_i} f, \qquad (i = 1, 2, \cdots, n)$$

where c is a fixed real or complex number. As a consequence of the definition the following properties of the operators D_{x_i} may be deduced:

$$(6) \qquad D_{x_j}\phi = 0 \text{ if } \phi \text{ is a function of } x_j \text{ only,}$$
$$(7) \qquad D_{x_i}(f \pm g) = D_{x_i} f \pm D_{x_i} g,$$
$$(8) \qquad D_{x_i} fg = f D_{x_i} g + (D_{x_i} f) g.$$

It is clear from these properties that if $D_{x_i} x_j$ $(i, j = 1, 2, \cdots, n)$ are given then $D_{x_i} f$ is uniquely determined, where f is any polynomial in the x's.

We shall understand by $D_{x_i} D_{x_j} f$ the expression obtained by first operating [illegible]

That is

$$c^2 D_{x_i} D_{x_j} f = (f x_j - x_j f) x_i - x_i (f x_j - x_j f).$$

In like manner we define $D_{x_i}{}^2$, $D_{x_i}{}^3$, $D_{x_i}D_{x_j}D_{x_k}$ and so on. For convenience we let $D_{x_i}{}^0 f$ denote f itself, that is $D_{x_i}{}^0$ is the unit operator.

In what follows we let f represent an arbitrary polynomial in the x's. We now prove that *

$$fx_i^n - x_i^n f = \sum_{s=1}^{n} c^s \binom{n}{s} x_i^{n-s} D_{x_i}{}^s f. \tag{9}$$

This relation is seen to reduce to (5) if $n = 1$ and hence is true for this case by definition. We accordingly assume it to be valid for a given n and show that it holds also for $n+1$. We have

$$\begin{aligned} fx_i^{n+1} - x_i^{n+1} f &= (fx_i^n - x_i^n f)x_i + x_i^n(fx_i - x_i f) \\ &= \left[\sum_{s=1}^{n} c^s \binom{n}{s} x_i^{n-s} D_{x_i}{}^s f\right] x_i + c x_i^n D_{x_i} f. \end{aligned}$$

But $(D_{x_i}{}^s f)x_i - x_i(D_{x_i}{}^s f) = cD_{x_i}{}^{s+1} f$, by (5) and we get

$$fx_i^{n+1} - x_i^{n+1} f = \sum_{s=1}^{n} c^s \binom{n}{s} x_i^{n-s+1} D_{x_i}{}^s f + \sum_{s=1}^{n} c^{s+1} \binom{n}{s} x_i^{n-s} D_{x_i}{}^{s+1} f + c x_i^n D_{x_i} f.$$

By making use of the fact that

$$\binom{n}{s} + \binom{n}{s-1} = \binom{n+1}{s}, \tag{10}$$

it is easily verified that this is relation (9) with n replaced by $n+1$ which completes the proof of this formula. As a generalization of this result we have the theorem:

THEOREM I. *Let x_i $(i = 1, 2, \cdots, n)$ be any class of distinct elements satisfying the usual laws of algebra except that multiplication is not necessarily commutative. Let f be any polynomial in these x's and g a polynomial of the form $\sum a_{m_1 m_2 \ldots m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, each with real or complex coefficients. If the operators D_{x_i} $(i = 1, 2, \cdots, n)$ are defined by relation* (5) *then* (a)

$$fg - gf = \sum_{s=1} c^s \sum_{i_1+i_2+\cdots+i_n=s} \frac{1}{i_1!\, i_2! \cdots i_n!} \frac{\partial^s g}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \times D_{x_n}{}^{i_n} \cdots D_{x_2}{}^{i_2} D_{x_1}{}^{i_1} f, \tag{11}$$

and (b),

$$fg - gf = -\sum_{s=1} (-c)^s \sum_{i_1+i_2+\cdots+i_n=s} \frac{1}{i_1!\, i_2! \cdots i_n!} (D_{x_1}{}^{i_1} D_{x_2}{}^{i_2} \cdots D_{x_n}{}^{i_n} f) \frac{\partial^s g}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}, \tag{12}$$

the sum in each case being taken over all non-vanishing terms.

* See Wentzel, *loc. cit.* The proof given here is essentially that given by Wentzel.

The derivatives used in these formulas are to be taken according to the usual rules for differentiating polynomials. We consider first the proof of part (a) of the theorem. Clearly if formula (11) is true for two functions g_1 and g_2 of the type there considered it will also be true for their sum. It is then sufficient to establish it for the case of a single term. We shall show that,

$$(13)\quad fx_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}-x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}f$$
$$=\sum_{s=1} c^s \sum_{i_1+i_2+\cdots+i_n=s}\binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n}x_1^{m_1-i_1}x_2^{m_2-i_2}\cdots x_n^{m_n-i_n}\,D_{x_n}^{i_n}\cdots D_{x_2}^{i_2}D_{x_1}^{i_1}f.$$

When written in this form the result remains true if the x's are not all distinct. For example we may have $x_1=x_3=x_4$, $x_2=x_5$ and so on. The proof of relation (13) is by induction. It is true by equation (9) for the special case where all the m_i but one are zero and f is any polynomial. We assume then that the relation gives a true expression for $fx_i^nx_j^m-x_i^nx_j^mf$, and show that it remains true if n is replaced by $n+1$.

For convenience let $g'=x_i^nx_j^m$ and let

$$H^{(s)}(f,g')=\sum_{i_1+i_2=s}\binom{n}{i_1}\binom{m}{i_2}x_i^{n-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f.$$

Then equation (13) with g' in place of g can be written in the form,

$$(13')\qquad fg'-g'f=\sum_{s=1}c^sH^{(s)}(f,g'),$$

and we wish to show that this remains valid if g' is replaced by x_ig'. Now

$$H^{(s)}(f,x_ig')=\sum_{i_1+i_2=s}\binom{n+1}{i_1}\binom{m}{i_2}x_i^{n+1-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f$$
$$=x_iH^{(s)}(f,g')+\sum_{i_1+i_2=s}\binom{n}{i_1-1}\binom{m}{i_2}x_i^{n+1-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f$$

by formula (10). From this we find that

$$H^{(s)}(f,x_ig')=x_iH^{(s)}(f,g')+H^{(s-1)}(D_{x_i}f,g').$$

Hence we wish to verify that

$$fx_ig'-x_ig'f=x_i\sum_{s=1}c^sH^{(s)}(f,g')+\sum_{s=1}c^sH^{(s-1)}(D_{x_i}f,g').$$

But we have

$$fx_ig'-x_ig'f=x_i(fg'-g'f)+(fx_i-x_if)g'$$
$$=x_i\sum_{s=1}c^sH^{(s)}(f,g')+c(D_{x_i}f)g'.$$

Hence we only need to show that

$$(D_{x_i}f)g' = \sum_{s=1} c^{s-1}H^{(s-1)}(D_{x_i}f, g')$$

or

$$(D_{x_i}f)g' - g'(D_{x_i}f) = \sum_{s=1} c^{s}H^{(s)}(D_{x_i}f, g')$$

which is true as we have made no restrictions on the polynomial f and relation (13′) thus remains true if f is replaced by $D_{x_i}f$. By going through the argument just given for the case $n = 0$ it is seen to be valid for that case also. Hence we have by induction that relation (13) is true if two of the m_i are different from zero and the others vanish. A repetition of this argument proves the general case.

In order to prove relation (12) we need to show that

$$(14)\quad fx_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} - x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}f = -\sum_{s=1}(-c)^s \sum_{i_1+i_2+\cdots+i_n=s} \binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n}(D_{x_1}^{i_1}D_{x_2}^{i_2}\cdots D_{x_n}^{i_n}f) \times x_1^{m_1-i_1}x_2^{m_2-i_2}\cdots x_n^{m_n-i_n}.$$

We shall derive this identity from relation (13) which has been established.

The identity (13) is a formal identity, that is if the D's are replaced by their expressions from (5) everything will cancel out. Hence if we apply any kind of transformation to every term the result will still be an identity. Suppose then in (13) we reverse the order of all factors and change the sign of c. If ϕ is any function of the x's denote by $\bar{\phi}$ the function obtained from ϕ by this transformation. For example if

$$\phi = x_1x_3^2x_1x_2 + cx_1x_2x_3^2x_1^3,$$

then

$$\bar{\phi} = x_2x_1x_3^2x_1 - cx_1^3x_3^2x_2x_1.$$

From (5) it is seen that $\overline{D_{x_i}f} = D_{x_i}\bar{f}$, for

$$x_i\bar{f} - \bar{f}x_i = -c\overline{D_{x_i}f} = -cD_{x_i}\bar{f}.$$

Hence the result of applying this transformation to the identity (13) is to obtain the identity,

$$x_n^{m_n}\cdots x_2^{m_2}x_1^{m_1}\bar{f} - \bar{f}x_n^{m_n}\cdots x_2^{m_2}x_1^{m_1} = \sum_{s=1}(-c)^s \sum_{i_1+i_2+\cdots+i_n=s} \binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n}(D_{x_n}^{i_n}\cdots D_{x_2}^{i_2}D_{x_1}^{i_1}\bar{f}) \times x_n^{m_n-i_n}\cdots x_2^{m_2-i_2}x_1^{m_1-i_1}.$$

This becomes relation (14) if we replace the general polynomial $\bar{f}$ by f and in the subscripts replace n by 1, $n-1$ by 2 and so on, which obviously does not affect the truth of the relation. This completes the proof of Theorem I.

It is interesting to note that relation (11), for example, may be written symbolically as

$$fg - gf = \sum_{s=1} (c^s/s!)\,[(\partial g/\partial x_1) D_{x_1} f + (\partial g/\partial x_2) D_{x_2} f + \cdots + (\partial g/\partial x_n) D_{x_n} f]^s,$$

and similarly for relation (12).

2. *The algebra of quantum mechanics.* We now take up an application of the identities obtained above to a special algebra arising in quantum mechanics. Consider first a single pair of variables, p and q, satisfying the relation,

$$pq - qp = c, \tag{15}$$

where c is a real or complex number and thus commutes with any function of p and q.

It is found that in this case

$$D_p f = -\,\partial f/\partial q, \qquad D_q f = \partial f/\partial p,$$

by equations (2). We have further that

$$D_p D_q f = D_q D_p f, \tag{16}$$

a result which will greatly simplify the formulas obtained. The result (16) may be verified by substituting for D_p and D_q from (2). We have by making this substitution,

$$(fq - qf)p - p(fq - qf) = (fp - pf)q - q(fp - pf),$$

which may be easily verified by use of relation (15).

Let f be an arbitrary polynomial in p and q and let g take, for example, the special form, $p^{m_1} q^{m_2} p^{m_3} q^{m_4}$. We have then from formula (13),

$$fg - gf = \sum_{s-1} c^s \sum_{i_1+i_2+i_3+i_4=s} (-1)^{i_1+i_3} \binom{m_1}{i_1}\binom{m_2}{i_2}\binom{m_3}{i_3}\binom{m_4}{i_4} \times p^{m_1-i_1} q^{m_2-i_2} p^{m_3-i_3} q^{m_4-i_4} \frac{\partial^s f}{\partial p^{i_2+i_4} \partial q^{i_1+i_3}}.$$

But by Leibniz' formula for the k_1-th derivative of a product it is easily seen that,

$$\sum_{i_1+i_3=k_1} \binom{m_1}{i_1}\binom{m_3}{i_3} p^{m_1-i_1} q^{m_2} p^{m_3-i_3} q^{m_4} = \frac{1}{k_1!}\,\frac{\partial^{k_1} g}{\partial p^{k_1}},$$

[illegible]

$$\sum [illegible] \sum \binom{m_1}{i_1}\binom{m_2}{i_2}\binom{m_3}{i_3}\binom{m_4}{i_4} [illegible]$$

[illegible] theorem:

THEOREM II. *If f and g are arbitrary polynomials in p and q, then* (a)*

$$(17)\qquad fg - gf = \sum_{s=1} c^s \sum_{k_1+k_2=s} \frac{(-1)^{k_1}}{k_1!\,k_2!} \frac{\partial^s g}{\partial p^{k_1}\partial q^{k_2}} \frac{\partial^s f}{\partial q^{k_1}\partial p^{k_2}},$$

and (b),

$$(18)\qquad fg - gf = -\sum_{s=1} c^s \sum_{k_1+k_2=s} \frac{(-1)^{k_2}}{k_1!\,k_2!} \frac{\partial^s f}{\partial q^{k_1}\partial p^{k_2}} \frac{\partial^s g}{\partial p^{k_1}\partial q^{k_2}},$$

the sum in each case being taken over all non-null derivatives of f and g.

The first part of this theorem is seen by the argument just given to be valid if $g = p^{m_1}q^{m_2}p^{m_3}q^{m_4}$. The argument is however quite general and will apply to any term of a polynomial by simply using the formula for the n-th derivative of a product of any number of functions. Further if it is true for any two polynomials it is true for their sum which completes the proof of formula (17). Formula (18) is obtained in like manner from relation (14).

Let us consider polynomials in the $2n$ variables $p_1, q_1, \cdots, p_n, q_n$ satisfying the relations (3). In this case we have

$$p_r f - f p_r = c\partial f/\partial q_r = -cD_{p_r}f$$

and

$$fq_r - q_r f = c\frac{\partial f}{\partial p_r} = cD_{q_r}f.$$

It is again easily verified that $D_{p_r}D_{q_s}f = D_{q_s}D_{p_r}f$ $(r, s = 1, 2, \cdots, n)$. The following theorem may therefore be proved in a way similar to the proof of Theorem II.

THEOREM III. *If f and g are arbitrary polynomials in the $2n$ variables, $p_1, q_1, p_2, q_2, \cdots, p_n, q_n$ satisfying the relations* (3), *then* (a)

$$(19)\qquad fg - gf = \sum_{s=1} c^s \sum_{k_1+k_2+\cdots+k_{2n}=s} \frac{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}{k_1!\,k_2!\cdots k_{2n}} \frac{\partial^s g}{\partial p_1^{k_1}\partial q_1^{k_2}\cdots\partial p_n^{k_{2n-1}}\partial q_n^{k_{2n}}} \times \frac{\partial^s f}{\partial q_1^{k_1}\partial p_1^{k_2}\cdots\partial q_n^{k_{2n-1}}\partial p_n^{k_{2n}}},$$

and (b)

* In the previous paper referred to above the following expressions were obtained:

$$fg - gf = \sum_{s=1} \frac{c^s}{s!}\left[\frac{\partial^s g}{\partial q^s}\frac{\partial^s f}{\partial p^s} - \frac{\partial^s f}{\partial q^s}\frac{\partial^s g}{\partial p^s}\right]$$

and

$$fg - gf = \sum_{s=1} \frac{(-c)^s}{s!}\left[\frac{\partial^s g}{\partial p^s}\frac{\partial^s f}{\partial q^s} - \frac{\partial^s f}{\partial p^s}\frac{\partial^s g}{\partial q^s}\right].$$

We thus have four different forms for the commutator of any two polynomials in p and q

$$(20)\quad fg - gf$$
$$= -\sum_{s=1} c^s \sum_{k_1+k_2+\cdots+k_{2n}=s} \frac{(-1)^{k_2+k_4+\cdots+k_{2n}}}{k_1!\,k_2!\cdots k_{2n}!} \frac{\partial^s f}{\partial q_1^{k_1}\partial p_1^{k_2}\cdots\partial q_n^{k_{2n-1}}\partial p_n^{k_{2n}}}$$
$$\times \frac{\partial^s g}{\partial p_1^{k_1}\partial q_1^{k_2}\cdots\partial p_n^{k_{2n-1}}\partial q_n^{k_{2n}}},$$

the sum in each case being taken over all non-null derivatives of f and g.

3. *Another special non-cummutative algebra.* Let us consider functions of three elements or variables, α, β, γ which are subject to the conditions,

$$(21)\qquad \alpha\beta - \beta\alpha = c\gamma, \qquad \gamma\alpha - \alpha\gamma = c\beta, \qquad \beta\gamma - \gamma\beta = c\alpha.$$

We prove first the following theorem:

THEOREM IV. *Any identity in α, β, γ remains an identity if the order of all factors is reversed and c is replaced by $-c$.*

An identity of the form $\phi(\alpha,\beta,\gamma) - \phi(\alpha,\beta,\gamma) = 0$, will be called a formal identity. We here consider identities which may be obtained from formal identities by a finite number of substitutions from (21), which is true for all polynomial identities. The theorem is clearly true for formal identities. Hence we need only to show that if it is true for a given identity it remains true after making any one of the substitutions of (21). If ϕ is any function of α, β, γ, c let $\bar{\phi}$ denote the function obtained from ϕ by reversing the order of all factors and changing the sign of c. Let $f = 0$ be an identity such that $\bar{f} = 0$ is also true. In any term of f replace, for example, $\alpha\beta$ by $\beta\alpha + c\gamma$ from the first of relations (21) and denote by $f' = 0$ the resulting identity. Now $\bar{f}'$ differs from $\bar{f}$ only in that we have replaced in one term $\beta\alpha$ by $\alpha\beta - c\gamma$ and these are equivalent. Hence we have also $\bar{f}' = 0$. A similar argument holds for any of the substitutions obtainable from (21). In thus building up a given identity from a formal identity the theorem is true at each step and hence for the final identity.

THEOREM V. *In any identity replace α, β, γ, c by α', β', γ', kc respectively, where α', β', γ' are obtained from α, β, γ by the non-singular transformation with real or complex coefficients*

$$\alpha' = a_{11}\alpha + a_{12}\beta + a_{13}\gamma, \quad \beta' = a_{21}\alpha + a_{22}\beta + a_{23}\gamma, \quad \gamma' = a_{31}\alpha + a_{32}\beta + a_{33}\gamma$$

of determinant Δ, and k is a real or complex number. The result will be another identity if and only if [illegible] *where A_{ij} denotes the co-factor of a_{ij} in Δ.*

We have

$$\begin{aligned}\alpha'\beta'-\beta'\alpha' &= (a_{11}a_{22}-a_{21}a_{12})(\alpha\beta-\beta\alpha)\\ &\quad -(a_{11}a_{23}-a_{21}a_{13})(\gamma\alpha-\alpha\gamma)+(a_{12}a_{23}-a_{22}a_{13})(\beta\gamma-\gamma\beta)\\ &= c\left[(a_{11}a_{22}-a_{21}a_{12})\gamma+(a_{21}a_{13}-a_{11}a_{23})\beta+(a_{12}a_{23}-a_{22}a_{13})\alpha\right]\\ &= ck(a_{33}\gamma+a_{32}\beta+a_{31}\alpha)=ck\gamma',\end{aligned}$$

if the conditions of the theorem are satisfied. In like manner we find that

$$\gamma'\alpha'-\alpha'\gamma'=ck\beta' \quad \text{and} \quad \beta'\gamma'-\gamma'\beta'=ck\alpha'.$$

Thus α', β', γ' satisfy the same identities as α, β, γ with c replaced by kc which proves that the condition of the theorem is sufficient. If the matrix of the transformation is not of the prescribed form the theorem will fail when applied to the fundamental identities (21). It is easily seen that if $a_{ij}=kA_{ij}$, then k must be a cube root of $1/\Delta$.

As special cases of this theorem we find that from any identity we may obtain another (a) by cyclic permutation of the letters α, β, γ or (b) by interchanging any two letters and changing the sign of c. These may be seen by considering the identities (21) directly.

It may be shown by induction that

$$D_\gamma\alpha^n=-\sum_{s=0}^{n-1}\alpha^{n-s-1}\beta\alpha^s, \quad \text{and} \quad D_\gamma\beta^n=\sum_{s=0}^{n-1}\beta^{n-s-1}\alpha\beta^s. \tag{22}$$

The other operators are obtained from these by cyclic permutation. Let us prove the second of equations (22). Assuming it holds for n we find:

$$\begin{aligned}cD_\gamma\beta^{n+1} &= \beta^{n+1}\gamma-\gamma\beta^{n+1}=\beta(\beta^n\gamma-\gamma\beta^n)+(\beta\gamma-\gamma\beta)\beta^n\\ &= c\,\{\sum_{s=0}^{n-1}\beta^{n-s}\alpha\beta^s+\alpha\beta^n\}=c\sum_{s=0}^{n}\beta^{n-s}\alpha\beta^s.\end{aligned}$$

But the relation is easily seen to be true for $n=1$ which completes the proof. The formulas (22) together with (6), (7) and (8) may be taken as the definition of $D_\gamma f$ where f is any polynomial in α, β, γ. Other expressions for these operators will be given later.

We find that $D_\alpha D_\beta f\neq D_\beta D_\alpha f$ but that

$$\begin{aligned}c^2(D_\alpha D_\beta f-D_\beta D_\alpha f) &= (f\beta-\beta f)\alpha-\alpha(f\beta-\beta f)\\ &\quad -(f\alpha-\alpha f)\beta+\beta(f\alpha-\alpha f)\\ &= f(\beta\alpha-\alpha\beta)+(\alpha\beta-\beta\alpha)f\\ &= -c(f\gamma-\gamma f)=-c^2D_\gamma f.\end{aligned}$$

In like manner each of the following relations may be verified:

$$(D_\alpha D_\beta-D_\beta D_\alpha)f=-D_\gamma f, \qquad (D_\gamma D_\alpha-D_\alpha D_\gamma)f=-D_\beta f,$$
$$(D_\beta D_\gamma-D_\gamma D_\beta)f=-D_\alpha f.$$

Thus the operators D_α, D_β, D_γ satisfy relations of the type (21) with c replaced by -1. It follows that corresponding to any identity expressed in terms of α, β, γ, c, there corresponds another in D_α, D_β, D_γ, -1, which may be interpreted as operating on an arbitrary function of α, β, γ.

Two general commutation formulas for polynomials in these variables are given in the following theorem which is an immediate consequence of Theorem I. Other formulas may be obtained by applying the transformations of Theorems IV and V to those given here.

THEOREM VI. *If f is an arbitrary polynomial in α, β, γ; g is a polynomial of the form $\sum a_{lmn}\alpha^l\beta^m\gamma^n$, and the operators D are defined by* (22), (6), (7) *and* (8), *then* (a)

$$(23) \qquad fg - gf = \sum_{s=1} c^s \sum_{i+j+k=s} \frac{1}{i!\,j!\,k!} \frac{\partial^s g}{\partial\alpha^i\partial\beta^j\partial\gamma^k} D_\gamma{}^k D_\beta{}^j D_\alpha{}^i f,$$

and (b)

$$(24) \qquad fg - gf = -\sum_{s=1} (-c)^s \sum_{i+j+k=s} \frac{1}{i!\,j!\,k!} (D_\alpha{}^i D_\beta{}^j D_\gamma{}^k f) \frac{\partial^s g}{\partial\alpha^i\partial\beta^j\partial\gamma^k}.$$

As an interesting special case of formula (24) let $f=\gamma$. Then $D_\gamma{}^k\gamma = 0$ $(k>0)$ and hence we only need to calculate $D_\alpha{}^i D_\beta{}^j\gamma$. The following table gives the values of this expression for $i, j = 0, 1, 2, \cdots, 5$.

$i \backslash j$	0	1	2	3	4	5
0	γ	$-\alpha$	$-\gamma$	α	γ	$-\alpha$
1	β	0	$-\beta$	0	β	0
2	$-\gamma$	0	γ	0	$-\gamma$	0
3	$-\beta$	0	β	0	$-\beta$	0
4	γ	0	$-\gamma$	0	γ	0
5	β	0	$-\beta$	0	β	0

$D_\alpha{}^i D_\beta{}^j \gamma$.

It is also seen that $D_\alpha{}^i D_\beta{}^j\gamma = D_\alpha{}^{i+4} D_\beta{}^j\gamma = D_\alpha{}^i D_\beta{}^{j+4}\gamma$ for $i, j > 0$. We thus find from (24),

$$\begin{aligned}(25) \quad \gamma g - g\gamma &= c(\beta\partial g/\partial\alpha - \alpha\partial g/\partial\beta) + (c^2/2!)(\gamma\partial^2 g/\partial\alpha^2 + \gamma\partial^2 g/\partial\beta^2) \\ &+ (c^3/3!)(-\beta\partial^3 g/\partial\alpha^3 - 3\beta\partial^3 g/\partial\alpha\partial\beta^2 + \alpha\partial^3 g/\partial\beta^3) \\ &+ (c^4/4!)(-\gamma\partial^4 g/\partial\alpha^4 - 6\gamma\partial^4 g/\partial\alpha^2\partial\beta^2 - \gamma\partial^4 f/\partial\beta^4) \\ &+ (c^5/5!)(\beta\partial^5 g/\partial\alpha^5 + 10\beta\partial^5 g/\partial\alpha^3\partial\beta^2 + 10\beta\partial^5 g/\partial\alpha\partial\beta^4 - \alpha\partial^5 g/\partial\beta^5) + \cdots.\end{aligned}$$

If ϕ is a polynomial in α alone we get as a special case from (25),

$$\begin{aligned}(26) \quad \gamma\phi - \phi\gamma &= \beta[c\,d\phi/d\alpha - (c^3/3!)d^3\phi/d\alpha^3 + (c^5/5!)d^5\phi/d\alpha^5 - \cdots] \\ &+ \gamma[(c^2/2!)d^2\phi/d\alpha^2 - (c^4/4!)d^4\phi/d\alpha^4 + (c^6/6!)d^6\phi/d\alpha^6 - \cdots].\end{aligned}$$

From this it follows that

$$-cD_\gamma \alpha^n = \sum_{s=1} (-1)^{s+1} c^{2s-1} \binom{n}{2s-1} \beta \alpha^{n-2s+1} + \sum_{s=1} (-1)^{s+1} c^{2s} \binom{n}{2s} \gamma \alpha^{n-2s}.$$

Now as in the ordinary case we have

$$\phi(\alpha + a) = \phi(\alpha) + a\partial\phi(\alpha)/\partial\alpha + (a^2/2!)\partial^2\phi(\alpha)/\partial\alpha^2 + \cdots$$

where a is a real or complex number. By making use of this fact it may be verified from (26) that if $i = (-1)^{1/2}$, then

$$\begin{aligned}\gamma\phi(\alpha) - \phi(\alpha)\gamma = (\beta/2i)\,[\phi(\alpha + ci) - \phi(\alpha - ci)]\\ -(\gamma/2)\,[\phi(\alpha + ci) + \phi(\alpha - ci) - 2\phi(\alpha)].\end{aligned}$$

From each of the relations here deduced one may obtain others by cyclic permutation of the letters or by interchanging two letters and changing the sign of c.

PRINCETON UNIVERSITY.

E H Moore June 1930

NEW FOUNDATION OF EUCLIDEAN GEOMETRY.

By Karl Menger.

My second paper on metrical geometry * contains a characterisation of the n-dimensional euclidean space among general semi-metrical spaces in terms of relations between the distances of its points. In courses on metrical geometry at American universities I have considerably shortened and revised my original proofs and generalized the formulations by introducing the concept of congruence order. The following paper contains these new proofs. In the first part we prove that every semi-metrical space, each $n+3$ points of which are congruent with $n+3$ points of the n-dimensional euclidean space, is congruent with a subset of the n-dimensional euclidean space. This is expressed by saying that the n-dimensional euclidean space has the congruence order $n+3$. In the second part we prove that each semi-metrical space containing more than $n+3$ points each $n+2$ points of which are congruent with $n+2$ points of the n-dimensional euclidean space, is congruent with a subset of the n-dimensional euclidean space. This fact is expressed by saying that the R_n has the quasi-congruence order $n+2$. It is proved by a systematic study of those sets which contain exactly $n+3$ points and are not congruent with $n+3$ points of the n-dimensional euclidean space whereas each $n+2$ of them are congruent with $n+2$ points of the n-dimensional euclidean space. These sets are called pseudo-euclidean sets. By means of these results the problem is reduced to the question: under what conditions are $n+2$ points congruent with $n+2$ points of the R_n and by what distance relations are the pseudo-euclidean $(n+3)$-tuples characterized. These purely algebraic problems are solved in the third part.†

I. Congruence Systems and Congruence Order of the R_n. By a *congruence system* is meant a system $\mathfrak{S}$ of sets and a relation $\simeq$ (called the congruence relation) that satisfy the following five postulates:

Postulate 1. If p, q are two points of a set M of $\mathfrak{S}$, and p', q' are two points of a set M' of $\mathfrak{S}$ (M' not necessarily distinct from M) then, either $p, q \simeq p', q'$ or $p, q \not\simeq p', q'$, where the relation $\not\simeq$ is the negative of the

* Cp. [illegible] Vol. [illegible]

† I wish to express my thanks [illegible] for [illegible] remarks concerning [illegible]

Postulate 2. For each two points p, q of a set of $\mathfrak{S}$, $p, q \approx q, p$.

Postulate 3. For each point p of a set of $\mathfrak{S}$ and each two points q and r of a set of $\mathfrak{S}$, $p, p \approx q, r$ if and only if $q = r$.

Postulate 4. If for two pairs of points p, q; p', q' of two sets of $\mathfrak{S}$, $p, q \approx p', q'$ then $p', q' \approx p, q$.

Postulate 5. If for three pairs of points p, q; p', q'; p'', q'' of three sets of $\mathfrak{S}$, $p, q \approx p', q'$ and $p, q \approx p'', q''$, then $p', q' \approx p'', q''$.

A set S is called a *semi-metrical space* provided that to each two elements p, q of S there corresponds a not negative real number, called the distance between the points p and q, such that, if we denote this number by pq, we have $pq = qp$, and $pq = 0$ if and only if the points p and q are identical.

In particular, for each integer n the R_n (i. e., the n-dimensional Euclidean space) is a semi-metrical space. The system of all semi-metrical spaces is a congruence system if the relation $p, q \approx p', q'$ is valid if and only if for the distances pq, $p'q'$ the equality $pq = p'q'$ subsists.

A mapping of the set S of the congruence system $\mathfrak{S}$ upon the set S' of $\mathfrak{S}$ is called a *congruent mapping* if to each pair of points of S there corresponds a congruent pair of points of S'. (A congruent mapping is, *eo ipso*, one-to-one). Two sets of a congruence system are called *congruent* if there exists a congruent mapping of one upon the other.

A set M of a congruence system $\mathfrak{S}$ has the *congruence order* n provided that *each set of $\mathfrak{S}$, each n points of which are congruent to n points of M, is congruent to a subset of M.* Evidently, each set of the congruence order n has also congruence order m, if m is an integer greater than n. For example, the R_0 (i. e., the space consisting of a single point) has the congruence order 2. In general, each set R of a congruence system $\mathfrak{S}$, consisting of n points has the congruence order $n+1$, for if S is a set of $\mathfrak{S}$, each $n+1$ points of which are congruent to $n+1$ points of R, then S cannot contain more than n points.

Let S and R be two sets of a congruence system. If R has the congruence order n, the set S is said to be *super-ordered* with respect to R if the following conditions are satisfied:

(a) S contains a subset congruent to R.

(b) Each set of n points of S, every $(n-1)$ of which are congruent to $(n-1)$ points of R, is congruent to n points of R.

(c) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of S, and $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$

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are $(n-1)$ points of S congruent to the first set of $(n-1)$ points, then to each point s of S there exists at least one point $\bar{s}$ of S such that

$$p_1, p_2, \cdots, p_{n-1}, s \simeq \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}, \bar{s}.$$

(*d*) If $p_1, p_2, \cdots, p_{n-1}$ and $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ are congruent sets of points of S, the first set not congruent to $(n-1)$ points of R, and if s, t and $\bar{s}, \bar{t}$ are two pairs of points of S such that

$$s, p_1, p_2, \cdots, p_{n-1} \simeq \bar{s}, \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$$
$$t, p_1, p_2, \cdots, p_{n-1} \simeq \bar{t}, \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$$

then $s, t \simeq \bar{s}, \bar{t}$.

It is to be observed that applying condition (d) to the case in which the points s and t are identical we see that the point s of condition (*c*) is uniquely determined. This remark we shall denote by (c'). It is of great use in what follows.

Remark 1. If R is the $(n-3)$-dimensional Euclidean space, and S is the $(n-2)$-dimensional Euclidean space, then R and S satisfy the conditions (*a*) — (*d*).*

(*a*) Each $(n-3)$-dimensional plane of the R_{n-2} is congruent to R_{n-3}.

(*b*) Each set of n points of the R_{n-2}, every $(n-1)$ of which are congruent to $(n-1)$ points of the R_{n-3} is congruent to n points of R_{n-3}.

(*c*) In applying a congruent self-transformation to the R_{n-2} which transforms $p_1, p_2, \cdots, p_{n-1}$ into $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ respectively, the point s is transformed into a point $\bar{s}$ satisfying the condition.

(*d*) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of the R_{n-2} not lying in an $(n-3)$-dimensional plane, then if $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ are congruent to $p_1, p_2, \cdots, p_{n-1}$ there exists only one congruent self-transformation of the R_{n-2} transforming $p_1, p_2, \cdots, p_{n-1}$ into $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. Applying this self-transformation to the R_{n-2} the points s and t are necessarily transformed into $\bar{s}$ and $\bar{t}$ respectively; for the points into which s and t are transformed have the same distances from $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$ as $\bar{s}$ and $\bar{t}$ have, and the R_{n-2} contains at most one point with $(n-1)$ given distances from $(n-1)$ points not lying in an $(n-3)$-dimensional plane.

Let us denote by S_n the n-dimensional sphere; for example, S_0 is a pair of points, S_1 is the circumference of a circle, S_2 is the surface of the sphere in three-space, etc. The section of an S_n by an m-dimensional plane $(m \leq n)$

* As soon as we shall have proved that the R_{n-3} has the congruence order n, it will follow from this remark that the R_{n-2} is super-ordered with respect to the R_{n-3}.

through the center of the $(n+1)$-dimensional sphere whose surface is S_n is called an $(m-1)$-dimensional great sphere of S_n.

Remark 2. If R is the S_{n-3}, and S the S_{n-2}, then the conditions (*a*)—(*d*) are satisfied.

(*a*) Each $(n-3)$-dimensional great sphere of S_{n-2} is congruent to S_{n-3}.

(*b*) Each set of n points of S_{n-2}, every $(n-1)$ of which are congruent to $(n-1)$ points of S_{n-3}, is congruent to n points of S_{n-3}.

(*c*) We may transform the sphere S_{n-2} congruently so that the points $p_1, p_2, \cdots, p_{n-1}$ are transformed into the congruent set $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. (If the two sets of points form two symmetric polygons, this transformation may be accomplished by means of a perspectivity with the center of the sphere as center of perspectivity). Then the point s is transformed into a point $\bar{s}$ satisfying the condition.

(*d*) If $p_1, p_2, \cdots, p_{n-1}$ are $(n-1)$ points of the S_{n-2} not lying in a $(n-3)$-dimensional great sphere, there exists only one congruent self-transformation of the S_{n-2} transforming the points $p_1, p_2, \cdots, p_{n-1}$ into the congruent set $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n-1}$. The points s and t are transformed by this transformation into points $\bar{s}$ and $\bar{t}$ respectively, with $s, t \approx \bar{s}, \bar{t}$.

We now prove the

THEOREM. *If R has the congruence order n and S is super-ordered with respect to R, then S has the congruence order $n+1$.*

Given a set S', each $n+1$ points of which are congruent to $n+1$ points of S, we must show that S' is congruent to a subset of S. Now, since each $n+1$ points of S' are congruent to $n+1$ points of S, *a fortiori*, each n points of S' are congruent to n points of S.

Suppose, first, that each n points of S' are congruent with n points of R. Then since R has the congruence order n, S' is congruent with a subset of R. But S contains a subset congruent to R (condition (*a*)). Hence, S' is congruent to a subset of S, and the theorem is proved for this case.

Suppose, now, that there exist n points of S' that are not congruent to n points of R. There exist, however, n points of S congruent to these n points of S'. Among these n points of S there are $(n-1)$ points that are not congruent to $(n-1)$ points of R (since otherwise, by condition (*b*), the n points of S would be congruent to n points of R, and the n points of S' would be congruent to n points of R).

We may label the n points of S congruent to these n points of S' so that

$$(*) \qquad s_1, s_2, \cdots, s_{n-1}$$

are not congruent to $(n-1)$ points of R. Let $s'_1, s'_2, \cdots, s'_{n-1}$ be the $(n-1)$ points of S' congruent to the points $(*)$ of S.

We construct, now, a mapping of S' upon a subset of S. Let s' be an arbitrarily chosen point of S'. Consider the set $s'_1, s'_2, \cdots, s'_{n-1}, s'$. These points are congruent to some n points of S, say, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$. Now the set $s'_1, s'_2, \cdots, s'_{n-1}$ is congruent also to the set $(*)$. Hence

$$s_1, s_2, \cdots, s_{n-1} \approx \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}$$

where the first $(n-1)$-tuple is not congruent to $(n-1)$ points of R.

Now, according to the remark (c'), there is exactly one point s of S such that

$$s_1, s_2, \cdots, s_{n-1}, s \approx \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$$

and hence

$$s_1, s_2, \cdots, s_{n-1}, s \approx s'_1, s'_2, \cdots, s'_{n-1}, s'.$$

The point s determined in this way we take as the image in S of the point s' for each point s' of S'.

We propose that this mapping of S' on a subset of S is a congruent one. To show this, let s' and t' be two points of S', and let s and t be their images in S. We prove that $s, t \approx s', t'$. Since each $n+1$ points of S are congruent to $n+1$ points of S', there exist some $n+1$ points of S, say, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}, \bar{t}$ congruent with the $n+1$ points, $s'_1, s'_2, \cdots, s'_{n-1}, s', t'$ of S'. Now, $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1} \approx s_1, s_2, \cdots, s_{n-1}$. Furthermore, since $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}\bar{s} \approx s'_1, s'_2, \cdots, s'_{n-1}, s'$, there subsists the relation $s_1, s_2, \cdots, s_{n-1}, s \approx \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{s}$, where the first $(n-1)$-tuple is not congruent to $(n-1)$ points of R.

In entirely analogous fashion, we show that $s_1, s_2, \cdots, s_{n-1}, t \approx \bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{n-1}, \bar{t}$. Hence, we conclude from condition (d) that $s, t \approx \bar{s}, \bar{t}$ and as $\bar{s}, \bar{t} \approx s', t'$, it follows that $s, t \approx s', t'$.

Thus we have shown that the mapping of S' upon a subset of S is a congruent one, and the theorem is proved.

We shall prove now:

First fundamental theorem. *The R_n and the S_n have the congruence order $n+3$ but, except in the case of the R_0, not the congruence order $n+2$.*

As previously remarked, the R_0 has the congruence order 2 and, therefore, the congruence order 3. In order to prove that the R_n has the congruence order $n+3$ we show: If the R_{n-3} has the congruence order n, then the R_{n-2} has the congruence order $n+1$. But if we assume that the R_{n-3} has the congruence order n it follows from Remark 1 that the R_{n-2} is superordered

with respect to the R_{n-3}. Hence according to the Theorem we have proved, the R_{n-2} has the congruence order $n+1$. In the same way utilizing Remark 2 we conclude that the S_{n-2} has the congruence order $n+1$ if the S_{n-3} has the congruence order n. As the S_0 which consists of two points has the congruence order 3 it follows that the S_n has the congruence order $n+3$.

Let us now prove that the S_n has not a congruence order less than $n+3$. It is easy to prove that S_n contains $n+2$ points each two of which have the same distance, let us say, d, whereas S_n does not contain $n+3$ points, each two of which have the same distance. If, now, T is a set consisting of k points each two of which have the same distance d, then each $n+2$ points of T are congruent with $n+2$ points of S_n, whereas T, if $k \geq n+3$ is not congruent to a subset of S_n. Hence S_n has not the congruence order $n+2$.

Let us denote by $p_1, p_2, \cdots, p_{n+1}$ $n+1$ points of the R_n, each two of which have the same distance, say, d. We denote by d' the distance of the point p_{n+2} which is the center of the $(n-1)$-dimensional sphere that circumscribes the points $p_1, p_2, \cdots, p_{n+1}$, and by d'' the distance of p_{n+2} from the plane through $p_1, p_2, \cdots, p_n$. Then, let P be a metrical space consisting of $n+3$ points, $\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3}$ such that

$$\bar{p}_i\bar{p}_j = d, \qquad (i, j \leq n+1)$$
$$\bar{p}_{n+2}\bar{p}_i = \bar{p}_{n+3}\bar{p}_i = d' \qquad (i = 1, 2, \cdots, n+1).$$
$$\bar{p}_{n+2}\bar{p}_{n+3} = 2d''.$$

It is easy to prove that each $n+2$ points of P are congruent with $n+2$ points of the R_n, whereas P is not congruent with a subset of the R_n. Hence the R_n has not the congruence order $n+2$.

2. Pseudo-Euclidean Sets and Quasi-congruence Order. We have seen that there exists a set P which is not congruent to a subset of R_n whereas each $n+2$ points of P are congruent to $n+2$ points of the R_n. This set consisted of exactly $n+3$ points. We shall prove now in general:

Each set containing more than $n+3$ points, each $n+2$ of which are congruent to $n+2$ points of the R_n, is congruent to a subset of the R_n. We could say that a set S of a congruence system has *the quasi-congruence order k if each set containing more than $k+1$ points, each k points of which are congruent to k points of S, is congruent to a subset of S.* Then evidently, each set of quasi-congruence order k has the congruence order $k+1$. Conversely, however, the S_n is a space of congruence order $n+3$ which has not the quasi-congruence order $n+2$, for we have seen that for each integer k there exists a set T containing k points which is not congruent with a

subset of S_n whereas each $n+2$ points of T are congruent with $n+2$ points of S_n. Utilizing the concept of quasi-congruence order, we shall prove the

SECOND FUNDAMENTAL THEOREM. *The R_n but not the S_n has the quasi-congruence order $n+2$.*

The R_0 has the congruence order 2 and hence, a fortiori, the quasi-congruence order 2. We shall prove the theorem by complete induction, i. e. under the assumption of its truth for the R_{n-1}.

The following terminology will be useful: We call $n+1$ points of the R_n *independent* if and only if they do not lie in a $(n-1)$-dimensional plane. We say in this case also that each of these $n+1$ points is independent of the n others. More generally we shall call a set of a congruence system which is congruent with $n+1$ independent points of the R_n an independent $(n+1)$-tuple. A set of a congruence system that consists of $n+1$ points which are congruent with $n+1$ points of a R_{n-1} will be called a *dependent* $(n+1)$-tuple. We say, furthermore, that n points $p_1, \cdots, p_n$ of the R_n are independent if and only if there exists a point p_{n+1} of the R_n such that the points $p_1, \cdots, p_n, p_{n+1}$ are independent. In this notation the properties (a) — (d) of the R_n which guaranteed its congruence order $n+3$ can be formulated in the following way:

(a) The $(n-1)$-dimensional planes of the R_n are congruent with the R_{n-1}.

(b) $n+2$ points of the R_n each $n+1$ of which are dependent lie in a $(n-1)$-dimensional plane.

(c) If in the R_n the two $(n+1)$-tuples $p_1, \cdots, p_{n+1}$ and $\bar{p}_1, \cdots, \bar{p}_{n+1}$ are congruent then there exists for each point p of the R_n at least one point $\bar{p}$ of the R_n such that $p_1, \cdots, p_{n+1}, p \approx \bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}$.

(c') If the points $p_1, \cdots, p_{n+1}$ are independent than the point $\bar{p}$ in condition (c) is uniquely determined.

(d) If the points $\bar{p}_1, \cdots, \bar{p}_{n+1}$ of the R_n are independent and $\bar{p}_{n+2}$, $\bar{p}_{n+3}$ and $p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+3}$ are points of the R_n such that

$$\bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \approx p_1, \cdots, p_{n+1}, p_{n+2}$$
$$\bar{p}_1, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \approx p_1, \cdots, p_{n+1}, p_{n+3}$$

then
$$\bar{p}_{n+2}\bar{p}_{n+3} \approx p_{n+2}p_{n+3}.$$

An immediate consequence of (c) is besides

(c'') If in the R_n the two n-tuples $p_1, \cdots, p_n$ and $\bar{p}_1, \cdots, \bar{p}_n$ are congruent then there exists for each two points p and q of the R_n at least one point $\bar{p}$ and at least one point $\bar{q}$ such that

$$p_1, \cdots, p_n, p, q \approx \bar{p}_1, \cdots, \bar{p}_n, \bar{p}, \bar{q}.$$

Besides these properties of the R_n we shall make use in what follows of some other properties viz:

(e) Two congruent $(n+1)$-tuples of the R_n are either both independent or both dependent.

(f) If the points $p_1, \cdots, p_n$ of the R_n are independent (i. e. do not lie in a $(n-2)$-dimensional plane) then there exists for each point p which is independent of them at least one point $p^0 \neq p$ such that

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0.$$

(g) If $p_1, \cdots, p_n$ are independent points of the R_n and

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0$$
$$p_1, \cdots, p_n, q \approx p_1, \cdots, p_n, q^0$$

where p^0 is either dependent on $p_1, \cdots, p_n$ or distinct from p, and where q^0 is either dependent on $p_1, \cdots, p_n$ or distinct from q, then

$$p_1, \cdots, p_n, p, q \approx p_1, \cdots, p_n, p^0, q^0.$$

If in condition (g) p and q are identical it follows that condition (f) can be sharpened to

(f') The point p^0 in condition (f) is uniquely determined.

(h) If in the R_n the points $p_1, \cdots, p_n$ are independent and p is dependent on them, then from

$$p_1, \cdots, p_n, p \approx p_1, \cdots, p_n, p^0$$

it follows that $p = p^0$.

(i) * If in the R_n the points $p_1, \cdots, p_{n+1}$ are independent and p is some other point and if we denote for $k = 1, 2, \cdots, n+1$ by p^k a point such that

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p$$

and such that $p^k \neq p$ if p is independent of $p_1, \cdots, p_{n+1}$ then there exists at most one point c of the R_n such that

$$c, p^1 \approx c, p^2 \approx \cdots \approx c, p^{n+1}.$$

Proof: The $(n+1)$ independent points $p_1, p_2, \cdots, p_{n+1}$ in R_n determine a simplex. The $(n+1)$ points $p^1, p^2, \cdots, p^{n+1}$ obtained as the mirror images of an $(n+2)$ *nd* point, p, in the $(n+1)$ planes of the simplex lie, in general,

* Property (i) has been found and proved by Leonard M. Blumenthal.

in R_n, and not in a space of lower dimensions. There is then but one point, c, of the R_n satisfying the relation

$$(*) \qquad c, p^1 \approx c, p^2 \approx \cdots \approx c, p^{n+1}.$$

It may happen, however, that the points $p^1, p^2, \cdots, p^{n+1}$ lie in an $(n-1)$-dimensional plane (the generalization of the Simpson Line that occurs in the R_2). If the points do not lie in an $(n-2)$-dimensional plane, then there is no point, c, satisfying $(*)$, in the $(n-1)$-dimensional plane bearing the points; for if such a point did exist, it is clear that the planes in which p is reflected would all pass through a point (the center of the n-dimensional sphere determined by the points $p^1, p^2, \cdots, p^{n+1}$ and the point p). It follows in this case that no point in R_n has the property $(*)$ since the orthogonal projection of such a point upon the plane containing $p^1, p^2, \cdots, p^{n+1}$ would be a point in this plane satisfying $(*)$.

If now, we let $k \leq n-1$ be the smallest dimension of the plane containing $p^1, p^2, \cdots, p^{n+1}$, then for $k < n-1$, the mirror planes are all orthogonal to this plane and hence do not form a simplex.†

(j) If in the R_n the points $p_1, \cdots, p_n, p_{n+1}$ are independent and p is a point such that for each integer $k = 1, 2, \cdots n$ the points $p_1, \cdots, p_{k-1}, p_{k+1} \cdots, p_n, p_{n+1}, p$ are dependent, then $p = p_{n+1}$.

(k) If q, r, s are three points of the R_n such that

$$q, r \approx r, s \approx s, q$$

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† An analytic proof may be given as follows: Let $(x_1{}^i, x_2{}^i, \cdots, x^i{}_{n-1})$ be the coördinates of the points p^i, $(i = 1, 2, \cdots, n+1)$, assumed to lie in an at most $(n-1)$-dimensional plane whose equation we may write as $X_n = 0$. We suppose, further, that the points lie on an at most $(n-1)$-dimensional sphere with equation $\sum_{k=1}^{n-1} X_k{}^2 = a^2$.

We show that the mirror planes do not, then, form a simplex. If $(x_1, x_2, \cdots, x_n)$ are the coördinates of the point p, the equations of the planes are

$$(x_1 - x_1{}^i)[X_1 - \tfrac{1}{2}(x_1 + x_1{}^i)] + (x_2 - x_2{}^i)[X_2 - \tfrac{1}{2}(x_2 + x_2{}^i)] + \cdots + x_n[X_n - \tfrac{1}{2}x_n] = 0 \qquad (i = 1, 2, \cdots, n+1)$$

and since the points p^i, $(i = 1, 2, \cdots, n+1)$ lie on the sphere $\sum_{k=1}^{n-1} X_k{}^2 = a^2$ the equations may be written

[illegible]

To prove that these planes do not form an n dimensional simplex it is sufficient to note that the determinant of these equations is zero.

and if $p_1, \cdots, p_n$ are n points of the R_n such that

$$p_i, q \approx p_i, r \approx p_i, s \qquad (i = 1, 2, \cdots n)$$

then the n points $p_1, \cdots, p_n$ are dependent, i. e. lie in a $(n-2)$ dimensional plane.

To prove this property we remark that the n points p_i lie on the surfaces of three spheres. If they would not lie in a $(n-2)$-dimensional plane the centres of these three spheres would lie on a straight line. The centres q, r, s of the three spheres form, however, according to the hypothesis an equilateral triangle and hence do not lie on a straight line.

We shall call a set S of a congruence system *equilateral* if each two couples of points p, q and r, s of S are congruent. Then we see

(l) The R_n does not contain any equilateral $(n+2)$-tuple.

An important rôle in the metrical theory of the euclidean space is played by the concept of *pseudo-euclidean sets.* A set S of a congruence system consisting of $n+3$ points is called a pseudo-euclidean $(n+3)$-tuple *if S is not congruent with a subset of the R_n whereas each $n+2$ points of S are congruent with $n+2$ points of the R_n.* We prove first of all the following:

LEMMA 1. *If the pseudo-euclidean $(n+3)$-tuple P' contains $n+2$ points, $n+1$ points of which are independent* (i. e. congruent with $n+1$ points of the R_n but not with $n+1$ points of the R_{n-1}) *and such that the $n+2$ points are congruent with $n+2$ points of the pseudo-euclidean $(n+3)$-tuple Q' then P' and Q' are congruent.*

Let $p'_1, \cdots, p'_{n+3}$ be the $n+3$ points of P' and $q'_1, \cdots, q'_{n+3}$ the points of Q'. Let us suppose that $p'_1, \cdots, p'_{n+2} \approx q'_1, \cdots, q'_{n+2}$ and that $p'_1, \cdots, p'_{n+1}$ are independent. We have then to prove that $p'_i, p'_{n+3} \approx q'_i, q'_{n+3}$ $(i = 1, \cdots, n+2)$. We study first some properties of P'.

As the set P' is pseudo-euclidean there exist $n+2$ points $p_1, \cdots, p_{n+1}, p_{n+2}$ of the R_n such that

$$(1) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+2} \approx p_1, \cdots, p_{n+1}, p_{n+2}.$$

According to the hypothesis the points $p_1, \cdots, p'_{n+1}$ are independent, i. e. do not lie in an $(n-1)$-dimensional plane. As the points $p'_1, \cdots, p'_{n+1}, p'_{n+3}$ are congruent with $n+2$ points of the R_n there exists according to property (c') of the R_n exactly one point p_{n+3} of the R_n such that

$$(2) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+3} \approx p_1, \cdots, p_{n+1}, p_{n+3}.$$

If p'_{n+2}, p'_{n+3} would be congruent with p_{n+2}, p_{n+3} the $n+3$ points of P' would be congruent with the points $p_1, \cdots, p_{n+2}, p_{n+3}$. As P' is a pseudo-euclidean set this is impossible. Hence we have

$$(3) \qquad p'_{n+2}, p'_{n+3} \not\simeq p_{n+2}, p_{n+3}.$$

Let k be one of the integers $1, 2, \cdots, n, n+1$. There exist $n+2$ points $\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3}$ of the R_n such that

$$(4_k) \qquad \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \\ \simeq p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}.$$

We have $\bar{p}_1, \cdots, \bar{p}_{n+1} \simeq p_1, \cdots, p_{n+1}$ as either of these $(n+1)$-tuples is congruent with $p'_1, \cdots, p'_{n+1}$, and these $(n+1)$-tuples are independent.

We state now the following preliminary proposition: If p is any point such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p,$$

then p is independent of the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$.

In order to prove this proposition we assume that the point p satisfying the above mentioned condition is dependent on $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ and we deduce a contradiction from this assumption. According to (c'') there exists at least one point x and at least one point y such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, x, y.$$

As p is dependent on $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ it follows from property (h) of the R_n that $x = p$. As

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$

it follows from property (h), furthermore, that $p = p_{n+2}$. We have thus

$$(5) \qquad \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \\ \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, y.$$

On the other hand, we have

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, y \simeq \bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3}$$

and hence according to (4_k) and (2)

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, y \simeq p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+3}.$$

As p is dependent on the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ it follows from property (g) of the R_n that

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, y \simeq p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, p_{n+3}$$

thus

(6) $$p_{n+2}, y \approx p_{n+2}, p_{n+3}.$$

On the other hand it follows from (5) that $p_{n+2}, y \approx \bar{p}_{n+2}, \bar{p}_{n+3}$ and thus, in consequence of (4_k)

(7) $$p_{n+2}, y \approx p'_{n+1}, p'_{n+3}.$$

(6) and (7) contradict (3). We have thus proved our preliminary proposition.

Exactly in the same fashion we can prove: If q is a point such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, q$$

then q is independent of $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$.

In consequence of (1) and (2) we have

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$
$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+3} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+3}$$

The points p_{n+2} and p_{n+3} are thus independent of $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$. According to ($f'$) there exists exactly one point which may be called $p^k{}_{n+2}$ and exactly one point which may be called $p^k{}_{n+3}$ satisfying the conditions

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+2}, \quad p^k{}_{n+2} \neq p_{n+2}$$
$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+3} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+3}, \quad p^k{}_{n+3} \neq p_{n+3}.$$

According to property (c'') there exists at least one point p and at least one point q such that

$$\bar{p}_1, \cdots, \bar{p}_{k-1}, \bar{p}_{k+1}, \cdots, \bar{p}_{n+1}, \bar{p}_{n+2}, \bar{p}_{n+3} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p, q.$$

According to (g) the point p is identical with either p_{n+2} or $p^k{}_{n+2}$, and q is identical with either p_{n+3} or $p^k{}_{n+3}$. As

$$p_{n+2}, p_{n+3} \approx p^k{}_{n+2}, p^k{}_{n+3} \not\approx p'_{n+2}, p'_{n+3} \text{ and } \bar{p}_{n+2}, \bar{p}_{n+3} \approx p_{n+2}, p_{n+3}$$

it follows that either $p = p_{n+2}$ and $q = p^k{}_{n+3}$ or $p = p^k{}_{n+2}$ and $q = p_{n+3}$. According to (g)

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}, p^k{}_{n+3} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k{}_{n+2}, p_{n+3}.$$

We have thus

$$p^k{}_{n+2}, p_{n+3} \approx p_{n+2}, p^k{}_{n+3} \approx p'_{n+2}, p'_{n+3}.$$

Hence it is proved for each k ($k = 1, 2, \cdots n+1$) that

$$p'_{n+2}, p'_{n+3} \approx p^k_{n+2}, p_{n+3}.$$

We consider now the pseudo-euclidean $(n+3)$-tuple Q'. According to the hypothesis we have

$$q'_1, \cdots, q'_{n+2} \approx p'_1, \cdots, p'_{n+2} \approx p_1, \cdots, p_{n+2}.$$

Also the $n+2$ points $q'_1, \cdots, q'_{n+2}$ of Q' are congruent with $n+2$ points of the R_n and there exists exactly one point q_{n+3} of the R_n such that

$$q'_1, \cdots, q'_{n+1}, q'_{n+3} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3}.$$

The same arguments which proved that

$$p'_{n+2}, p'_{n+3} \approx p^k_{n+2}, p_{n+3} \qquad (k = 1, 2, \cdots, n+1)$$

prove that also

$$q'_{n+2}, q'_{n+3} \approx p^k_{n+2}, q_{n+3}. \qquad (n = 1, 2, \cdots, n+1).$$

From property (i) of the R_n it follows that $q_{n+3} = p_{n+3}$. Therefore we have

$$q'_i, q'_{n+3} \approx p_i, p_{n+3} \approx p'_i, p'_{n+3} \qquad (i = 1, 2, \cdots, n+1)$$

$$q'_{n+2}, q'_{n+3} \approx p^k_{n+2}, p_{n+3} \approx p'_{n+2}, p'_{n+3}.$$

This is the proposition of the Lemma 1.

We shall prove now the following:

THEOREM I ON PSEUDO-EUCLIDEAN SETS. *Each $n+1$ points of a pseudo-euclidean $(n+3)$-tuple are independent,* i. e. congruent with $n+1$ points of the R_n but not congruent with $n+1$ points of the R_{n-1}.

We prove first of all: If P' is a pseudo-euclidean $(n+3)$-tuple then P' contains at least *one* independent $(n+1)$-tuple. If all $(n+1)$-tuples of points of P' would be dependent then P' would be congruent with a subset of the R_{n-1} as the R_{n-1} by our inductive hypothesis has the quasi-congruence order $n+1$. Then P' would bê a fortiori congruent with a subset of the R_n and this is not the case.

We label the points of P' so that $p'_1, \cdots, p'_{n+1}$ are independent. Let $p_1, \cdots, p_{n+1}$ $n+1$ be congruent points of the R_n. They do not lie in a $(n-1)$-dimensional plane. We form the points p^k_{n+2} ($k = 1, \cdots, n+1$) which were considered in the proof of Lemma 1, for which

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2}, p_{n+3}$$
$$\approx p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}.$$

Let us now suppose that the points $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ are dependent. Then the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2}$ are dependent, also. As the points $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}$ are independent it follows in consequence of property (h) from

$$p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p^k_{n+2} \approx p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{n+1}, p_{n+2}$$

that $p^k_{n+2} = p_{n+2}$. Then $p_{n+2}, p_{n+3} \approx p^k_{n+2}, p_{n+3} \approx p'_{n+2}, p'_{n+3}$ and this contradicts (3). The assumption has led to a contradiction and we have thus proved:

If the points $p'_1, \cdots, p'_{n+1}$ of the pseudo-euclidean $(n+3)$-tuple $p'_1, \cdots, p'_{n+3}$ are independent then each of the $(n+1)$-tuples $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ $(k = 1, 2, \cdots, n+1)$ is independent. For reasons of symmetry it follows that also each of the $(n+1)$-tuples $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+3}$ $(k = 1, 2, \cdots, n+1)$ is independent.

The independence of the $(n+1)$-tuple $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}$ follows from the independence of the points $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+2}$ in the same way as the independence of $p'_1, \cdots, p'_{k-1}, p'_{k+1}, \cdots, p'_{n+1}, p'_{n+2}$ follows from the independence of $p'_1, \cdots, p'_{n+1}$. Theorem I is thus proved.

Based on Theorem I we may formulate Lemma 1 as the following

THEOREM II ON PSEUDO-EUCLIDEAN SETS. *Two pseudo-euclidean $(n+3)$-tuples such that one contains a $(n+2)$-tuple congruent with a $(n+2)$-tuple of the other are congruent.*

We prove now the following:

Lemma 2. If a set consisting of $n+4$ distinct points each $n+2$ of which are congruent with $n+2$ points of the R_n, contains one pseudo-euclidean $(n+3)$-tuple it contains at least three pseudo-euclidean $(n+3)$-tuples $(n \geqq 1)$.

Let us call $p'_1, \cdots, p_{n+4}$ the $n+4$ distinct points of the set P' and assume that the points $p'_1, \cdots, p'_{n+3}$ form a pseudo-euclidean set. P' contains $n+4$ sets each consisting of $n+3$ points. If $n+2$ of these sets are pseudo-euclidean then, as $n \geqq 1$, the lemma is true. Otherwise there are at least two $(n+3)$-tuples each of which is congruent with $n+3$ points of the R_n. We may label the points of P' so that $p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4}$ and $p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}$ are congruent to euclidean $(n+3)$-tuples. In this case we shall prove that there exist besides $p'_1, \cdots, p'_{n+3}$ at least two pseudo-euclidean $(n+3)$-tuples in P'.

First of all, there correspond to the pseudo-euclidean set $p'_1, \cdots, p'_{n+3}$, $n+3$ points $p_1, \cdots, p_{n+3}$ of the R_n such that

$$p_1, \cdots, p_{n+1}, p_{n+2} \approx p'_1, \cdots, p'_{n+1}, p'_{n+2} \tag{1}$$
$$p_1, \cdots, p_{n+1}, p_{n+3} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3} \tag{2}$$
$$p_{n+2}, p_{n+3} \not\approx p'_{n+2}, p'_{n+3}. \tag{3}$$

The points $p_1, \cdots, p_{n+1}$ are congruent to $n+1$ points of a pseudo-euclidean $(n+3)$-tuple. According to theorem I on pseudo-euclidean sets they do not lie in an $(n-1)$-dimensional plane. There exists therefore according to (c') exactly one point p_{n+4} in the R_n such that

$$p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+4} \approx p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4} \tag{4}$$
$$p_1, \cdots, p_{n+1}, p_{n+3}, p_{n+4} \approx p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}. \tag{5}$$

From (4) and (5) it follows that

$$p'_{n+4}, p'_i \approx p_{n+4}, p_i \qquad (i = 1, \cdots, n+3).$$

As the point p'_{n+4} is distinct from the $n+3$ points p'_i it follows thus that p_{n+4} is distinct from the $n+3$ points p_i $(i = 1, \cdots, n+3)$. According to property (j) of the R_n there exist therefore at least two integers i and j between 1 and $n+1$ such that neither $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}, p_{n+4}$ nor $p_1, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{n+1}, p_{n+4}$ lie in a $(n-1)$-dimensional plane. We propose now that

$$p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}, p'_{n+4} \tag{6}$$
$$p'_1, \cdots, p'_{j-1}, p'_{j+1}, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}, p'_{n+4} \tag{7}$$

are the two desired pseudo-euclidean $(n+3)$-tuples.

Let us show this for the system (6). As each $(n+2)$-tuple of (6) is congruent with $n+2$ points of the R_n we have merely to show that (6) is not congruent with $n+3$ points of the R_n. If we map $p'_1, \cdots, p'_{i-1}, p'_{i+1}, \cdots, p'_{n+1}, p'_{n+4}$ on $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}, p_{n+4}$ then, as these points do not lie in an $(n-1)$-dimensional plane the points p'_{n+2} and p'_{n+3} in consequence of (c') must be necessarily mapped on p_{n+2} and p_{n+3} respectively. Hence it is according to (3) impossible to map the $n+3$ considered points on $n+3$ points of the R_n. In the same way it can be shown that (7) is a pseudo-euclidean $(n+3)$-tuple. This completes the proof of lemma 2.

Lemma 3. *If P' is a set consisting of $n+4$ distinct points each $n+2$ [illegible] congruent with $n+2$ points of the R_n and which contains at least three pseudo-euclidean $(n+3)$-tuples then each $(n+3)$-tuple of points of P' is pseudo euclidean.*

Let us denote by $p'_1, \cdots, p'_{n+4}$ the $n+4$ distinct points of P' and let us label them so that

$$(1) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+3}$$
$$(2) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+2}, p'_{n+4}$$
$$(3) \qquad p'_1, \cdots, p'_{n+1}, p'_{n+3}, p'_{n+4}$$

are pseudo-euclidean sets. As the $(n+3)$-tuples (1) and (2) both contain the $n+2$ points $p'_1, \cdots, p'_{n+2}$ it follows from theorem I on pseudo-euclidean sets that the $(n+3)$-tuples (1) and (2) are congruent. In the same way one sees that (1) and (3) are congruent. Hence we have

$$(4) \qquad p'_{n+2}, p'_{n+3} \approx p'_{n+2}, p'_{n+4} \approx p'_{n+3}, p'_{n+4}$$
$$(5) \qquad p'_{n+2}, p'_i \approx p'_{n+3}, p'_i \approx p'_{n+4}, p'_i \qquad (i = 1, \cdots, n+1).$$

We prove now that the $n+3$ points $p'_2, \cdots, p'_{n+4}$ form a pseudo-euclidean set. We assume that they are not pseudo-euclidean and deduce a contradiction from this assumption. As each $n+2$ of the points are congruent with $n+2$ points of the R_n it follows from the assumption that there exist $n+3$ points $p_2, \cdots, p_{n+4}$ of the R_n which are congruent with the points $p'_2, \cdots, p'_{n+4}$. In consequence of (4) and (5) the n points $p_2, \cdots, p_{n+1}$ according to property (k) of the R_n are dependent. This, however, is impossible as they are congruent with the n points of the pseudo-euclidean $(n+3)$-tuple (1) and, as no $n+1$ points of a pseudo-euclidean $(n+3)$-tuple are independent, also each n-tuple of points of (1) is independent.

Exactly in the same way we could deduce a contradiction from the assumption that any other $(n+3)$-tuple of P' is not pseudo-euclidean. Lemma 3 is thus proved.

We are now in the position to prove the third fundamental theorem. We have to prove that a set S containing more then $n+3$ points each $n+2$ of which are congruent with $n+2$ points of the R_n is congruent with a subset of the R_n. We shall assume that S is a set containing at least $n+4$ distinct points each $n+2$ of which are congruent with $n+2$ points of the R_n and deduce a contradiction from the hypothesis that S is not congruent with a subset of the R_n. As the R_n has the congruence order $n+3$ it follows from the hypothesis that S contains a $(n+3)$-tuple of points $p'_1, \cdots, p'_{n+3}$ which are not congruent with $(n+3)$ points of the R_n. Since each $n+2$ points of this $(n+3)$-tuple are congruent with $n+2$ points of the R_n it follows that they form a pseudo-euclidean $(n+3)$-tuple. As S contains at least $n+4$ distinct points there exists a point p'_{n+4} of S distinct from the points $p', \cdots, p'_{n+3}$. The set P' of the $n+4$ points $p'_1, \cdots, p'_{n+3}, p'_{n+4}$ contains at least one pseudo-euclidean $(n+3)$-tuple. According to lemma

2 the set P' contains at least three pseudo-euclidean $(n+3)$-tuples. According to lemma 3 each $(n+3)$-tuple of points of P' is pseudo-euclidean.

We prove now that the set P' is equilateral, i. e. each two pairs of points of P' are congruent. In order to prove that two pairs of points which have one point in common are congruent, e. g. that $p'_{n+2}, p'_{n+3} \approx p'_{n+3}, p'_{n+4}$ we remark that the two pseudo-euclidean $(n+3)$-tuples

$$p'_1 \cdots, p'_{n+2}, p'_{n+3} \text{ and } p'_1, \cdots, p'_{n+2}, p'_{n+4}$$

have the $n+2$ points $p'_1, \cdots, p'_{n+2}$ in common and hence, according to theorem II on pseudo-euclidean sets, are congruent. This shows that in particular $p'_{n+2}, p'_{n+3} \approx p'_{n+3}, p'_{n+4}$. In the same fashion it can be proved e. g. that $p'_{n+1}, p'_{n+2} \approx p'_{n+2}, p'_{n+3}$. We have thus $p'_{n+1}, p'_{n+2} \approx p'_{n+3}, p'_{n+4}$. In the same way each two pairs of points of P' whether or not they have a point in common can be proved to be congruent. The set P' is thus equilateral. In particular each subset of P' consisting of $n+2$ points is equilateral. According to property (1) of the R_n this contradicts the assumption that each $n+2$ points of S' are congruent with $n+2$ points of the R_n. The hypothesis leads thus to a contradiction and this completes the proof of the second fundamental theorem.

3. Metrical characterization of the euclidean spaces. In this last part we deal with subsets of semi-metrical spaces. To each two points p, q there corresponds a real number $pq = qp > 0$ if $p \neq q$ whereas $pp = 0$. For each k points $p_1, p_2, \cdots, p_k$ of a semi-metrical space we denote by $D(p_1, p_2, \cdots, p_k)$ the symmetric determinant

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & (p_1p_2)^2 & \cdots & (p_1p_k)^2 \\ 1 & (p_2p_1)^2 & 0 & \cdots & (p_2p_k)^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & (p_kp_1)^2 & (p_kp_2)^2 & \cdots & 0 \end{vmatrix}$$

which we also shall denote symbolically by

$$\begin{vmatrix} 0 & 1 \\ 1 & (p_ip_j)^2 \end{vmatrix} \qquad (i, j = 1, 2, \cdots, k).$$

The importance of this determinant lies in the fact that if $p_1, p_2, \cdots, p_k$ [illegible] the volume $V(p_1, p_2, \cdots, p_k)$ of the $(k-1)$-dimensional simplex determined by these k points is given by the *volume formula*

$$v^2(p_1, p_2, \cdots, p_k) = \frac{(-1)^k}{(k-1)!^2 2^{k-1}} \cdot D(p_1, p_2, \cdots, p_k).$$

Evidently the determinant of k points is *congruence invariant,* i.e. if $p'_1, p'_2, \cdots, p'_k$ are congruent with $p_1, p_2, \cdots, p_k$ then

$$D(p_1, p_2, \cdots, p_k) = D(p'_1, p'_2, \cdots, p'_k).$$

We denote by $D(p_1, p_2, \cdots, p_{k-2}, p_{k-1}, p_k; x)$ the function of x which we obtain by substituting x in place of the terms $(p_{k-1}p_k)^2$ and $(p_kp_{k-1})^2$ in the determinant $D(p_1, p_2, \cdots, p_k)$. We may symbolize this function also by

$$D(p_1, \cdots, p_{k-2}, p_{k-1}, p_k; x) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & (p_ip_j)^2 & (p_ip_{k-1})^2 & (p_ip_k)^2 \\ 1 & (p_{k-1}p_j)^2 & 0 & x \\ 1 & (p_kp_j)^2 & x & 0 \end{vmatrix}$$

$$(i, j = 1, 2, \cdots, k-2).$$

Remark a. The coefficient of x^2 in the quadratic polynomial $D(p_1, \cdots, p_k; x)$ is $-D(p_1, p_2, \cdots, p_{k-2})$. This follows immediately if we develop D according to Laplace.

Remark b. If $p_1, \cdots, p_{k-2} \approx p'_1, \cdots, p'_{k-2}$

$$p_ip_{k-1} = p'_ip'_{k-1}, \qquad p_ip_k = p'_ip'_k \qquad (i = 1, 2, \cdots, k-2)$$

then $\quad D(p_1, \cdots, p_{k-2}, p_{k-1}, p_k; x) \equiv D(p'_1, \cdots, p'_{k-2}, p'_{k-1}, p'_k; x).$

The main result of this third part will be the

THIRD FUNDAMENTAL THEOREM. *In order that $n+2$ points $p'_1, p'_2, \cdots, p'_{n+2}$ of a semi-metrical space each $n+1$ of which are congruent with $n+1$ points of the R_n be congruent with $n+2$ independent points of the R_{n+1} or with $n+2$ points of the R_n it is necessary and sufficient that*

$$\text{sign } D(p'_1, p'_2, \cdots, p'_{n+2}) = (-1)^n \textit{ or } D(p'_1, p'_2, \cdots, p'_{n+2}) = 0$$

respectively.

If $p'_1, p'_2, \cdots, p'_{n+2}$ are congruent with $n+2$ points $p_1, p_2, \cdots, p_{n+2}$ of the R_{n+1} then $D(p'_1, p'_2, \cdots, p'_{n+2}) = D(p_1, p_2, \cdots, p_{n+2})$. According to the volume formula $(-1)^{n+2}$. $D(p_1, p_2, \cdots, p_{n+2})$ is identical with the square of the volume of the $(n+1)$-dimensional simplex determined by the points $p_1, p_2, \cdots, p_{n+2}$ (if we neglect a positive factor). As the square of this volume is not negative, and is zero if and only if the $n+2$ points lie in

a n-dimensional plane of the R_{n+1}, it follows that the condition of the theorem is *necessary.*

In order to prove that the condition is *sufficient* let us assume that $p'_1, p'_2, \cdots, p'_{n+2}$ are $n+2$ points each $n+1$ of which are congruent with $n+1$ points of the R_n and whose determinant has the sign $(-1)^n$ or is zero. We have to prove the existence of $n+2$ congruent points of the R_{n+1} which are independent or dependent respectively.

This proposition is trivial in the case $n=0$. For $D(p'_1, p'_2) = 2(p'_1 p'_2)^2$. Hence p'_1 and p'_2 are identical if and only if $D(p'_1, p'_2) = 0$. In this case they are congruent with two points of the R_0 (consisting of one point). If sign $D(p'_1, p'_2) = (-1)^0$, i. e. is positive, then $p'_1 p'_2$ is a real positive number and p'_1 and p'_2 are congruent with two points of the R_1 which are independent (i. e. not identical).

We prove our proposition therefore under the inductive hypothesis of its truth for $n+1$ points, i. e. we make the assumption: If $n+1$ points, each n of which are congruent with n points of the R_{n-1}, have a determinant of the sign $(-1)^{n-1}$ or 0 then they are congruent with $n+2$ independent points of an n-dimensional plane of the R_{n+1} or with $n+2$ points of a $(n-2)$-dimensional plane of the R_n respectively. We deduce first of all some conclusions from this assumption.

Lemma 4. *If $p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}$ lie in an n-dimensional plane of the R_{n+1} and $p_1, p_2, \cdots, p_n$ do not lie in an $(n-2)$-dimensional plane then the equation*

$$D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}; x) = 0$$

has the double root $x = (p_{n+1} p_{n+2})^2$.

Let us denote the polynomial $D(p_1, \cdots, p_{n+2}; x)$ by $D(x)$. If we substitute the value $x = (p_{n+1} p_{n+2})^2$ in $D(x)$ it becomes $D(p_1, \cdots, p_n, p_{n+1}, p_{n+2})$. As the $n+2$ points $p_1, \cdots, p_{n+2}$ lie in an n-dimensional plane Π their determinant equals 0, according to the necessity of the condition of the third fundamental theorem which already has been proved. Hence $x = (p_{n+1} p_{n+2})^2$ is a root of the equation $D(x) = 0$. In order to prove the lemma we have thus merely to prove that the equation $D(x) = 0$ has a double root. In order to prove this we remark first of all that the R_{n+1} is divided by the n-dimensional plane Π into two open parts which may be distinguished as the left part and the right part. We determine a sequence of points of the left part converging towards the point p_{n+1}, say p'^k_{n+1} $(k = 1, 2, \cdots$ ad inf.), and a sequence of points of the right part converging towards p_{n+2}, say p'^k_{n+2} $(k = 1, 2, \cdots$ ad inf.). We consider for each integer

k the polynomial $D(p_1, \cdots, p_n, p^k_{n+1}, p^k_{n+2};\ x)$ which may be denoted by $D^k(x)$. For each k one root of the equation $D^k(x) = 0$ is $x_k = (p^k_{n+1}p^k_{n+2})^2$. If we denote by $\bar{p}^k_{n+2}$ the mirror image of the point p^k_{n+2} in Π, then, in consequence of remark b, we have

$$D(p_1, \cdots, p_n, p^k_{n+1}, p^k_{n+2}) \equiv D(p_1, \cdots, p_n, p^k_{n+1}, \bar{p}^k_{n+2}).$$

Hence each zero of the second polynomial is also a root of the equation $D^k(x) = 0$. A zero of the second polynomial is $\bar{x}_k = (p^k_{n+1}\bar{p}^k_{n+2})^2$. As $p^k_{n+1}p^k_{n+2} \neq p^k_{n+1}\bar{p}^k_{n+2}$ we know then, two roots of the equation $D^k(x) = 0$. This equation does not vanish identically as x^2 has, according to remark a, the coefficient $D(p_1, \cdots, p_n)$ which, is $\neq 0$ as $p_1, \cdots, p_n$ are supposed not to be in a $(n-2)$-dimensional plane. The values $x_k = (p^k_{n+1}p^k_{n+2})^2$ and $\bar{x}_k = (p^k_{n+1}\bar{p}^k_{n+2})^2$ are thus the two roots of the equation $D^k(x) = 0$. As the points p^k_{n+2} tend towards the point p_{n+2} of Π their images in Π, i. e. the points $\bar{p}^k_{n+2}$ tend toward the same point. It follows that $\lim (x_k - \bar{x}_k) = 0$. If we denote by D_k the discriminant of the equation $D^k(x) = 0$ it follows that $\lim_{k=\infty} D_k = 0$. As the points p^k_{n+1} and p^k_{n+2} tend toward p_{n+1} and p_{n+2} respectively it follows that the polynomials $D^k(x)$ tend toward the polynomial $D(x)$. As the coefficient of x^2 in $D(x)$ equals $D(p_1, p_2, \cdots, p_n)$ and hence is not zero it follows that the discriminant of $D(x) = 0$ is the limit of the discriminants D_k. This limit, as we saw, is 0. Hence $D(x) = 0$ is a not identically vanishing quadratic equation whose discriminant equals zero and, therefore, has a double root. Lemma 4 is thus proved.

LEMMA 5. *If $p'_1, \cdots, p'_{n+2}$ is a pseudo-euclidean $(n+2)$-tuple then sign* $D(p'_1, \cdots, p'_{n+2}) = (-1)^{n+1}$.

There exist, as we know, $n+2$ points $p_1, \cdots, p_n, p_{n+1}, p_{n+2}$ in the R_n such that

$$p_1, \cdots, p_n, p_{n+1} \approx p'_1, \cdots, p'_n, p'_{n+1}$$
$$p_1, \cdots, p_n, p_{n+2} \approx p'_1, \cdots, p'_n, p'_{n+2}$$
$$p_{n+1}p_{n+2} \neq p'_{n+1}p'_{n+2}.$$

According to remark b it follows from these relations that

$$D(p'_1, p'_2, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x) \equiv D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}; x).$$

These polynomials will be denoted by $D'(x)$ and $D(x)$ respectively. The points $p_1, \cdots, p_n$ are congruent with n points of a pseudo-euclidean $(n+2)$-tuple and hence, according to theorem II on pseudo-euclidean sets, do not lie in a $(n-2)$-dimensional plane. According to lemma 4 the equation

$D(x)=0$ has thus the double root $(p_{n+1}p_{n+2})^2$. This value is thus a double root of the equation $D'(x)=0$, also. For each other value the quadratic polynomial $D'(x)\equiv D(x)$ has the same sign. For large values of x the sign of $D(x)$ is identical with the sign of the coefficient of x^2 in $D(x)$. This coefficient according to remark a equals $-D(p_1,\cdots,p_n)$. As the points $p_1,\cdots,p_n$ do not lie in a $(n-2)$-dimensional plane it follows from the necessity of the condition of the third fundamental theorem which already has been proved that sign $D(p_1,\cdots,p_n)=(-1)^n$. The sign of $D(x)$ and $D'(x)$ for each value of x that is different from $(p_{n+1}p_{n+2})^2$ is thus $(-1)^{n+1}$. As $p'_{n+1}p'_{n+2}\neq p_{n+1}p_{n+2}$ the lemma 5 is proved.

We are now in the position to prove the sufficiency of the conditions of the third fundamental theorem. Let us assume that $p'_1,\cdots,p'_{n+2}$ are $n+2$ points each $n+1$ of which are congruent to $n+1$ points of the R_n and such that

$$\text{sign } D(p'_1,\cdots,p'_{n+2}) = (-1)^n \text{ or } 0.$$

We have to prove that there exist $n+2$ congruent points in the R_{n+1} independent or lying in a n-dimensional plane respectively.

According to the hypothesis on $D(p'_1,\cdots,p'_{n+2})$ it is impossible by the lemma that the $n+2$ given points form a pseudo-euclidean $(n+2)$-tuple. If each $n+1$ of our $n+2$ points are congruent with $n+1$ points of the R_{n-1} then the $n+2$ points are thus congruent with $n+2$ points of the R_{n-1} and a fortiori with $n+2$ points of the R_{n+1}. (This case evidently can only happen if $D(p_1,\cdots,p_{n+2})=0$.)

We may thus asssume that $n+1$ of the $n+2$ points are not congruent with $n+1$ points of the R_{n-1}, let us say $p'_1,\cdots,p'_{n+1}$. There exist $n+1$ congruent points $p_1,\cdots,p_{n+1}$ in an n-dimensional plane Π of the R_{n+1}. The points $p_1,\cdots,p_n$ lie in a $(n-1)$-dimensional plane Π^* but not in a $(n-2)$-dimensional plane. As each $n+1$ of our $n+2$ points are congruent with $n+1$ points of the R_n there exists at least one point p in Π such that

$$p_1, p_2,\cdots,p_n, p \approx p'_1, p'_2,\cdots,p'_n, p'_{n+2}.$$

If we rotate p in the R_{n+1} around the $(n-1)$-dimensional plane Π^* the path of p is a circle perpendicular to Π^*. The circle has the radius O, i.e. degenerates in a single point, if and only if the point lies in Π^*, and hence if and only if the points $p'_1,\cdots,p'_n, p'_{n+2}$ are congruent with $n+1$ points of the R_{n-1}. If this is not the case then the circle has exactly two points p_{n+2} and $\bar{p}_{n+2}$ in common with Π. As p_{n+1} does not lie in Π^* we have $p_{n+1}p_{n+2}\neq p_{n+1}\bar{p}_{n+2}$ and we assume the notation such that

$$p_{n+1}\bar{p}_{n+2} < p_{n+1}\bar{\bar{p}}_{n+2}.$$

Let us now assume that $D(p'_1, \cdots, p'_{n+2}) = 0$ and prove that there exist $n+2$ points congruent with $p'_1, \cdots, p'_{n+2}$ in the n-dimensional plane Π. We distinguish two cases. First it may be that the points $p'_1, \cdots, p'_n, p'_{n+2}$ are congruent with $n+1$ points of the R_{n-1}. We form then the above mentioned points $p_1, \cdots, p_n, p_{n+1}, p$ and know that p lies in Π^*. Thus $p_1, \cdots, p_n, p_{n+1}, p$ lie in the n-dimensional plane Π whereas $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane. According to lemma 4 the equation $D(p_1, \cdots, p_n, p_{n+1}, p; x) = 0$ has the double root $x = (p_{n+1}p)^2$. According to remark b the polynomials $D(p'_1, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x)$ and $D(p_1, \cdots, p_n, p_{n+1}, p_{n+2}; x)$ are identical. Hence $x = (p_{n+1}p)^2$ is also the double root of the first of these polynomials which may be denoted by $D'(x)$. From the hypothesis that $D(p'_1, \cdots, p'_{n+1}) = 0$ it follows that $x = (p'_n p'_{n+1})^2$ is a root of the equation $D'(x) = 0$. The equation $D(x) = 0$ does not vanish indentically as the coefficient of x^2 is $-D(p'_1, \cdots, p'_n) = -D(p_1, \cdots, p_n)$ and this is not zero as the points $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane. It follows that $p'_{n+1}p'_{n+2} = p_{n+1}p$ and hence the $n+2$ points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$ are congruent with the $n+2$ points $p_1, \cdots, p_{n+1}, p$ of the n-dimensional plane Π.

The second case is that the points $p'_1, \cdots, p'_{n+2}$ are not congruent with $n+1$ points of the R_{n-1}. In this case $x = (p_{n+1}\bar{p}_{n+2})^2$ and $x = (p_{n+1}\bar{\bar{p}}_{n+2})^2$ are two solutions of the equation $D(p_1, \cdots, p_n, p_{n+1}, p; x) = 0$. For if we substitute these two values we get the determinant of $n+2$ points of the n-dimensional plane Π, viz. $D(p_1, \cdots, p_n, p_{n+1}, \bar{p}_{n+2})$ and $D(p_1, \cdots, p_n, p_{n+1}, \bar{\bar{p}}_{n+2})$ respectively, which are both equal to zero. According to remark b the above mentioned equation is identical with the equation $D(p'_1, \cdots, p'_n, p'_{n+1}, p'_{n+2}; x) = 0$. A root of this equation is $x = (p'_{n+1}p'_{n+2})^2$, for if we substitute this value we get $D(p'_1, \cdots, p'_{n+2})$ which is zero, according to the hypothesis. It follows that $p'_{n+1}p'_{n+2}$ equals either $p_{n+1}p_{n+2}$ or $p_{n+1}\bar{p}_{n+2}$. Hence the $n+2$ points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$ are congruent with either $p_1, \cdots, p_{n+1}, \bar{p}_{n+2}$ or with $p_1, \cdots, p_{n+1}\bar{\bar{p}}_{n+2}$ and thus in either case, congruent with $n+2$ points of the n-dimensional plane Π.

Let us assume now that sign $D(p'_1, \cdots, p'_{n+2}) = (-1)^n$ and prove that there exist $n+2$ independent points of the R_{n+1} which are congruent with $p'_1, \cdots, p'_{n+2}$. We form again the points $p_1, \cdots, p_{n+1}, \bar{p}_{n+2}, \bar{\bar{p}}_{n+2}$ and consider the polynomial $D(p'_1, \cdots, p'_{n+1}, p'_{n+2}; x)$ which may be denoted by $D'(x)$. According to remark b the three polynomials

$$D'(x),\ D(p_1, \cdots, p_{n+1}, \bar{p}_{n+2};\ x);\quad D(p_1, \cdots, p_{n+1}, \bar{\bar{p}}_{n+2};\ x)$$

are identical. A zero of the second is $x = (p_{n+1}\bar{p}_{n+2})^2$, a zero of the third is $x = (p_{n+1}\bar{\bar{p}}_{n+2})^2$. These two numbers, the first of which has been assumed to be smaller than the second, are thus roots of the equation $D'(x) = 0$, also. The coefficient of x^2 in $D'(x)$ is $-D(p'_1, \cdots, p'_n)$ and thus equals $-D(p_1, \cdots, p_n)$. As the n points $p_1, \cdots, p_n$ do not lie in a $(n-2)$-dimensional plane their determinant is not zero and has the sign $(-1)^n$. The coefficient of x^2 in $D'(x)$ has, therefore, the sign $(-1)^{n+1}$ and this is the sign of $D'(x)$ for large values of x. As $D'(x)$ is a quadratic polynomial with the two different zeros $(p_{n+1}\bar{p}_{n+2})^2$ and $(p_{n+1}\bar{\bar{p}}_{n+2})^2$ we conclude:

For $x \leqq (p_{n+1}\bar{p}_{n+2})^2$, sign $D'(x) = (-1)^{n+1}$ or 0.

For $(p_{n+1}\bar{p}_{n+2})^2 < x < (p_{n+1}\bar{\bar{p}}_{n+2})^2$, sign $D'(x) = (-1)^n$.

For $x \geqq (p_{n+1}\bar{\bar{p}}_{n+2})^2$, sign $D'(x) = (-1)^{n+1}$ or 0.

It follows thus from sign $D'(x) = (-1)^n$ that $(p_{n+1}\bar{p}_{n+2})^2 < x < (p_{n+1}p_{n+2})$. We have supposed that sign $D(p'_1, \cdots, p'_{n+1}, p'_{n+2}) = (-1)^n$. That means that $D'(x)$ for $x = (p'_{n+1}p'_{n+2})^2$ has the sign $(-1)^n$. We have thus proved:

$$(*) \qquad (p_{n+1}\bar{p}_{n+2})^2 < (p'_{n+1}p'_{n+2})^2 < (p_{n+1}\bar{\bar{p}}_{n+2})^2$$

and hence

$$p_{n+1}\bar{p}_{n+2} < p'_{n+1}p'_{n+2} < p_{n+1}\bar{\bar{p}}_{n+2}.$$

Let us now consider the above mentioned circle obtained by the rotation of p around the $(n-1)$-dimensional plane Π^* and perpendicular to Π. The nearest point to p_{n+1} on this circle is the point $\bar{p}_{n+2}$. If we move on the circle from this point toward the point $\bar{\bar{p}}_{n+2}$ then the distance from p_{n+1} increases continuously till it attains its maximum in the point $\bar{\bar{p}}_{n+2}$ which is the remotest point from p_{n+1} on the circle. In consequence of the inequality (*) there exists a point on the circle which we shall call p_{n+2}, such that $p_{n+1}p_{n+2} = p'_{n+1}p'_{n+2}$. This point is identical neither with $\bar{p}_{n+2}$ not $\bar{\bar{p}}_{n+2}$ and hence does not lie in the n-dimensional plane Π. The $n+2$ points $p_1, p_2, \cdots p_{n+1}, p_{n+2}$ are thus $n+2$ independent points of the R_{n+1} congruent with the $n+2$ given points $p'_1, \cdots, p'_{n+1}, p'_{n+2}$. The proof of the third fundamental theorem is thus completed.

It follows from the proof that under the hypothesis, sign $D(p'_1, \cdots, p'_{n+2}) = (-1)^n$ there exist $n+2$ independent points of the R_{n+1} congruent with the points $p'_1, \cdots, p'_{n+2}$ if we merely suppose that *two* of the $(n+1)$-tuples which have in common n points not congruent with n points of the R_n are congruent with $(n+1)$-tuples of the R_n.

We have proved lemma 5 for pseudo-euclidean $(n+2)$-tuples under the

hypothesis that the third fundamental theorem is true for $(n+1)$-tuples. From the third fundamental theorem we may conclude, therefore, the following

THEOREM 3 ON PSEUDO-EUCLIDEAN SETS. *The sign of the determinant of a pseudo-euclidean $(n+3)$-tuple is $(-1)^n$.*

A set consisting of $n+3$ points each $n+2$ of which are congruent with $n+2$ points of the R_n has been called pseudo-euclidean if it is not congruent with a subset of the R_n. From theorem 3 it follows immediately that such a pseudo-euclidean set is not congruent with a subset of the R_{n+1} or the R_{n+2} either. We see thus, that *a pseudo-euclidean set is not congruent with a subset of any euclidean space or even of Hilbert space.* Another corollary of theorem 3 is that *no proper subset of a pseudo-euclidean set is pseudo-euclidean.*

An immediate consequence of the second and third fundamental theorem is the following

METRICAL CHARACTERISATION OF EUCLIDEAN SETS. *In order that a semimetrical space S be congruent with a subset of the R_n it is necessary and sufficient that, if S contains more than $n+3$ points, for each integer $k \leqq n+1$ and for each k-tuple of points $p_1, \cdots, p_k$ of S*

$$\operatorname{sign} D(p_1, \cdots, p_k) = (-1)^k \text{ or } 0$$

and that the determinant of each $n+2$ points of S vanishes. If S contains exactly $n+3$ points $p_1, \cdots, p_{n+3}$ it is necessary and sufficient for the congruence of S with a subset of the R_n that besides these conditions $D(p_1, \cdots, p_{n+3}) = 0$.

Some further details whose proofs have been given in other papers may be mentioned in terminating this article. As to the pseudo-euclidean quadruples it has been proved * that they are characterized among the semimetrical spaces consisting of four distinct points p_1, p_2, p_3, p_4 by the relations

$$p_1p_2 = p_3p_4, \qquad p_1p_3 = p_2p_4, \qquad p_1p_4 = p_2p_3 = p_1p_2 + p_1p_3.$$

It has been shown,† furthermore, that the facts concerning pseudo-euclidean quadruples as well as concerning the congruence order 4 and the quasi congruence order 3 of the R_1 are merely special cases of theorems of abstract groups.

* Cp. Menger, *Mathematische Annalen*, Vol. 100, p. 126.

† Cp. Menger, *Mathematische Zeitschrift*, Vol. 33, p. 126.

The theory of pseudo-euclidean $(n+3)$-tuples in general is connected * with the theory of certain Cremona transformations of the R_n. In the case $n=2$ the theory of the isogonal transformations of the plane is applied. It follows from facts concerning isogonal transformations in particular † that in a pseudo-euclidean quintupel not only no three points are congruent with three points of the straight line but, furthermore, that no four points are congruent with four points of a circle.

In order that a metrical space S be congruent with a subset of the Hilbert space it is necessary and sufficient ‡ that S be separable (i. e. contains a dense subset which is denumerable) and that for each integer k and for each k points $p_1, \cdots, p_k$ of S

$$\operatorname{sgn} D(p_1, \cdots, p_k) = (-1)^k \text{ or } 0.$$

It is remarkable that the condition of separability cannot be omitted whereas it was not necessary to postulate it in the euclidean case.

The condition sign $D(p_1, p_2, p_3) = -1$ or 0 is equivalent to the so called triangle-inequality. Its being valid for each three points of a semi-metrical space is thus characteristic for the metrical spaces. It has been proved § that in a complete metrical space which is convex ¶ and exteriorly convex there exists for each two distinct points p and q a subset containing p and q which is congruent with the straight line. It follows ‖ that among the semi-metrical spaces satisfying the conditions which are characteristic for the subsets of the R_n the R_n itself is characterized by the properties of being complete, convex, exteriorly convex and containing $n+1$ points which are not congruent with $n+1$ points of the R_{n-1}.

The R_1 has been proved ** to be the only metrical space which is convex, exteriorly convex, complete, and has the quasi-congruence order 3.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

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* Cp. *loc. cit.* *, p. 126.

† Cp. *loc. cit.* *, p. 127.

‡ Cp. Menger, *Wien. akad. Anzeiger*, 1928.

§ Cp. *loc. cit.* *, p. 87.

¶ Cp. *loc. cit.* *, p. 81. A metrical space is called convex or exteriorly convex [illegible] for each two distinct points p and q a point s distinct from p and q and such that [illegible] respectively.

‖ Cp. *loc. cit.* *, p. 140.

** Cp. Menger, *Wien. akad. Anzeiger*, 1931.

NOTE ON THE GREEN FUNCTION OF A STAR-SHAPED THREE DIMENSIONAL REGION.*

By J. J. Gergen.

1. *Introduction.* The object in this note is to prove the following theorem.

Theorem. *Suppose that the region* † D *in the* (x, y, z) *space is star-shaped* ‡ *with regard to the origin* O, *and that* $g(x, y, z)$ *is the Green function* § *of* D *with pole at* O; *then at every point in* $D - O$ *the square of the gradient*

$$(\nabla g)^2 = g_x^2 + g_y^2 + g_z^2$$

of g *satisfies the condition*

$$(\nabla g)^2 \geqq g^2/r^2 > 0 \tag{1.1}$$

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* Preliminary report presented to the American Mathematical Society, December 30, 1930. The numerals in bold-faced type in this paper refer to the following treatises and papers:

1. Carathéodory, C., *Vorlesungen über Reele Funktionen*, Leipzig (1927).

2. Gergen, J. J., "Mapping of a General Type of Three Dimensional Region on a Sphere," *American Journal of Mathematics*, Vol. 52 (1930), pp. 197-224.

3. Kellogg, O. D., "On the Classical Dirichlet Problem for General Domains," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), pp. 397-406.

4. ———, "Recent Progress with the Dirichlet Problem," *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 601-625.

5. ———, "On the Derivatives of Harmonic Functions on the Boundary," *Transactions of the American Mathematical Society*, Vol. 33 (1931), pp. 486-510.

6. Liapounoff, M. A., "Sur certaines questions qui se rattachent au problème de Dirichlet," *Journal de Mathématiques (Liouville)*, Vol. 40 (1898), pp. 241-311.

7. Neumann, E. R., *Studien über die Methoden von C. Neumann und G. Robin fur Lösung der beiden Randwertaufgaben der Potentialtheorie*, Leipzig (1905).

8. Osgood, W. F., *Lehrbuch der Funktionentheorie*, Vol. 1, Berlin (1907).

† By a region is meant an open continuum.

‡ A region or, in general, any set of points E is star-shaped with regard to a point O when, and only when, each line segment joining O to a point in E lies entirely in E.

§ The Green function of an arbitrary star-shaped region may not exist in the ordinary sense. In the theorem we understand by g the generalized Green function of D. The generalized Green function is, of course, identical with the ordinary one in case the latter exists. For the definitions and properties of the generalized Green function, and also for a discussion and set of references on the Dirichlet problem, see Kellogg **3** and **4**. In these papers Kellogg considers, in general, only regions whose boundaries are bounded point sets; but no such restriction is necessary in his definition and proof of the uniqueness of the generalized Green function.

where $r = (x^2 + y^2 + z^2)^{1/2}$, *the inequality holding throughout* $D - O$ *unless* $g - 1/r$ *vanishes identically there.** *Furthermore, for every positive constant* c, *the region* † *determined by the inequality* $g > c$ *is star-shaped with regard to* O.

This theorem has, as corollary, that the Green function corresponding to a convex, three dimensional region, and arbitrarily assigned pole has a non-vanishing gradient. This fact was conjectured and mentioned to the author by Professor Morse; and it was in an attempt to prove this conjecture tenable that the theorem was established.

In connection with these results it should perhaps be recalled that, in contrast with the corresponding situation in two dimensions,‡ the gradient of the Green function of a simply-connected three dimensional region may vanish at points in the region.§ The importance of these results is, then, that they furnish a partial answer to the general question as to what conditions on a region and the position of the pole are sufficient to insure the absence of equilibrium points of the Green function corresponding to that region and pole. The theorem gives us, of course, more information about the Green function of a star-shaped region than that its gradient is non-vanishing. Incidentally, the inequality (1.1) is the best possible, in the sense that, if D is the whole of space, so that $g = 1/r$, we have

$$(\nabla g)^2 = g^2/r^2$$

throughout $D - O$.

The proof of the theorem depends largely on three facts:

(i) If the frontier d of D is a bounded, regular, analytic surface, then each of the partial derivatives g_x, g_y, g_z, of g coincides in D with a function which is continuous in $D + d - O$.

(ii) The function

$$u(x, y, z) = r\partial g/\partial r = xg_x + yg_y + zg_z$$

is harmonic in $D - O$.

(iii) If $f(Q)$ is defined ¶ and lower semi-continuous for Q on a sphere σ, and if

$$f(Q) \geq R,$$

* In this exceptional case D is, of course, equivalent to the whole of space, in the [illegible] that its generalized Green function is the same as the Green function of the [illegible].

† This region, by definition, [illegible] the point O at which $g = \infty$.

‡ Cf. Osgood, 8, pp. 588-590.

§ For an example illustrating this fact, see Gergen, 2, pp. 198-200.

¶ [illegible] here as a possible value of f the value ∞.

where R is independent of Q, then a sequence $\{\phi_n(Q)\}$, $n = 1, 2, \cdots$, of functions $\phi_n(Q)$, continuous on σ, can be so defined that

$$R \leq \phi_1(Q) \leq \phi_2(Q) \leq \cdots \leq \phi_n(Q) \cdots \leq f(Q), \tag{1.2}$$

$$\lim_{n\to\infty} \phi_n(Q) = f(Q). \tag{1.3}$$

The truth of (i)* and (iii)† is well known, while (ii) can be verified by differentiation.

2.1. *Lemmas.* To establish the theorem we first prove two lemmas relative to D, g, and u.

LEMMA 1. *If D is bounded, and star-shaped with regard to O and if the boundary d of D is a regular, analytic surface, then*

$$u(x, y, z) \leq -g(x, y, z) \tag{2.11}$$

in $D - O$.‡

To prove this, consider the function $w(x, y, z)$ defined as

$$w = u + g$$

in $D - O$. By (ii), u is harmonic in $D - O$, and thus, since g is harmonic there, w has the same property. Further, by (i), u is continuous in $D + d - O$, when defined on d by means of its limiting values. Hence, since g vanishes continuously on d, w is continuous in $D + d - O$, when properly defined on d. Finally, since g has the form

$$g = 1/r + v(x, y, z),$$

where v is harmonic in D, we have

$$w = v + xv_x + yv_y + zv_z;$$

and it follows that w is harmonic at O, when defined as $v(0, 0, 0)$ there. Thus w is harmonic in D and continuous in $D + d$. To show, then, that (2.11) holds, it is enough to prove that w is never positive on d.

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* Cf. Liapounoff, **6**, pp. 305-311, and Neumann, **7**, p. 178. Liapounoff proves that the partial derivatives of g have the property stated for a much more general type of boundary, provided also that the Neumann principle is applicable to the boundary. Neumann proves that the principle in question is applicable to boundaries of a very general character, in particular, to boundaries of the type specified above. For a statement of the Neumann principle, see Liapounoff, p. 268. See also Kellogg, **5**. Kellogg considers the boundary values of the partial derivatives of all orders of harmonic functions. He obtains results which include (i) as a particular case.

† For a proof of (iii) see Carathéodory, **1**.

‡ It is classical that the ordinary Green function exists for regions of this type.

Let $P_1(x_1, y_1, z_1)$ be any point on d. Then, since D is star-shaped with regard to O, and d is an analytic surface, the segment OP_1 lies entirely in D, except for its extremity P_1. Thus we have

$$w(x_1, y_1, z_1) = \lim_{\eta \to 1-0} w(\eta x_1, \eta y_1, \eta z_1).$$

But this is evidently

$$w(x_1, y_1, z_1) = \lim_{\eta \to 1-0} u(\eta x_1, \eta y_1, \eta z_1),$$

since g vanishes on d. Accordingly,

$$\begin{aligned} w(x_1, y_1, z_1) &= \lim_{\eta \to 1-0} \left\{ \eta \mid OP_1 \mid [\partial g/\partial r]_{\substack{x=\eta x_1\\ y=\eta y_1\\ z=\eta z_1}} \right\} \\ &= \mid OP_1 \mid \lim_{\eta \to 1-0} \left\{ [\partial g/\partial r]_{\substack{x=\eta x_1\\ y=\eta y_1\\ z=\eta z_1}} \right\}. \end{aligned}$$

Now, since the limit on the right here exists, and since g is positive in D and vanishes continuously on d, we must have $w(x_1, y_1, z_1) \leq 0$. This completes the proof of the lemma.

2.2. LEMMA 2. *An infinite sequence* $\{D_n\}$, $n = 1, 2, \cdots$, *of bounded regions* D_n *can be so constructed that*

(a) $$D_1 \subset D_2 \subset \cdots \subset D_n \cdots \subset D,$$

(b) $$D_1 + D_2 + \cdots + D_n + \cdots = D,$$

(c) *the boundary* d_n *of* D_n *is a regular, analytic surface,*

(d) D_n *is star-shaped with regard to* O.

The proof rests on (iii). Consider a point Q on the unit sphere σ about O. The ray $\gamma(Q)$ through Q, issuing from O, either fails to pierce d or else, since d is closed, there is a first point, starting from O, of intersection of $\gamma(Q)$ with d. At each point Q of σ, then, let $f(Q)$ be defined as the distance from O to this first point of intersection if $\gamma(Q)$ pierces d, and as ∞ otherwise.

The function f, we observe, is lower semi-continuous on σ, for d is closed. Moreover, since O is a point of D and D is open, there is a number $R > 0$ such that

$$f(Q) \geq R$$

for Q on σ. Hence, by (iii), we can find a sequence $\{\phi_n(Q)\}$, $n = 1, 2, \cdots$, of functions $\phi_n(Q) = \phi_n(x, y, z)$, continuous on σ, such that (1.2) and (1.3) hold.

Now, since $\phi_n(Q)$ is continuous, there exists a function

$$\psi_n(P) = \psi_n(x,\, y,\, z)$$

which is harmonic in the interior $\sum$ of σ, continuous in $\sigma + \sum$, and which assumes the value $n\phi_n(Q)/(n+1)$ at each point Q of σ.

By means of ψ_n we define D_n. We choose $r_n < 1$ so near to one that, if Q is any point on σ and P the point of intersection of OQ with the sphere σ_n of radius r_n about O,

$$|\, \psi_n(P) - \psi_n(Q) \,| < R/(4n)^2; \tag{2.21}$$

and we define D_n as the set of points determined by the inequality

$$r < \psi_n(xr_n/r,\, yr_n/r,\, zr_n/r).$$

A point P belongs to D_n, then, when, and only when,

$$|\, OP \,| < \psi_n(P') \tag{2.22}$$

where P' is the point of intersection of OP, or OP extended, with σ_n. The set D_n is, we observe, not null, for, if $P(x,\, y,\, z)$ be any point on σ_n,

$$\begin{aligned} \psi_n(P) &> \psi_n(x/r_n,\, y/r_n,\, z/r_n) - R/(4n)^2 \\ &= n\phi_n(x/r_n,\, y/r_n,\, z/r_n)/(n+1) - R/(4n)^2 \\ &> 7R/16, \end{aligned}$$

so that D_n contains all the points in the interior of the sphere of radius $7R/16$ about O.

Now plainly D_n is bounded, open, and star-shaped with regard to O. Moreover, since the boundary d_n of D_n is represented by the equation

$$h_n(x, y, z) = r - \psi_n(xr_n/r,\, yr_n/r,\, zr_n/r) = 0,$$

and since $\partial h_n/\partial r = 1$ and ψ_n is harmonic and therefore analytic in $\sum$, d_n is a regular analytic surface. It remains, then, to prove that (a) and (b) hold.

Consider (a). Let P be any point of D_n, and P_n, Q the points of intersection of OP, or OP extended, with σ_n and σ, respectively. By (1.2), (2.21) and (2.22),

$$|\, OP \,| < \psi_n(P_n) < n\phi_n(Q)/(n+1) + R/(4n)^2 < \phi_n(Q),$$

whence

$$|\, OP \,| < f(Q), \tag{2.23}$$

and

$$\begin{aligned} |\, OP \,| &< n\phi_{n+1}(Q)/(n+1) + R/(4n)^2 \\ &< (n+1)\phi_{n+1}(Q)/(n+2) - R/4(n+1)^2, \end{aligned}$$

whence

$$|\, OP \,| < \psi_{n+1}(P_{n+1}). \tag{2.24}$$

From (2. 23) it follows that D_n is contained in D, and from (2. 34) that D_n is contained in D_{n+1}.

As for (b), we have, if P is any point of D and Q the point of intersection of OP, or OP extended, with σ,

$$| OP | < f(Q),$$

since D is star-shaped with regard to O. But, denoting by P_k the point of intersection of OQ with σ_k, we have also, by (1. 3),

$$\lim_{n \to \infty} \psi_n(P_n) = \lim_{n \to \infty} \phi_n(Q) = f(P).$$

Hence, for n sufficiently large,

$$| OP | < \psi_n(P_n),$$

and so D_n contains P. This completes the proof.

3. *Proof of the theorem.* As a consequence of Lemmas 1 and 2 the proof of the theorem is almost immediate. Let $g_n(x, y, z)$ be Green's function of D_n with pole at O. Then $g_n - g$ tends to zero with $1/n$ in $D - O$,* the convergence being uniform in any closed subset of $D - O$. Moreover, because of this uniform convergence,

$$u_n = xg_{nx} + yg_{ny} + zg_{nz}$$

tends to

$$u = xg_x + yg_y + zg_z$$

in $D - O$. Accordingly, by Lemma 1,

$$u \leq - g$$

in $D - O$. But, by the reasoning of Lemma 1, $u + g$ is harmonic in D and thus

(3. 1) $$u < - g$$

in $D - O$ unless

$$u + g = 0$$

throughout $D - O$. This equation is, however, equivalent to the equation

$$u + g = v + xv_x + yv_y + zv_z = 0,$$

where v is harmonic in D and given by

$$v = g - 1/r$$

* Cf. Kellogg, **3**, p. 598, and the fourth footnote of this paper.

in $D-O$; thus (3.1) holds unless

$$v(0, 0, 0)=0.$$

This is possible only when v vanishes throughout D,* and thus (3.1) holds unless g reduces to $1/r$.

Now

$$g>0 \tag{3.2}$$

in $D-O$, and

$$u^2=(xg_x+yg_y+zg_z)^2 \leq r^2(\nabla g)^2$$

there; hence

$$(\nabla g)^2 \geq g^2/r^2>0,$$

the inequality holding unless g reduces to $1/r$. This is (1.1).

To complete the proof, we observe that (3.1) and (3.2) imply that the directional derivative of g in the direction of O at any point in $D-O$ is positive. Hence, if c is any positive constant and P_1 any point in the region $g>c$, $g(P)>c$ on the segment OP_1. Thus the region $g>c$ is star-shaped with regard to O; and the theorem is proved.

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* See Kellogg, 5. We have

$$g_n=1/r+v_n,$$

where $v_n<0$ in D_n,

$$v=\lim_{n\to\infty} v_n.$$

Hence $v \leq 0$ in D, and therefore cannot vanish in D unless it vanishes throughout D.

A JUNCTION PROPERTY OF LOCALLY CONNECTED SETS.

By G. T. Whyburn.

1. Let M denote a connected and locally connected metric space. In this paper it will be shown that if K is any compact subset of such a set M, there exists a subset H of M which is dense in K (i. e. every point of K either belongs to H or is a limit point of $H \cdot K$) and is the image under a *uniformily continuous* transformation of the set of all dyadic rational numbers on an interval. This proposition yields a number of important consequences, some of which will be discussed in the concluding section (§ 4) of this paper. In particular, it yields as a corollary the celebrated theorem of Hahn-Mazurkiewicz that any *compact* space M is the continuous image of an interval, and it enables one to deduce the recent Moore-Menger theorem on the arcwise connectivity of *complete* spaces M from the well known theorem that *compact* spaces M are arcwise connected.

2. Lemma. *If F is any finite subset of M, any two points a and b of F can be joined, for every $\epsilon > 0$, by an ϵ-chain* of points containing F and each pair of successive points of which lie together in an ϵ-region* of M.*

For let the points of F be $a = p_1, p_2, \cdots, p_n = b$. Now we can join p_1 and p_2 by an ϵ-chain each pair of successive points of which lie together in an ϵ-region of M. For this purpose it is only necessary to join p_1 and p_2 by a simple chain † of $\epsilon/2$-regions and take one point from each link in this chain. Similarly we join p_2 to p_3 by such a chain of points, and then join p_3 to p_4, and so on until we reach p_n. Then clearly the sequence of points taken in the order in which they are obtained in this complete process is an ϵ-chain of points from a to b which contains F and is such that any two successive points lie together in an ϵ-region of M.

3. Theorem. *If K is any self-compact subset of M, there exists a subset H of M which is dense in K and which is the image under a uniformly continuous transformation of the set of all dyadic rational ‡ numbers on the unit interval.*

* An *ϵ-chain* of points joining a and b is a finite sequence $a = X_1, X_2, \cdots, X_n = b$ of points such that each pair of successive ones are at a distance $< \epsilon$ apart. A *region* in M is any connected open subset of M, and an *ϵ-region* is any region of diameter $< \epsilon$.

† See R. L. Moore, "On the Foundation of Plane Analysis Situs," *Transactions [illegible] Society*, Vol. 17 (1916), Theorem 10.

‡ [illegible] 0, 1 which can be written [illegible]

Proof. Since K is compact and metric it is therefore separable. Let $P = \sum_1^\infty p_i$ be a countable set of points in K which is dense in K and, for each integer n, let $P_n = \sum_1^n p_i$. Let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive numbers such that $\sum_1^\infty \epsilon_i$ converges.

It follows by the Borel Theorem that K is contained in the sum of a finite number of $\epsilon_1/2$-regions of M; and if we choose one point of P in each of these regions and take an integer n_1 greater than the subscript of any of the points so chosen, then every point of K lies in an $\epsilon_1/2$-region with some point of P_{n_1}. Now let a and b be any two points of M. By the lemma, § 2, there exists an $\epsilon_1/2$-chain C^1 of points from a to b which contains P_{n_1} and is such that any two successive points lie in an $\epsilon_1/2$-region and which, furthermore, we may suppose contains exactly $2^{v_1} + 1$ points

$$a = X_0^1,\ X_1^1, \cdots, X^1_{2^{v_1}} = b.$$

Similarly, with the aid of the Borel Theorem, it follows that there exists an integer $n_2 \geqq n_1$ such that every point of K lies in an $\epsilon_2/2$-region with some point of P_{n_2}. For each integer i, $0 \leqq i \leqq 2^{v_1}$, let F_i^1 be the set of all those points of P_{n_2} which lie together with X_i^1 in an $\epsilon_1/2$-region. Since C^1 contains P_{n_1}, it follows that every point of P_{n_2} belongs to some set F_i^1. Now for each i, there exists an ϵ_1-region M_i^1 which contains $X_i^1 + F_i^1 + X^1_{i+1}$, because each point of $F_i^1 + X^1_{i+1}$ lies together with X_i^1 in some $\epsilon_1/2$-region. Since the region M_i^1 is itself a connected and locally connected space, it follows by the lemma that M_i^1 contains an $\epsilon_2/2$-chain C_i^1 of points from X_i^1 to X^1_{i+1} which contains F_i^1 and is such that any two successive points lie together in a $\epsilon_2/2$-region. Furthermore, we may suppose that all these chains $[C_i^1]$ contain the same number, say $2^{v_2} + 1$, of points which we denote by

$$X_i^1 = X^2_{i \cdot 2^{v_2}},\ X^2_{i \cdot 2^{v_2}+1}, \cdots, X^2_{(i+1)2^{v_2}} = X^1_{i+1}.$$

Clearly the chains $[C_i^1]$ taken in order form an $\epsilon_2/2$-chain C^2 from a to b which contains all points of P_{n_2} and is such that any two successive points lie together in an $\epsilon_2/2$-region.

Let us continue in this manner. In general, for each k, we choose an integer $n_k \geqq n_{k-1}$ such that every point of K lies in an $\epsilon_k/2$-region with some point of P_{n_k}. For each i, $0 \leqq i \leqq 2^{v_1+v_2+\cdots+v_{k-1}}$, let F_i^{k-1} be the set of all points of P_{n_k} which lie together with X_i^{k-1} in an $\epsilon_{k-1}/2$-region. Then there exists an ϵ_{k-1}-region M_i^{k-1} containing $X_i^{k-1} + F_i^{k-1} + X^{k-1}_{i+1}$. By the lemma,

M_i^{k-1} contains an $\epsilon_k/2$-chain C_i^{k-1} from X_i^{k-1} to X_{i+1}^{k-1} which contains all points of F_i^{k-1} and is such that any two successive points lie together in an $\epsilon_k/2$-region. We may suppose that all these chains $[C_i^{k-1}]$ contain the same number, say $2^{v_k}+1$, of points. Clearly, then, these chains $[C_i^{k-1}]$, taken in the order $C_1^{k-1}, C_2^{k-1}, \cdots, C_{2^{v_1+v_2+\cdots+v_{k-1}}}^{k-1}$ form an $\epsilon_k/2$-chain C^k of points from a to b which contains P_{n_k} and is such that any two successive points lie together in an $\epsilon_k/2$-region and which consists of exactly $2^{v_1+v_2+\cdots+v_k}+1$ points:

$$a = X_0^k, X_1^k, \cdots, X^k_{2^{v_1+v_2+\cdots+v_k}} = b.$$

Furthermore, in this notation we have always:

$$\text{(i)} \qquad X^k_{i\cdot 2^{v_k}} = X_i^{k-1}.$$

Now let D be the set of all dyadic rational numbers on the interval $(0,1)$ and let t be any number of D. Then for some k we can write

$$t = i/2^{v_1+v_2+\cdots+v_k}, \qquad (0 \leq i \leq 2^{v_1+v_2+\cdots+v_k}).$$

To this value of t we make correspond the point X_i^k and write $T(t) = X_i^k$. Thus we define the transformation T over D. It is a consequence of (i) that T is single valued on D. Furthermore T is uniformly continuous on D, as is seen by identically the same argument as given for a similar purpose in the author's paper "Concerning continuous images of the interval." *

Finally, let $H = T(D)$. Then since $H = \sum_1^\infty C^n$ and for each n, $C^n \supset P_{n_n} = \sum_1^n p_i$, it follows that $H \supset \sum_1^\infty p_i = P$; and since P is dense in K, therefore H is dense in K. This completes the proof.

4. *Consequences of* § 3. (a). Suppose that the set M as above treated is a subset of some complete space. In this case it follows that since D is compact, so also is H. And since T is uniformly continuous on D, it is well known that the definition of T can be extended to the limit points of D in one and only one way so that the extended transformation is single valued and continuous on $\bar{D} = (0,1)$. Thus $\bar{H}$ is the continuous image of an interval, and therefore it is a compact locally connected continuum. Obviously $\bar{H}$ contains K. Hence we have shown that *if the connected and locally connected set M is a subset of some complete space, then every self-*

* This Journal, Vol. 53 (1931), p. 673. Much similarity will be noted between [illegible] The latter in turn is very similar to an argument employed by Hahn in *Wiener Berichte*, Vol. 123 (1914), p. 2436.

compact subset K of M lies in a compact locally connected continuum which is a subset of $\bar{M}$.

(b). Suppose M itself is compact. Then we can take $K = M$ and thus we have $\bar{H} = M$. Then by the reasoning given under (a), we see that M is the continuous image of an interval. Thus we obtain the Hahn-Mazurkiewicz * theorem as a corollary to our proposition in § 3.

(c). Suppose M itself is complete. Then, treating M as the space, we have $\bar{H} \subset \bar{M} = M$. Thus by (a) we have that *any self-compact subset K of a connected and locally connected complete space M is contained in a compact locally connected continuum which is a subset of M.*

Thus, using the well known theorem that any compact locally connected continuum is arcwise connected (i. e. any two of its points are the extremities of a simple arc lying in it), we obtain the stronger proposition of Moore-Menger † that any connected and locally connected complete space is arcwise connected. Indeed, using (a), we obtain an even stronger result which may be stated as follows: *If M is any connected and locally connected subset of a complete space, then every two points of M can be joined by a simple continuous arc lying wholly in $\bar{M}$.*

It is to be noted, however, that this stronger result is a consequence also of some results of Menger's obtained in his article just referred to.

(d). By (a) we know that any complete space which contains a non-degenerate connected and locally connected set must contain compact locally connected continua and hence must contain simple continuous arcs. Thus in particular, a compact continuum which contains no arc ‡ *can contain no non-degenerate connected and locally connected set.* Since a continuum which is hereditarily locally connected, i. e., a continuum every subcontinuum of which is locally connected, has § the stronger property that *each of its connected subsets is locally connected,* we remark that the arcless continua and the hereditarily locally connected continua are diametric opposites not only as regards their subcontinua but also with respect to their non-degenerate connected subsets.

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* See Hahn, *loc. cit.*, and Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 1 (1920), p. 166.

† See R. L. Moore, *Bulletin of the American Mathematical Society*, abstract, Vol. 33 (1927), p. 141; and K. Menger, *Monatshefte für Mathematik und Physik*, Vol. 36 (1930), p. 210.

‡ For examples, see Knaster, *Fundamenta Mathematicae*, Vol. 3 (1922), p. 247, and Whyburn, this Journal, Vol. 52 (1930), p. 319.

§ See R. L. Wilder, *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), p. 616.

ON THE ANALYTICAL EXTENSION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.*

By Howard K. Hughes.

1. *Introduction.* In this paper we shall be concerned with the two kinds of infinite series known, respectively, as "factorial series of the first kind" and "factorial series of the second kind." A factorial series of the first kind is a series of the form

$$\sum_{n=0}^{\infty} a_n n!/z(z+1)\cdots(z+n), \tag{1}$$

while a factorial series of the second kind is one of the form

$$a_0+\sum_{n+1}^{\infty} a_n(z-1)(z-2)\cdots(z-n)/n! = \sum_{n=0}^{\infty} a_n\binom{z-1}{n}. \tag{2}$$

In each case z represents a variable, real or complex, and the coefficients $a_0, a_1, a_2, \cdots$ are independent of z.

Series of form (1) are of fundamental importance in the study of the solutions of difference equations, and play a rôle which is analogous to that played by power series in expressing the solutions of differential equations.

On the other hand, series (2) is of less value in function theory since the representation of a given function $f(z)$ in such a series is sometimes possible in more than one way.† However series (2) frequently enters in problems in interpolation, it being seen that the coefficients a_n are equal to the successive differences of the function $f(z)$ at the point $z=1$.

Factorial series apparently made their first appearance in the work of Newton and Stirling. They were used by Schlömilch about the middle of the nineteenth century in connection with studies concerning definite integrals. Nevertheless, it is not until recent years that a critical study has been made of the series in question, particularly as to their convergence and analytic properties. This has been accomplished in large measure by Bendixson, Nielsen, Landau and Nörlund. A list of papers relative to such investigations will be found in the Bibliography. Still more recently, H. Bohr has discovered some results concerning the summability of series (1). Nörlund has [illegible] further the possibility of developing a given function $f(z)$ in

* Presented to the American Mathematical Society at Chicago, April 1, 19[illegible].

† See, for example, Nörlund (5), pp. 224-226. ‡ Bibliography at end of article.

factorial series of both kinds, and has also discovered some noteworthy results on the analytical extension of functions defined by such series. He has employed extensively factorial series of the first kind in determining the solutions of linear difference equations. Moreover, certain generalized forms of series (1) and (2) have been studied by Pincherle, Carmichael, Fort, and others.

The object of this paper is to obtain new results pertaining to the analytical extension of a function such as is defined by series (1) or (2). As has been shown by Nielsen, the region of convergence of a factorial series is, in general, a half-plane, lying to the right of a vertical line in the z complex plane, this line being called the *boundary line of convergence*. Our problem is to extend analytically into regions to the left of the boundary line of convergence the function defined by such a series, the series itself being regarded as given.

Chapter I is expository, and sets forth some of the fundamental facts concerning factorial series as discovered by Nielsen, Landau, and others. We include also a brief discussion of the results already noted by Nörlund on the analytical extension of such series. Chapter II is devoted to results on analytical extension which are believed to be new, these results being embodied in three theorems. As to the methods employed, the proof of Theorem I rests upon the properties of a certain definite integral, while the proofs of Theorems II and III rest upon the calculus of residues. For this reason the chapter has been divided into two parts. Theorem I, including its proof, has been embodied in Part I, while Theorems II and III have been embodied in Part II. As a corollary to Theorem III, certain results pertaining to the asymptotic development of a function $W(z)$ defined by series (2) have been obtained. Several examples have been introduced to illustrate the results.

CHAPTER I.

Fundamental Theorems in the Theory of Factorial Series.

2. The purpose of this chapter is twofold. In the first place we shall present in concise form several theorems, now classical, concerning the convergence of the two kinds of series mentioned in the Introduction, namely,

$$\sum_{n=0}^{\infty} a_n n!/z(z+1)\cdots(z+n) \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_n \binom{z-1}{n}. \tag{2}$$

These theorems are fundamental in what follows. In the second place we shall indicate briefly the results which have been obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2).

The domain of convergence of factorial series has been studied by Nielsen, Pincherle, Landau, and others. Moreover, Nörlund has deduced a noteworthy result concerning the region of uniform convergence. The more important results pertaining to both convergence and uniform convergence, and the consequences thereof, and those concerning analytical extension will now be briefly stated, the proofs, however, being omitted.

I* (a). If series (1) converges for $z = z_0$, then it converges for all values of z for which $R(z) > R(z_0)$† except at the points $z = 0, -1, -2, \cdots$.

(b). If series (2) converges for $z = z_0$, where z_0 is not a positive integer, then it converges for all values of z for which $R(z) > R(z_0)$.

In case the series converges neither everywhere nor nowhere, it follows from the foregoing theorem that the region of convergence of series (1) or (2) is a half-plane, bounded on the left by a line of the form $R(z) = \lambda$, the number λ being called the *abscissa of convergence*. The series converges everywhere to the right of the line $R(z) = \lambda$, and diverges everywhere to the left of the same line. On the line the series may either converge or diverge. In the case of series (1), the points $z = 0, -1, -2, \cdots$, should any such points lie to the right of the boundary line of convergence, are excluded from the domain of convergence, since at these points the terms of the series become infinite. Series (2) may also be said to converge for positive integral values of z, should any such points lie to the left of the line $R(z) = \lambda$, since at these points the series reduces to a polynomial. Theorems will be stated presently showing how λ may actually be determined for any given series of form (1) or (2).

Landau has proved the following theorem concerning the uniform convergence of series (1) and (2):

II. Each of the two series (1) and (2) converges uniformly in any region R which, together with its boundary, lies interior to the half-plane of convergence of the series, provided in the case of series (1) that R does not include any of the points $z = 0, -1, -2, \cdots$.

Moreover Nörlund has shown that if series (1) or (2) converges at $z = z_0$ [illegible] of series (2), $z_0 \neq$ positive integer), then it converges uniformly in any sector δ whose vertex is at z_0 and which [illegible] inequalities

* See Nielsen (2), pp. 416-429; (8), p. 124, 235; also Bendixson (1), pp. 15-20.

† [illegible] $R(z)$ occurs, it is understood to mean the *real part of* z.

$$0 \leqq |z - z_0| \leqq G, \qquad -\pi/2 + \epsilon \leqq \arg(z - z_0) \leqq \pi/2 - \epsilon,$$

where G is any positive number, and ϵ is an arbitrarily small positive quantity. In case of series (1), the neighborhoods of the points $z = 0, -1, -2, \cdots$, should any such points lie in δ, are excepted.

As an immediate consequence of II, we have

III. Each of the two series (1) and (2) represents an analytic function in its half-plane of convergence, the neighborhoods of the points $z = 0, -1, -2, \cdots$ being excepted in the case of series (1).

We next note the manner in which the actual value of λ pertaining to any given series (1) or (2) may be determined. For this purpose, let us set

$$\alpha = \overline{\lim_{n\to\infty}} \log |\sum_{0}^{n} a_n| / \log n, \qquad \beta = \overline{\lim_{n\to\infty}} \log |\sum_{0}^{n-1} (-1)^n a_n| / \log n,$$
$$\alpha' = \overline{\lim_{n\to\infty}} \log |\sum_{n+1}^{\infty} a_n| / \log n, \qquad \beta' = \overline{\lim_{n\to\infty}} \log |\sum_{n}^{\infty} (-1)^n a_n| / \log n.$$

Then we have the following theorem: *

IV. The abscissa λ of convergence of series (1) is given by α or α' according as the series $\sum_{n=0}^{\infty} a_n$ is divergent or convergent; and the abscissa λ of convergence of series (2) is given by β or β' according as the series itself is divergent or convergent for $z = 0$.

As limiting cases we may have $\lambda = -\infty$ or $\lambda = +\infty$, corresponding to which cases the series converges or diverges respectively for all values of z.

3. We shall now mention briefly some results obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2). We have the following four theorems: †

V. Any function $\Omega(z)$ which can be developed in a series of form (1) can also be developed in the form

$$\Omega(z) = \sum_{n=0}^{\infty} c_n n!/(z+\omega)(z+2\omega)\cdots(z+n\omega) \tag{3}$$

where ω is any given constant. Let λ_ω denote the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω such that $\omega_1 > 1$, and let ω' be any variable value such that $\omega' > \omega_1$. Then $\lim_{\omega'\to\infty} \lambda_{\omega'}$ exists, and is such that

$$\lambda_{\omega'} \leqq \lambda_{\omega_1} \leqq \lambda.$$

* Landau (1), p. 156, 195; Nörlund (5), p. 223, 257.

† Nörlund (1); (2), pp. 341-379; (5), pp. 233-240, and 262-266.

As an immediate consequence of V, we have the following:

VI. Series (3) when the (variable) quantity ω is real and greater than unity, furnishes the analytical extension of the function $\Omega(z)$ defined by (1) throughout the strip

$$\lambda_\omega < R(z) \leqq \lambda$$

except in the neighborhoods of the points $z = 0, -1, -2$, etc.

VII. Any function $W(z)$ which can be developed in a series of form (2) can also be developed in the form

$$W(z) = \sum_{n=0}^{\infty} c_n (z-\omega)(z-2\omega)\cdots(z-n\omega)/n! \tag{4}$$

where ω is any given constant. Let λ_ω represent the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω, and ω' any variable value such that $0 < \omega' < \omega_1 < 1$, then $\lim\limits_{\omega'\to 0} \lambda_{\omega'}$ exists and is such that

$$\lambda_{\omega'} \leqq \lambda_{\omega 1} \leqq \lambda.$$

Hence we have, corresponding to VI above, the following:

VIII. Series (4), when the (variable) quantity ω is real and such that $0 < \omega < 1$, furnishes the analytical extension of the function $W(z)$ defined by (2) throughout the strip

$$\lambda\omega < R(z) \leqq \lambda.$$

CHAPTER II.

Analytical Extension of Functions Defined by Factorial Series.

I. *Analytical Extension By Use of a Loop Integral.*

4. As has been stated in the Introduction, it is the purpose of this paper to derive new results concerning the analytical extension of functions defined by factorial series. In the results of Nörlund, mentioned in the preceding chapter, the given series is transformed into another factorial series whose line of convergence, in general, lies to the left of the line of convergence, $R(z) = \lambda$, of the given series. Thus, these results furnish a means of ob- [illegible] analytical extension of the function defined by the given series [illegible]

[illegible] satisfy certain conditions.

For our present purpose, let us now write series (1) of the Introduction in the form

$$(1) \qquad \sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n)$$

where it is understood that z is a variable, and that the coefficients $g(n)$ depend only on n.

We start with a result shown incidentally by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane,"* namely that the (entire) function $f(t,z)$ defined by the special series

$$f(t,z) = \sum_{n=0}^{\infty} t^n/\Gamma(z+n+1)$$

may be written in the following form for any value (real or complex) of t, and for any value of z such that $R(z) > 0$:

$$(2) \qquad f(t,z) = \frac{e^t t^{-z}}{\Gamma(z)} \int_0^t e^{-t} t^{z-1} dt.$$

If we restrict ourselves to values of t such that $R(t) > 0$, we may evidently write (2) in the form

$$(3) \qquad f(t,z) = \frac{e^t t^{-z}}{\Gamma(z)} \left\{ \int_0^{\infty} e^{-t} t^{z-1} dt - \int_t^{\infty} e^{-t} t^{z-1} dt \right\},$$

where it is understood that in the first integral the integration takes place along the positive axis of reals from $t=0$ to $t=\infty$, while in the second integral it takes place in the direction of the positive real axis from $t=t$ to $t=\infty$. We have now only to recall that

$$\int_0^{\infty} e^{-t} t^{z-1} dt = \Gamma(z)$$

in order to write (3) in the form

$$(4) \qquad f(t,z) = e^t t^{-z} \left\{ 1 - \frac{1}{\Gamma(z)} \int_1^{\infty} e^{-t} t^{z-1} dt \right\}$$

But the right hand member here appearing is defined not only when $R(z) > 0$, but is seen to have a meaning whatever be the value of z. Moreover in the neighborhood of any finite point z it is seen to be analytic.

If in particular we take t equal to unity in (4), and multiply both members by $\Gamma(z)$, the product $\Gamma(z)\cdot f(1,z)$ occurring on the left is seen to be the function $G(z)$ defined by the special factorial series

* *Bulletin of American Mathematical Society*, Vol. 34 (1928), p. 91. We here change Ford's notation slightly in order to accommodate his result more aptly to our subsequent discussion.

$$\sum_{n=0}^{\infty} 1/z(z+1)\cdots(z+n).$$

Thus, for the function $G(z)$ defined by this special series, we may write

$$G(z) = e\{\Gamma(z) - \int_1^{\infty} e^{-t}t^{z-1}dt\}.$$

Since the right member of this equation preserves a meaning and is an analytic function of z throughout any region of the z plane which does not include any of the points $z = 0, -1, -2, \cdots$, it furnishes the value of $G(z)$ throughout any such region.

We shall now show that this special result may be generalized so as to obtain an analogous one which will apply to factorial series of the first kind in general, at least under some restrictions on the coefficients. In fact, such a generalization is possible as follows:

THEOREM I. *Having given the factorial series*

$$(1) \qquad \sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n)$$

let the following power series be formed:

$$g(0) + \frac{\Delta g(0)}{1!}(1-t) + \frac{\Delta^2 g(0)}{2!}(1-t)^2 + \cdots$$

where

$$\Delta g(0) = g(1) - g(0), \qquad \Delta^2 g(0) = g(2) - 2g(1) + g(0),$$

etc. Suppose that the function $h(t)$ defined by this series can be expanded about the origin in a series whose radius of convergence exceeds unity. Suppose further that $h(t)$ is analytic throughout an infinite strip of the t plane which shall include the positive real t axis together with the origin, and which may be of arbitrarily small width. Furthermore suppose that the function $h(t)$ is such that the improper integral

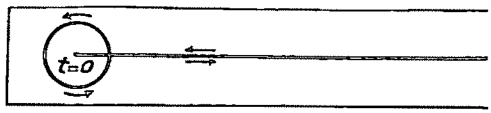
$$\int_0^{\infty} h(t)e^{-t}t^{z-1}dt$$

exists, at least for $R(z) > 0$. Then the function $\Omega(z)$ defined by (1) may be expressed for all values of z in the form*

$$(5) \qquad \Omega(z) = \frac{e}{e^{2\pi iz}-1}\int_L h(t)e^{-t}t^{z-1}dt - e\int_1^{\infty} h(t)e^{-t}t^{z-1}dt.$$

* For further comments on these conditions, see Article 5.

where the first integral is the loop integral obtained by an integration about the origin from the point at infinity along the upper side of the positive real t axis to the neighborhood of the origin, and returning to infinity along the lower side of the same axis, as indicated in the figure.



An immediate corollary of this theorem is as follows: If the given series (1) has a finite abscissa of convergence, then relation (5) furnishes the analytical extension of the function $\Omega(z)$ throughout all finite portions of the z plane which do not contain any of the points $z = 0, -1, -2, \cdots$.

In order to prove the theorem, let us start by considering the expression

$$F(t, z) = e^t \cdot t^{-z} \int_0^t h(t) e^{-t} t^{z-1} dt \tag{6}$$

where z has any value (real or complex) such that $R(z) > 0$, and where we shall regard $h(t)$ as any function which satisfies the conditions imposed in the hypothesis. Under such conditions the integrand in (6) can be expanded in powers of t, and the series can be integrated between the limits 0 and t provided $|t| \leqq 1$. It follows that we may write the function $F(t, z)$ in the form

$$F(t, z) = c_0 + c_1 t + c_2 t^2 + \cdots, \tag{7}$$

where the coefficients c_n are as follows:

$$c_0 = h(0)/z$$
$$c_1 = h(0)/z + [h'(0) - h(0)]/(z+1)$$
$$c_2 = h(0)/z + [h'(0) - h(0)]/1!(z+1) + [h''(0) - 2h'(0) + h(0)]/2!(z+2)$$
$$\cdots\cdots\cdots\cdots$$
$$c_n = \sum_{s=0}^{n} G(s)/(n-s)!\,(z+s),$$
$$\cdots\cdots\cdots\cdots$$

where

$$G(s) = \sum_{r=0}^{s} (-1)^r \binom{s}{r} h^{(s-r)}(0).$$

But if we write c_1 in the form

$$c_1 = A/z + B/z(z+1),$$

upon clearing of fractions and equating coefficients of like powers of z, thus determining the values of A and B, we find

$$c_1 = h'(0)/z + [h(0) - h'(0)]/z(z+1).$$

If we do the same for c_2, c_3, etc., the coefficients in (7) are found to take the following form:

$$
\begin{aligned}
&c_0 = h(0)/z \\
&c_1 = h'(0)/z + [h(0) - h'(0)]/z(z+1) \\
&c_2 = h''(0)/2!\,z + [h'(0) - h''(0)]/1!\,z(z+1) \\
(8)\qquad &\qquad\qquad + [h(0) - 2h'(0) + h''(0)]/z(z+1)(z+2) \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&c_n = \sum_{s=0}^{n} H(s)/(n-s)!\,z(z+1)\cdots(z+n),
\end{aligned}
$$

where

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$H(s) = \sum_{r=0}^{s} (-1)^r \binom{s}{r} h^{(n+r-s)}(0).$$

For our present purpose, let us now set t in (7) equal to unity. The result is

$$F(1, z) = c_0 + c_1 + c_2 + \cdots$$

where the values of c_n are given by (8). We shall next collect the terms in $1/z$, $1/z(z+1)$, etc. The coefficient of $1/z$ is

$$h(0) + h'(0) + h''(0)/2! + \cdots,$$

or, simply $h(1)$. The coefficient of $1/z(z+1)$ is

$$\{h(0) + h'(0) + h''(0)/2! + \cdots\} - \{h'(0) + h''(0) + h'''(0)/2! + \cdots\}$$

or $h(1) - h'(1)$. Having continued indefinitely, we may now write

$$
\begin{aligned}
(9)\qquad F(1, z) &= h(1)/z + [h(1) - h'(1)]/z(z+1) \\
&\qquad + [h(1) - 2h'(1) + h''(1)]/z(z+1)(z+2) + \cdots \\
&\cdots + \sum_{s=0}^{n} (-1)^s \binom{n}{s} h^{(s)}(1)/z(z+1)\cdots(z+n) + \cdots,
\end{aligned}
$$

where it is to be noted that the right member is a factorial series of the first kind. But by (6) we see that, by placing $t = 1$, the function defined by series (9) is given by the integral

$$(10)\qquad F(1, z) = e\int_0^1 h(t)e^{-t}t^{z-1}dt$$

provided that $R(z) > 0$.

[illegible]

$$(11)\qquad \sum_{n=0}^{\infty} g(n)/z(z+1)\cdots(z+n)$$

with the purpose of determining the function $h(t)$ so that the two series (1) and (9) shall coincide, thus causing the function $\Omega(z)$ defined by (1) to be given by the integral (10). In order that the two series shall coincide, it is necessary and sufficient that

$$\begin{aligned} g(0) &= h(1), \\ g(1) &= h(1) - h'(1), \\ g(2) &= h(1) - 2h'(1) + h''(1), \\ &\cdots\cdots\cdots \\ g(n) &= \sum_{n=0}^{\infty} (-1)^s \binom{n}{s} h^{(s)}(1), \\ &\cdots\cdots\cdots \end{aligned}$$

These equations are equivalent to the following:

$$\begin{aligned} g(0) &= h(1), \\ \Delta g(0) &= -h'(1), \\ \Delta^2 g(0) &= h''(1), \\ &\cdots\cdots\cdots \\ \Delta^n g(0) &= (-1)^n h^{(n)}(1), \\ &\cdots\cdots\cdots \end{aligned}$$

Hence we must have

$$h(t) = g(0) + \Delta g(0)\ (1-t) + \Delta^2 g(0)(1-t)^2/2! + \cdots.$$

If this function $h(t)$ satisfies the conditions indicated in the hypothesis of the theorem, we can now assert that the function $\Omega(z)$ defined by series (1) may be written in the form

$$\Omega(z) = e \int_0^1 h(t) e^{-t} t^{z-1} dt. \tag{11}$$

provided $R(z) > 0$. If the abscissa of convergence of (1) is greater than zero, then it follows that (11) furnishes the analytical extension of the function $\Omega(z)$ throughout that portion of the z plane lying between the boundary line of convergence and the axis of pure imaginaries.

In order to extend this result so as to obtain the analytical extension of the function $\Omega(z)$ over the whole finite z plane, let us now write the integral appearing in (11) in the form

$$\int_0^1 h(t) e^{-t} t^{z-1} dt = \int_0^\infty h(t) e^{-t} t^{z-1} dt - \int_1^\infty h(t) e^{-t} t^{z-1} dt. \tag{12}$$

By hypothesis, the first integral on the right exists when $R(z) > 0$. When regarded as a function of z this integral is seen to be analytic throughout

any portion of the half-plane $R(z) > 0$, while the second integral is seen at once to be analytic for all values of z. Inasmuch as the first integral may fail to preserve a meaning for $R(z) < 0$, we proceed to study the function $J(z)$ defined by this integral. In this connection let us consider the loop integral

$$l(z) = \int_L h(t) e^{-t} t^{z-1} dt, \tag{13}$$

the contour L extending from infinity along the upper side of the positive real t axis to the point $t = \epsilon$, (ϵ being an arbitrarily small positive number), thence around the circle of radius ϵ about the origin, and returning to infinity along the lower side of the same axis. It will be sufficient for the moment to consider z as real and positive. If we let $t = r(\cos\theta + i\sin\theta)$, then we have $t^{z-1} = \exp[(z-1)(\log r + i\theta)]$, $0 \leqq \theta \leqq 2\pi$. Along the upper side of the positive real t axis, the angle θ is 0, while along the lower side we have $\theta = 2\pi$. The contribution to $l(z)$ arising from integrating along the upper side of the axis from $t = \infty$ to $t = \epsilon$ is given by the integral

$$-\int_\epsilon^\infty h(t) e^{-t} t^{z-1} dt$$

while the contribution arising from integrating along the lower side from $t = \epsilon$ to $t = \infty$ is given by

$$e^{2\pi i z} \int_\epsilon^\infty h(t) e^{-t} t^{z-1} dt.$$

As ϵ approaches zero, the sum of these two contributions approaches as a limit the expression

$$(e^{2\pi i z} - 1) J(z)$$

where

$$J(z) = \int_0^\infty h(t) e^{-t} t^{z-1} dt.$$

Upon the circle, we have $t^{z-1} = (\epsilon e^{i\theta})^{z-1}$. Moreover the product $h(t) \cdot e^{-t}$, being analytic at the origin, remains less than a constant k in this neighborhood. Hence the contribution to $l(z)$ arising from integrating around the circle is in absolute value less than

$$k\epsilon^z \left| \int_0^\pi e^{iz\theta} d\theta \right| = 2\pi k \epsilon^z.$$

Therefore, as ϵ approaches zero, the contribution in question vanishes. Consequently, we have

$$J(z) = l(z)/(e^{2\pi i z} - 1). \tag{14}$$

Suppose now we let z take on complex values. The left member of (14) is an analytic function of z for $R(z) > 0$. But the right member preserves a meaning and is analytic for all values of z except $z = 0, -1, -2, \cdots$. Hence (14) must hold for all values of z except zero and negative integers. Taking account of (12), and (13), we have, finally,

$$(15) \qquad \Omega(z) = e/(e^{2\pi iz} - 1) \int_L h(t) e^{-t} t^{z-1} dt - e \int_1^\infty h(t) \cdot e^{-t} t^{z-1} dt$$

Hence the proof of the theorem is complete.

5. *Remarks.* The following remarks relative to the foregoing theorem are now to be noted.

1′. We have required that the function $h(t)$ shall admit an expansion in powers of t which shall have a radius of convergence exceeding unity. If, however, the radius is unity, but the series can be integrated termwise from 0 to 1, then Theorem I still holds provided that the other conditions required of $h(t)$ are satisfied.

2′. We require $h(t)$ to be analytic throughout an infinite strip of arbitrarily small width which shall include the positive real t axis and the origin. This condition will evidently be satisfied by any entire function.

3′. We have required further that the function $h(t)$ be such that the integral

$$\int_0^\infty h(t) e^{-t} t^{z-t} dt$$

shall exist for $R(z) > 0$. If the condition stated in remark 2′ is satisfied, then this integral will evidently exist whenever the following statement can be made: Corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that for all values of t sufficiently large, we may write

$$| h(t) | < K_\epsilon e^{t(1-\epsilon)}.$$

We note in this connection that all the conditions imposed on $h(t)$ are satisfied in the special case in which the general coefficient $g(n)$ is a polynomial, for then the function $h(t)$ is also a polynomial.

4′. Nielsen* has shown that if series (1) of the Introduction converges, then the function $\Omega(z)$ defined by that series may be expressed in the form

$$(16) \qquad \Omega(z) = \int_0^1 \phi(t) t^{z-1} dt,$$

* (8), pp. 239-241.

where $\phi(t)$ is the function defined by the power series

$$a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots.$$

Moreover Pincherle* has shown that an integral of the type here appearing exists and is an analytic function of z throughout a half-plane $R(z) > \sigma$, where the abscissa σ may be greater than, equal to, or less than the abscissa λ of convergence of the corresponding factorial series.

That the general result obtained in Theorem I gives the particular result of Nielsen when $R(z) > \max(0, \sigma)$ may be shown as follows: We have defined $h(t)$ in the form

$$h(t) = g(0) + \Delta g(0)(1-t) + \Delta^2 g(0)(1-t)^2/2! + \cdots.$$

Furthermore we may write

$$e^{-t} = (1/e[1 + (1-t) + (1-t)^2/2! + \cdots].$$

Upon multiplying these two series, we obtain

$$\begin{aligned}(17)\quad h(t)\cdot e^{-t} = (1/e)[g(0) &+ \{g(0) + \Delta g(0)\}(1-t) \\ &+ \{\Delta^2 g(0)/2! + \Delta g(0) + g(0)/2!\}(1-t)^2 + \cdots \\ &+ [(\Delta+1)^n g(0)/n!](1-t)^n + \cdots].\end{aligned}$$

But

$$\begin{aligned}&g(0) + \Delta g(0) = g(1) = a_1 \\ &\Delta^2 g(0)/2! + \Delta g'(0) + g(0)/2! = g(2)/2! = a_2,\end{aligned}$$

and in general

$$(\Delta + 1)^n g(0) = g(n)/n! = a_n.$$

Therefore we may write

$$h(t)e^{-t} = (1/e)[a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots],$$

or

$$h(t)e^{-t} = 1/e \cdot \phi(t).$$

Consequently, for $R(z) > \max(0, \sigma)$, we have

$$(18)\qquad e\int_0^1 h(t)e^{-t}t^{z-1}dt = \int_0^1 \phi(t)t^{z-1}dt.$$

Now when $R(z) > 0$, the quantity

$$\frac{1}{e^{-2\pi iz}-1}\int_L h(t)e^{-t}t^{z-1}dt$$

---

* See Ref. (3).

occurring in the right member of (15) reduces to the integral

$$\int_0^\infty h(t)e^{-t}t^{z-1}dt$$

and therefore for $R(z) > 0$ the whole right member reduces at once to the left member of (18). The equivalence of the two results for $R(z) > \max(0, \sigma)$ is thus established. It appears, therefore, that the result of Theorem I goes further than Nielsen's result since the former furnishes the value of the function $\Omega(z)$ throughout the whole finite z plane while the latter furnishes the value of the same function only, in general, for values of z in a certain half-plane.

Examples. 1. Let $\Omega(z)$ be defined by the series

$$\sum_{n=0}^\infty n^2/z(z+1)\cdots(z+n).$$

Then we have $h(t) = t^2 - 3t + 2$. Hence we have

$$\Omega(z) = \int_0^1 e^{1-t}(t^2 - 3t + 2)\cdot t^{z-1}dt.$$

This integral fails to have a meaning for $R(z) < 0$, since the integrand becomes infinite at the lower limit. But by Theorem I we have, for any finite z,

$$\Omega(z) = \frac{e}{e^{2\pi iz}-1}\int_L (t^2 - 3t + 2)e^{-t}t^{z-1}dt - e\int_1^\infty (t^2 - 3t + 2)e^{-t}t^{z-1}dt.$$

2. Consider the series

$$\sum_{n=0}^\infty 2^n/z(z+1)\cdots(z+n).$$

Then we have

$$\begin{aligned} h(t) &= 1 + (1-t) + (1/2!)(1-t)^2 + \cdots, \\ &= e^{1-t}. \end{aligned}$$

and therefore we may express $\Omega(z)$ in the form

$$\Omega(z) = e^2\int_0^1 e^{-2t}t^{z-1}dt.$$

However, this integral has no meaning for $R(z) < 0$, but applying Theorem I we have for any finite value of z,

$$\Omega(z) = \frac{e^2}{e^{2\pi iz}-1}\int_L e^{-2t}t^{z-1}dt - e^2\int_1^\infty e^{-2t}t^{z-1}dt.$$

II. *Analytical Extension by the Calculus of Residues.*

6. In this portion of the present chapter we shall employ a method which differs essentially from that used above. In fact we shall employ the calculus of residues to obtain the analytical extension of functions defined by both kinds of factorial series under the assumption that the coefficients alternate in sign, and satisfy certain further conditions.

The calculus of residues has been used by Barnes, Ford, and others in finding the analytical extension of functions defined by various types of power series. As consequences of the results obtained, certain further results pertaining to the asymptotic developments of such functions have been established. The question naturally presents itself as to whether the method mentioned can be used also to obtain analogous results for factorial series. We shall show, in fact, that some such results are possible.

Our investigations are based upon the following fundamental theorem which is a direct consequence of elementary considerations in the theory of functions of a complex variable: *

"If $P(w)$ and $Q(w)$ are any two functions of the complex variable w, both of which are single valued and analytic in a region R of the w complex plane, and of which the latter vanishes at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, which are zeroes of the first order, and if C_n denote any contour lying in R and inclosing the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, then we have

$$\frac{1}{2\pi i}\int_{C_n}\frac{P(w)}{Q(w)}\,dw = \sum_{t=1}^{n}\frac{P(\lambda_t)}{Q'(\lambda_t)}, \tag{19}$$

where the integration is performed in the positive sense."

Our first theorem is as follows:

THEOREM II. *Given any factorial series of the form*

$$\sum_{n=0}^{\infty}\frac{(-1)^n g(n)}{z(z+1)\cdots(z+n)} \tag{20}$$

whose abscissa λ of convergence is finite. If the function $g(n)$ occurring in the coefficient of this series is such that, when considered as a function $g(w)$ of the complex variable $w = x + iy$, (a) it is single valued and analytic throughout all portions of the w plane lying to the right of (or upon) the vertical line $w = -\tfrac{1}{2} + iy$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, ($\lambda_t \neq$ integer), and (b) to any arbitrarily small

* See, for example, Ford, "Studies on Divergent Series and Summability," *Michigan Science Series*, Vol. II, 1916, page 18[illegible]

positive number ϵ *there corresponds a positive constant* K_ϵ *such that for all values of* $x \geqq -\frac{1}{2}$ *and for all positive values of* y *sufficiently large we may write*

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)},$$

then the function $\Omega(z)$ *defined by series* (20) *when* $R(z) > \lambda$ *may be extended analytically throughout the whole finite* z *plane, except in neighborhoods of the points* $z = 0, -1, -2, \cdots$, *and throughout this region will be defined by the equation*

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2} + iy)\,dy}{\Gamma(z + \frac{1}{2} + iy)\cosh \pi y} - \sum_{t=1}^{n} r_t \tag{21}$$

where r_t *represents the residue of the function*

$$\pi\Gamma(z)g(w)/\Gamma(z + w + 1)\sin \pi w \tag{22}$$

at the point $w = \lambda_t$.

The proof of this theorem is analogous to that of the second part of the general theorem proved by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane," already mentioned in the first part of this chapter.

In order to prove the theorem, let us write series (20) in the form

$$\sum_{n=0}^{\infty} (-1)^n \Gamma(z)g(n)/\Gamma(z + n + 1). \tag{23}$$

Furthermore, for simplicity, let us assume first that the function $g(w)$ has no poles to the right of the line $w = -\frac{1}{2} + iy$. Moreover, let us regard z as having any fixed value which we shall take as real, positive, greater than λ, and not an integer. The theorem then results, as we shall show, from integrating the function (22) about the rectangular contour C_n formed in the w plane by the lines

$$w = -1/2 + iy, \qquad w = 1/2 + 2n + iy, \qquad w = x \pm ij,$$

where n is any positive integer, and j is an arbitrarily large positive number.

Taking

$$P(w) = \frac{\pi\Gamma(z)g(w)}{\Gamma(z + w + 1)}, \qquad Q(w) = \sin \pi w,$$

and applying (19), we arrive at the relation

$$\Gamma(z) \sum_{n=0}^{2n} \frac{(-1)^n g(n)}{\Gamma(z + n + 1)} = \frac{\Gamma(z)}{2i} \int_{C_n} \frac{g(w)\,dw}{\Gamma(z + w + 1)\sin \pi w}. \tag{24}$$

We now proceed to study the integral here appearing in further detail.

First, along that side of C_n upon which $w = x + ij$, we have $dw = dx$, $\sin \pi w = \sin \pi j \, (\sin \pi x \cosh \pi j + i \cos \pi x)$. Hence if we denote by I the contribution to the contour integral in (24) arising from integrating along this side, we have

$$I = \frac{\Gamma(z)}{2i \sin h\pi j} \int_{2n+\frac{1}{2}}^{-\frac{1}{2}} \frac{g(x + ij)\,dx}{\Gamma(z + x + 1 + ij)\,(\sin \pi x \cosh \pi j + i \cos \pi x)}.$$

Now it is a well known property of the Gamma function that if α and β are real, then we may write *

$$(25) \qquad |\,\Gamma(\alpha + i\beta)\,| = (2\pi)^{\frac{1}{2}}\,|\,\alpha + i\beta\,|^{\,\alpha - \frac{1}{2}} e^{-\alpha - \beta \tan^{-1} \beta/\alpha} \cdot (1 + \delta)$$

where δ approaches zero as α or β becomes infinite. Moreover as $j \rightarrow +\infty$, $\sinh \pi j$ becomes infinite like $e^{\pi j}$, and when we take account of condition (b) of the hypothesis, it appears that the absolute value of the integrand in I vanishes to as high an order as that of $e^{-\epsilon j}\,|\,z + x + 1 + ij\,|^{z+x+\frac{1}{2}}$ as $j \rightarrow +\infty$. Hence we have at once $\lim\limits_{j \rightarrow \infty} I = 0$.

Similarly, the contribution arising from integrating along that side of C_n upon which $w = x - ij$ will be seen to approach zero as $j \rightarrow +\infty$. For as the expression in question differs from I above only in that j is replaced by $-j$, and as $j \rightarrow +\infty$ the function $\sinh\,(-\pi j)$ becomes infinite like $e^{\pi j}$, the absolute value of the integrand vanishes like $e^{-\epsilon j}\,|\,z + x + 1 - ij\,|^{z+x+\frac{1}{2}}$ as $j \rightarrow +\infty$.

Next, along that side of C_n along which $w = 2n + \frac{1}{2} + iy$, we have $dw = i\,dy$, $\sin \pi w = \cosh \pi y$. If J represents the contribution to the contour integral (24) from the integration along this side, then we have

$$(26) \qquad J = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(\frac{1}{2} + 2n + iy)\,dy}{\Gamma(z + 2n + 3/2 + iy)\cosh \pi y}.$$

As $y \rightarrow +\infty$, $\cosh \pi y$ becomes infinite like $e^{\pi y}$, while as $y \rightarrow -\infty$ the same function becomes infinite like $e^{-\pi y}$. Upon noting relation (25) and condition (b) of the hypothesis, we see that as $y \rightarrow +\infty$, the absolute value of the integrand vanishes to as high an order as that of $e^{-\epsilon y}\,|\,z + 2n + \frac{3}{2} + iy\,|^{\,z+2n+1}$, while as $y \rightarrow -\infty$, it vanishes to as high an order as that of the quantity $e^{\epsilon y}\,|\,z + 2n + \frac{3}{2} - iy\,|^{z+2n+1}$. It follows, therefore, that the improper integral appearing in (26) exists. Moreover we may now show [illegible] J [illegible] 0. To do this, let us write J in the form

[illegible]

* Godefroy, *La Fonction Gamma*, Paris, 1901, page 45.

$$J = \tfrac{1}{2} \cdot \frac{\Gamma(z) \cdot g(2n)}{\Gamma(z+2n+1)} \int_{-\infty}^{+\infty} \frac{g(2n+\tfrac{1}{2}+iy)}{g(2n)} \cdot \frac{\Gamma(z+2n+1)}{\Gamma(z+2n+\tfrac{3}{2}+iy)} \cdot \frac{dy}{\cosh \pi y}.$$

Since the given series is assumed convergent for $R(z) > \lambda$, we have at once

$$\lim_{n\to\infty} \frac{\Gamma(z) g(2n)}{\Gamma(z+2n+1)} = 0.$$

Furthermore, from (25), we may write

$$\left| \frac{\Gamma(z+2n+1)}{\Gamma(z+2n+\tfrac{3}{2}+iy)} \right| < \frac{|\, z+2n+\tfrac{3}{2} \,|^{z+2n+1}}{|\, z+2n+\tfrac{3}{2}+iy \,|^{z+2n+1}}\, e^{-y \tan^{-1}[y/(2n+z+3/2)]} < e^{\pi/2 |\, y}$$

Finally, taking account of condition (b) of the hypothesis, we have $\lim_{n\to\infty} J = 0$.

Along the fourth side of C_n, we have $w = -\tfrac{1}{2} + iy$, $dw = idy$, $\sin \pi w = -\cosh \pi y$, and the integration takes place from $+\infty$ to $-\infty$. If we now take into consideration the contribution to the contour integral arising from integrating along this side, the fundamental relation (24) takes the following form (n having been allowed to become infinite as indicated above):

$$(27) \qquad \Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\tfrac{1}{2}+iy)\, dy}{\Gamma(z+\tfrac{1}{2}+iy) \cosh \pi y}.$$

Up to this point we have confined z to values which are real, positive, and greater than λ. Suppose now that z be allowed to take on any value, real or complex. The left member of (27) is an analytic function of z when $R(z) > \lambda$, except in neighborhoods of the points $z = 0, -1, -2, \cdots$. We may show, however, that the right member represents an analytic function of z throughout any finite portion T of the z plane which does not — any of the points $z = 0, -1, -2, \cdots$. To establish this fact it suffices to show that the improper integral involved converges uniformly throughout any such region. For this purpose, consider the integral

$$(28) \qquad \int_{l}^{\infty} \frac{g(-\tfrac{1}{2}+iy)\, dy}{\Gamma(z+\tfrac{1}{2}+iy) \cosh \pi y}.$$

From (25) and condition (b) of the hypothesis, it appears that the absolute value of the integrand for a sufficiently large y is less than

$$e^{-\epsilon y} \,|\, z_1 + \tfrac{1}{2} + iy \,|^{z_1}$$

where z_1 denotes the maximum absolute value of z in the proposed region T. But we may write

$$e^{-\epsilon y} \mid z_1 + \tfrac{1}{2} + iy \mid^{z_1} = e^{-(\epsilon/2)y} \cdot e^{-(\epsilon/2)y} \mid z_1 + \tfrac{1}{2} + iy \mid^{z_1} < K e^{-(\epsilon/2)y}$$

where K is a constant. Hence we have

$$\left| \int_l^\infty \frac{g(-\tfrac{1}{2} + iy)\,dy}{\Gamma(z + \tfrac{1}{2} + iy)\cosh \pi y} \right| < K \int_l^\infty e^{-(\epsilon/2)y} dy = (2/\epsilon) K e^{-\epsilon l/2}$$

provided l is a sufficiently large quantity independent of z. Consequently the integral (28) can be made less in absolute value than any arbitrarily small positive number η by proper choice of the lower limit l. In the same way, we can show that

$$\left| \int_{-\infty}^{-l} \frac{g(-\tfrac{1}{2} + iy)\,dy}{\Gamma(z + \tfrac{1}{2} + iy)\cosh \pi y} \right| < \eta$$

provided l is sufficiently large. Thus the uniform convergence of the improper integral in (27) is established. The relation must therefore hold for all values of z except zero and negative integers.

The proof for the special case in which the function $g(w)$ has no poles in the half-plane $R(w) \geqq -\tfrac{1}{2}$ is thus complete. That the theorem is true when we allow $g(w)$ to have poles as indicated in condition (a) of the hypothesis follows when we note that relation (24) continues to hold (n being sufficiently large) provided we add to its left member the expression

$$\sum_{t=0}^{n} r_t$$

as it is defined in the statement of the theorem.

7. *Remarks.* 1′. The line $w = -\tfrac{1}{2} + iy$ taken as the left side of the contour C_n can be replaced by any line $w = k + iy$ where $-1 < k < 0$ provided that $g(w)$ is analytic when $R(w) \geqq k$, except for a finite number of poles as indicated in condition (a).

2′. Should the given series converge throughout the whole z plane, then the function $\Omega(z)$ is expressed at all points z by (21).

3′. Suppose condition (a) is not satisfied, but the function $g(w)$ has a finite number of branch points situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$. Then (21) continues to hold provided one adds to the left member the sum of the loop integrals at these points. The p-th of these is described as follows: Draw the straight line extending from the point λ_p to infinity in the direction of the positive axis of imaginaries, and let this line be regarded as a cut in the w plane. Now let the loop consist of the two lines drawn on either side of and near the cut, the ends of these lines in the neighborhood

of the point λ_p being joined by a circular arc of small radius drawn about this point. The integration is to be performed in the positive sense.

Examples.

1. Consider the function $\Omega(z)$ defined by the series

$$\Omega(z) = \sum_{n=0}^{\infty} (-1)^n n!/z(z+1)\cdots(z+n).$$

This series converges when $R(z) > 0$, and may be shown to be represented by the integral

$$\int_0^1 [t^{z-1}/(2-t)]\,dt$$

which in turn preserves a meaning for $R(z) > 0$. Since the function $n!$ or $\Gamma(n+1)$ satisfies both conditions (a) and (b) of the hypothesis of Theorem II, then by that theorem we may write

$$\Omega(z) = \frac{\Gamma(z)}{2}\int_{-\infty}^{+\infty}\frac{\Gamma(\tfrac{1}{2}+iy)\,dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y},$$

which holds for all values of z except $0, -1, -2, \cdots$.

2. Let $g(n) = h(n)\, n!$, where the function $h(w)$ satisfies condition (a), while corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that

$$|\, h(x \pm iy)/h(x)\,| < K_\epsilon e^{y(\pi-\epsilon)}, \qquad x \geqq -\tfrac{1}{2}.$$

then the function $g(w)$ satisfies both conditions (a) and (b), and, assuming that the given series converges, we have

$$\Omega(z) = \frac{\Gamma(z)}{2}\int_{-\infty}^{+\infty}\frac{h(-\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}+iy)\,dy}{\Gamma(z+\tfrac{1}{2}+iy)\cosh \pi y};$$

this result holding for all finite values of z different from zero and negative integers.

8. So far our results have been concerned with the factorial series of the first kind. We shall now turn our attention to the second kind of factorial series. For this purpose, let us write series (2) of the Introduction in the form

$$g(0) + \sum_{n=1}^{\infty} g(n)(z-1)(z-2)\cdots(z-n)$$

where, as above, z denotes a variable, and where the general coefficient $g(n)$ depends only on n. Then we have the following theorem which is analogous to Theorem II:

THEOREM III. *Given any factorial series of the form*

$$(29) \qquad g(0) + \sum_{n=1}^{\infty} (-1)^n g(n)(z-1)(z-2)\cdots(z-n),$$

whose abscissa of convergence is finite. If the function $g(n)$ occurring in the general coefficient of this series is such that when considered as a function $g(w)$ of the complex variable $w = x + iy$ (a) it is single-valued and analytic throughout all portions of the half-plane $R(w) \geqq -\frac{1}{2}$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, none of which are integers, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for $x \geqq -\frac{1}{2}$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)}$$

then the function $W(z)$ defined by series (29) may be extended analytically throughout all finite portions of the z plane except in the neighborhoods of the points $z = 0, -1, -2, \cdots$, and throughout such regions will be defined by the equation

$$W(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2} + iy)\,dy}{\Gamma(z + \frac{1}{2} - iy)\cosh \pi y} - \sum_{t=1}^{n} r_t$$

where r_t represents the residue of the function

$$\frac{\pi g(w) \cdot \Gamma(z)}{\Gamma(z - w)\sin \pi w}$$

at the point $w = \lambda_t$.

In fact the given series may be written in the form

$$\sum_{n=0}^{\infty} (-1)^n g(n)\Gamma(z)/\Gamma(z-n)$$

and the proof of the theorem is analogous to the proof of Theorem I above. It will therefore be left to the reader.

9. If the function $g(w)$ is single valued and analytic throughout the half-plane $R(w) \geqq -q - \frac{1}{2}$, where q is any positive integer, then by taking the line $w = -q - \frac{1}{2} + iy$ as the left side of the contour C_n used in the proofs of the last two theorems, we may show that the following equation holds:

$$(30) \qquad W(z) = \sum_{n=-1}^{-q} \frac{(-1)^n g(n)\Gamma(z)}{\Gamma(z-n)} + \frac{(-1)^q \Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-q - \frac{1}{2} + iy)\,dy}{\Gamma(z + q + \frac{1}{2} - iy)\cosh \pi y}.$$

If $g(w)$ has a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots \lambda_n$, none of which are integers, then we must add to the left member of (30) the sum of the residues of the function

$$\pi g(w)\Gamma(z)/\Gamma(z-w)\sin \pi w$$

at the points λ_t.

As a consequence of (30), we have the following

COROLLARY. If the coefficient $g(n)$ occurring in the general coefficient of series (29), when considered as a function of the complex variable $w = x + iy$, is such that (a) it is single-valued and analytic throughout the finite w plane, except for a finite number of poles $\lambda_1, \lambda_2, \cdots \lambda_n$, none of which is an integer, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for all values of x and for all values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2-\epsilon)},$$

then the function $W(z)$ defined by (29) will be such that for all values of z lying in any sector (vertex at $z = 0$) which does not include the negative real axis, we may write

$$W(z) \sim -\sum_{t=1}^{n} r_t + \frac{g(-1)}{z} - \frac{g(-2)}{z(z+1)} + \frac{g(-3)}{z(z+1)(z+2)} - \cdots.$$

In fact the expression

$$\text{(31)} \qquad \Gamma(z) \int_{-\infty}^{+\infty} \frac{g(-q-\frac{1}{2}+iy)\,dy}{\Gamma(z+q+\frac{1}{2}-iy)\cosh \pi y}$$

may be written in the form

$$\frac{\Gamma(z)}{\Gamma(z+q+\frac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q-\frac{1}{2}+iy)\,\Gamma(z+q+\frac{1}{2})\,dy}{\Gamma(z+q+\frac{1}{2}-iy)\cosh \pi y}$$

or

$$\frac{\Gamma(z)}{\Gamma(z+q+\frac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q-\frac{1}{2}+iy)\,B(z+q-\frac{1}{2}, 1-iy)\,dy}{\Gamma(z-iy)\cosh \pi y}$$

where B is the customary symbol for the Beta function. Now we have from the definition of the Beta function

$$|B(z+q-\tfrac{1}{2}, 1-iy)| < \left| \int_0^1 x^{z+q-3/2}(1-x)^{-iy}dx \right|$$
$$< \left| \int_0^1 x^{z+q-3/2}dx \right| = 1/|z+q-\tfrac{1}{2}|.$$

Recalling that

$$(z+q-\tfrac{1}{2})\Gamma(z+q-\tfrac{1}{2})=\Gamma(z+q+\tfrac{1}{2}),$$

we see that the absolute value of the remainder (31) is less than

$$\left|\frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})}\right| \cdot \int_{-\infty}^{+\infty}\frac{|\,g(-q-\tfrac{1}{2}+iy)\,|\,dy}{|\,\Gamma(1-iy)\,|\cosh \pi y}\,.$$

By virtue of (25) and condition (b) of the hypothesis, the integral here appearing exists. Furthermore, by (25), the expression

$$\left|\frac{\Gamma(z)}{\Gamma(z+q+\tfrac{1}{2})}\right|$$

vanishes to as high an order as the $(q-\tfrac{1}{2})$ — th as $|\,z\,|\to\infty$, provided that z does not go to infinity along the negative real axis. Hence the corollary follows.

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ON SEPARATION, COMPARISON AND OSCILLATION THEOREMS FOR SELF-ADJOINT SYSTEMS OF LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS.†

By G. A. Bliss and I. J. Schoenberg.

In order to indicate the character of the theorems in this paper let us consider for a moment the differential equation

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

in which the coefficients $p(x)$ and $q(x)$ are real, single-valued, and continuous functions of x for $x_1 \leqq x \leqq x_2$. The complete set of zeros on x_1x_2 of a solution $y = y(x)$ of (1) which vanishes at $x = \xi$ but is not identically zero on x_1x_2 has no limit point on x_1x_2 and is therefore finite, and the point ξ defines uniquely the whole set of zeros, say $\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k$. Sturm ‡ proved the following

Separation Theorem. *The sets of zeros on x_1x_2 of two different solutions of (1) coincide or else separate each other, i. e. between two adjacent points of one set there is one and only one point of the second set.*

Let us consider further two differential equations of a more special type

$$y'' + A(x)y = 0, \tag{2}$$

$$y'' + A^*(x)y = 0, \tag{3}$$

in which $A(x)$ and $A^*(x)$ are continuous on x_1x_2 and such that

$$A(x) > A^*(x) \text{ for } x_1 \leqq x \leqq x_2. \tag{4}$$

A consequence of these assumptions is Sturm's §

Comparison Theorem. *The zeros of a solution of (2) are closer together than the zeros of a solution of (3). More precisely: let $\xi^*_{-l}, \cdots, \xi^*_{-1}, \xi, \xi^*_1, \cdots, \xi^*_k$, be a complete set of zeros of a solution of (3) on x_1x_2, then ξ*

† Presented to the Society June 13, 1931.

‡ C. Sturm, "Mémoire sur les équations différentielles du second ordre," *Journal de mathématiques pures et appliquées*, Vol. 1 (1836), pp. 106-186.

§ See C. Sturm, *loc. cit.*, p. 125, where a more general theorem is given.

defines a set of zeros for a solution of (2) *on* x_1x_2 *which has at least* $k+l+1$ *consecutive points* $\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k$, *and these are such that*

$$\xi_\alpha < \xi^*_\alpha \quad (\alpha = 1, 2, \cdots, k), \quad \xi_{-\beta} > \xi^*_{-\beta} (\beta = 1, 2, \cdots, l).$$

Finally let

$$y'' + A(x, \lambda) y = 0, \tag{5}$$

be a differential equation for which $A(x, \lambda)$ has the following properties:

1. $A(x, \lambda)$ is supposed to be a continuous function of (x, λ) for $x_1 \leqq x \leqq x_2$, and $\lambda \geqq \lambda_0$.

2. For values λ', λ'' satisfying $\lambda_0 \leqq \lambda' < \lambda''$ we suppose
$$A(x, \lambda') < A(x, \lambda''), \text{ for } x_1 \leqq x \leqq x_2.$$

3. For every $M > 0$ arbitrarily large, there exists a number $\lambda(M)$, such that $A(x, \lambda) > M$, for $x_1 \leqq x \leqq x_2$ and $\lambda > \lambda(M)$.

4. The interval x_1x_2 contains no two consecutive zeros of a solution of (5), taken for $\lambda = \lambda_0$.

The comparison theorem makes it possible to prove under these assumptions the following

OSCILLATION THEOREM.† *There is a uniquely determined sequence of numbers* $(\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots)$ *with*

$$\lim_{m \to \infty} \lambda_m = \infty,$$

such that

$$y'' + A(x, \lambda_m) y = 0, \qquad (m = 1, 2, \cdots),$$

has a solution vanishing at $x = x_1$ *and* $x = x_2$, *and which vanishes precisely* $m - 1$ *times between* x_1 *and* x_2.

The assumptions 1, 2, 3, and 4, are always satisfied for $A(x, \lambda) = \lambda A(x)$, $A(x)$ being continuous and positive on x_1x_2, and $\lambda_0 = 0$. In this case, the sequence λ_m has the following further property: *The series* $\sum_{m=1}^{\infty} \lambda_m^{-\alpha}$ *converges for* $\alpha > 1/2$ *and diverges for* $\alpha \leqq 1/2$.

This paper gives a generalization of these three theorems for a self-adjoint system of linear second order differential equations. The theorems

† This theorem goes back to Sturm. For the statement given here see L. Bieberbach, *Differentialgleichungen*, 2nd edition, Berlin (1926), pp. 155-156.

concern what we call a conjugate system of points with respect to such a system of equations. Our separation theorem includes results of J. Radon * on conjugate points in the calculus of variations. The comparison theorem, which is related to a comparison theorem of Morse ‡, will be established quite independently with the help only of methods which are classical in the calculus of variations. This theorem makes it possible to extend to this more general situation the well known continuity proof for the Sturm-Liouville oscillation theorem.

1. The Generalized Separation Theorem.

We consider the quadratic form

$$(1.1) \qquad 2\Omega(x, y, y') = P_{ik}(x) y_i y_k + 2Q_{ik}(x) y_i y_k' + R_{ik}(x) y_i' y_k',$$

its coefficients P_{ik}, Q_{ik}, R_{ik}, being functions of x of class C' † on x_1x_2, with $P_{ik} = P_{ki}$, $R_{ik} = R_{ki}$, and

$$(1.2) \qquad R_{ik}(x) v_i v_k > 0 \quad \text{for} \quad v_i v_i > 0, \quad x_1 \leqq x \leqq x_2.$$

We are to deal with the system

$$(1.3) \qquad J_i(y) \equiv (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0, \qquad (i = 1, 2, \cdots, n),$$

which is the most general self-adjoint system of second order linear differential equations. Condition (1.2) implies $|R_{ik}| \neq 0$ on x_1x_2 and therefore (1.3) can be solved for the second derivatives y_i'' and put into the normal form on the interval $x_1 \leqq x \leqq x_2$. Its solutions $y_i = y_i(x)$, which we are using later, are supposed to be of class C'' § and not identically zero on x_1x_2.

One readily verifies the following identities

$$(1.4) \qquad \begin{gathered} y_i\Omega_{y_i} + y_i'\Omega_{y_i'} = 2\Omega, \; y_i\Omega_{z_i} + y_i'\Omega_{z_i'} = z_i\Omega_{y_i} + z_i'\Omega_{y_i'}, \\ y_iJ_i(z) - z_iJ_i(y) = (d/dx)(y_i\Omega_{z_i'} - z_i\Omega_{y_i'}). \end{gathered}$$

For two solutions $y_i = y_i(x)$, $z_i = z_i(x)$ of (1.3), one therefore has $y_i\Omega_{z_i'} - z_i\Omega_{y_i'} = \text{const.}$ These two solutions are said to be conjugate if

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* J. Radon, "Zum Problem von Lagrange," *Abhandlungen aus dem Mathematischen Seminar Hamburg*, Vol. 6, pp. 298-299. The methods used in our § 1 may be applied without great change to the discussion of conjugate points for the Lagrange problem.

‡ M. Morse, "A Generalization of the Sturm Separation and Comparison Theorems in n-Space," *Mathematische Annalen*, Vol. 103, pp. 52-69. For geometric applications of this theorem see I. J. Schoenberg, "Some Applications of the Calculus of Variations [illegible] (to appear in the *Annals of Mathematics*).

§ A function is of class C' if it is continuous and has a continuous first derivative. The function is of class C'' if also the second derivative is continuous.

$y_i\Omega_{z_i'} - z_i\Omega_{y_i'} = 0$. We shall call *a conjugate system of solutions* of (1.3), a system of n solutions $u_{1k}, u_{2k}, \ldots, u_{nk}$, $(k = 1, 2, \cdots, n)$, which are conjugate to each other and linearly independent on x_1x_2.

The solution $y_i = y_i(x)$ is said to vanish at $x = \xi$ if $y_i(\xi) = 0$, $(i = 1, 2, \cdots, n)$. We shall prove the following

LEMMA 1.1.† *If u_{ik} is a conjugate system of solutions with $U \equiv | u_{ik} | \neq 0$ on x_1x_2, then no solution $y_i = y_i(x)$ of (1.3) can vanish twice on the interval x_1x_2.*

Let $y_i = y_i(x)$ be some solution of (1.3). Because of $U \neq 0$, $y_i = u_{ik}a_k$ defines uniquely a set of functions $a_k = a_k(x)$ on x_1x_2. We put

$$y_i = u_i = u_{ik}a_k, \qquad v_i = u_{ik}a_k'. \tag{1.5}$$

Primes attached to expressions involving u_i or v_i will always indicate derivatives of those expressions with respect to x calculated as if the coefficients a_k, a_k' were independent of x. One readily verifies, then, the relations

$$\begin{gathered}(\Omega_{u_i'})' = \Omega_{u_i}, \ (\Omega_{v_i'})' = \Omega_{v_i}, \ u_i\Omega_{v_i'} - v_i\Omega_{u_i'} = 0, \\ (d/dx)\Omega_{u_i'} = (\Omega_{u_i'})' + \Omega_{v_i'} = \Omega_{u_i} + \Omega_{v_i'}\end{gathered} \tag{1.6}$$

in which it is understood that the differentiation indicated by d/dx takes account of the fact that the coefficients a_k are functions of x. With the same convention concerning differentiation

$$y_i' = u_{ik}'a_k + u_{ik}a_k' = u_i' + v_i. \tag{1.7}$$

With the help of Taylor's formula, the first equation (1.4), and the equations (1.6) and (1.7), one readily verifies the further relations

$$\begin{aligned}2\Omega(x, y, y') &= 2\Omega(x, u, u' + v) \\ &= 2\Omega(x, u, u') + 2v_i\Omega_{u_i'} + R_{ik}v_iv_k \\ &= u_i\Omega_{u_i} + u_i'\Omega_{u_i'} + 2v_i\Omega_{u_i'} + R_{ik}v_iv_k \\ &= u_i(\Omega_{u_i} + \Omega_{v_i'}) + (u_i' + v_i)\Omega_{u_i'} + R_{ik}v_iv_k\end{aligned}$$

$$2\Omega(x, y, y') = (d/dx)(y_i\Omega_{u_i'}) + R_{ik}v_iv_k. \tag{1.8}$$

Hence with the help of the first equation (1.4)

$$y_i(\Omega_{y_i} - (d/dx)\,\Omega_{y_i'}) + (d/dx)y_i(\Omega_{y_i'} - \Omega_{u_i'}) = R_{ik}v_iv_k. \tag{1.9}$$

This relation and condition (1.2) show that $y_i = y_i(x)$ can vanish at $x = \xi$,

† Compare G. A. Bliss, "The Problem of Lagrange in the Calculus of Variations," *American Journal of Mathematics*, Vol. 52 (1930), § 32, where the same fact is proved for the general Lagrange problem.

$x = \xi'$, only if $v_i = u_{ik} a_k' \equiv 0$ for $\xi \leqq x \leqq \xi'$; therefore $a_k' \equiv 0$ and hence the a_k are constant on $\xi\xi'$ in which case $y_i = u_{ik} a_k$ can not vanish at all on $x_1 x_2$ since $U \neq 0$ on $x_1 x_2$.

LEMMA 1.2. *A conjugate system of solutions* (u_{ik}) *can not have* $U = | u_{ik} | \equiv 0$ *on any subinterval of* $x_1 x_2$.

If $U \equiv 0$ on some subinterval of $x_1 x_2$, then one can certainly choose a subinterval $\xi\xi'$ sufficiently small so that the same minor of U of maximum order shall be $\neq 0$ on $\xi\xi'$. This permits us to select functions $a_k = a_k(x)$ of class C' on $\xi\xi'$ with $y_i = u_{ik} a_k \equiv 0$ and not all $a_k(\xi) = 0$. Formula (1.9) gives $v_i \equiv 0$ on $\xi\xi'$ and from (1.7) one concludes that $u_{ik}' a_k \equiv 0$ on $\xi\xi'$. The set of functions $u_i = u_{ik} c_k$, with $c_k = a_k(\xi)$, is therefore a solution of (1.3) which has the properties $u_i(\xi) = 0$, $u_i'(\xi) = 0$, and which hence vanishes identically on $x_1 x_2$. But $u_{ik} c_k \equiv 0$ on $x_1 x_2$ contradicts the definition of a conjugate system of solutions.

LEMMA 1.3. *There exists a conjugate system of solutions whose elements all vanish at an arbitrarily selected point* $x = \xi$. *There also exists a conjugate system of solutions whose determinant* $U^* = 1$ *at an arbitrarily selected point* $x = \xi$.

The proof follows readily with the help of the canonical variables

$$z_i = \Omega_{y_i'} = Q_{ki} y_k + R_{ik} y_k'. \tag{1.10}$$

Solving with respect to y_k' we get

$$y_k' = G_k(x, y, z)$$

which is linear and homogeneous in y_i and z_i. In terms of the canonical variables x, y_i, z_i, (1.3) takes the normal form

$$\begin{aligned} dy_i/dx &= G_i(x, y, z), \\ dz_i/dx &= P_{ik} y_k + Q_{ik} G_k(x, y, z). \end{aligned} \tag{1.11}$$

A solution of (1.3) or (1.11) is uniquely determined by giving the values of y_i and z_i at some initial point $x = \xi$. The set of n solutions $y_i = u_{ik}$, $z_i = v_{ik}$, $(k = 1, 2, \cdots, n)$, defined by the initial values

$$u_{ik}(\xi) = 0, \quad v_{ik}(\xi) = \delta_{ik}, \tag{1.12}$$

is readily shown to be a conjugate system of solutions satisfying the requirements for the first conjugate system of our lemma. The initial values

$$u_{ik}(\xi) = \delta_{ik}, \quad v_{ik}(\xi) = 0, \tag{1.13}$$

similarly lead at once to the second conjugate system required by the lemma. We will designate further on by $U(\xi, x)$ and $U^*(\xi, x)$ the determinants of these special systems defined by the initial conditions (1.12) and (1.13) respectively.

LEMMA 1.4. *Let $y_i = y_i(x)$ be a solution of (1.3) with $y_i(\xi) = 0$, and let (u_{ik}) be the first conjugate system of solutions defined for $x = \xi$ in Lemma 1.3. The solution $y_i(x)$ is linearly expressible in the form $y_i = u_{ik}c_k$ with constant coefficients c_k.*

The solution $y_i = y_i(x)$ defines a set $z_i = z_i(x)$ by means of (1.10). Let (u_{ik}) be the conjugate system of solutions defined by (1.12). We determine the constants c_k from $z_i(\xi) = v_{ik}(\xi)c_k$. The solution $y_i = u_{ik}c_k$, $z_i = v_{ik}c_k$, of (1.11) has initial values all zero at $x = \xi$ and hence vanishes identically on x_1x_2. This proves our lemma.

Definition. The points ξ and ξ' on x_1x_2 are said to be associated if there is a solution $y_i = y_i(x)$ of (1.3) with not all $y_i(x) \equiv 0$ on $\xi\xi'$ and having $y_i(\xi) = 0, y_i(\xi') = 0, (i = 1, 2, \cdots, n)$.

THEOREM 1.1. *If on the interval x_1x_2 a point ξ has a following (preceding) associated point, then it also has a first following (preceding) associated point ξ_1. This point ξ_1, which we shall call the following (preceding) conjugate point to ξ, is given by the first root of the determinant $U(\xi, x) = 0$ following (preceding) the value $x = \xi$.*

It is sufficient to make the proof for the conjugate point following the point $x = \xi$. Let ξ' be a following associated point of ξ. We set up the conjugate system of solutions defined by (1.12). One readily proves that ξ' is a root of $U(\xi, x) = 0$, because by definition there is a solution $y_i = y_i(x)$ vanishing at ξ and ξ'. By lemma 1.4 there are constants c_k with $y_i = u_{ik}c_k$, and $y_i(\xi') = u_{ik}(\xi')c_k = 0$ gives $U(\xi, \xi') = 0$.

Let conversely ξ' be a point following ξ, with $U(\xi, \xi') = 0$. One determines constants c_k by means of the system $u_{ik}(\xi')c_k = 0$, and $y_i(x) = u_{ik}(x)c_k$ has the property $y_i(\xi) = y_i(\xi') = 0$, which shows that ξ and ξ' are associated.

All we have to show is the existence of a first root ξ_1 of $U(\xi, x) = 0$, following the point $x = \xi$. Because $U(\xi, x)$ is a continuous function of x, it is sufficient to show that $x = \xi$ is an isolated root of $U(\xi, x) = 0$. This last point follows from lemma 1.1 and lemma 1.3 and our arguments thus far. The conjugate system with $U^*(\xi, \xi) = 1$ (Lemma 1.3) defines some

[illegible] which $U(\xi, x) \neq 0$. By lemma 1.1 there are in this interval no two associated points and hence also no point associated to ξ. Hence $U(\xi, x) = 0$ has no root in this interval except $x = \xi$. This proves theorem 1.1.

THEOREM 1.2. *If the point ξ_1 is conjugate to the point ξ, then also ξ is conjugate to ξ_1.*

It is sufficient to make the proof for $\xi < \xi_1$. Let ξ_2 be the conjugate point to ξ_1 preceding ξ_1. Certainly $\xi \leqq \xi_2$ (because ξ and ξ_1 are associated) and all we have to show is that actually $\xi = \xi_2$. Let us assume $\xi < \xi_2 < \xi_1$. By theorem 1.1 ξ_1 is the first zero of $U(\xi, x) = 0$ following ξ, and ξ_2 is the first zero of $U(\xi_1, x) = 0$ preceding ξ_1. Let ξ' be any point with $\xi < \xi' < \xi_2$. The determinant $U(\xi', x)$ vanishes at $x = \xi'$ but can not vanish for $\xi' < x < \xi_1$, because the interval $\xi\xi_1$ can not contain two associated points (theorem 1.1, lemma 1.1). Furthermore $U(\xi', \xi_1) \neq 0$ would imply a contradiction with lemma 1.1, ξ_2 and ξ_1 being associated while $U(\xi', x)$ would be $\neq 0$ on $\xi_2\xi_1$. One therefore has $U(\xi', \xi_1) = 0$ for every ξ' with $\xi < \xi' < \xi_2$. Hence ξ_1 is associated with every such ξ' and it results (theorem 1.1) that $U(\xi_1, x) \equiv 0$ on $\xi \leqq x \leqq \xi_2$, which result contradicts lemma 1.2. Hence $\xi = \xi_2$ and ξ_2 is actually conjugate to ξ_1.

This theorem makes possible the following

Definition. A set of points $\xi_1 < \xi_2 < \cdots < \xi_k$ on the interval x_1x_2, such that each is conjugate to the adjacent points of the set while ξ_1 has no preceding and ξ_k no following conjugate point on x_1x_2, is said to form a conjugate system of points.

THEOREM 1.3. *Every point ξ on x_1x_2 has associated with it a unique conjugate system which is completely defined by any one of its points.*

Every point ξ defines a conjugate $\xi_1 > \xi$ or else none $> \xi$ on x_1x_2 (theorem 1.1). The point ξ_1 defines a conjugate $\xi_2 > \xi_1$ or else none $> \xi_1$ (theorem 1.1). In this way one defines the points of the conjugate system following ξ. There is only a finite number, because the set $\xi, \xi_1, \xi_2, \cdots$ so [illegible] have no limit point [illegible]. This last statement [illegible]

SEPARATION THEOREM 1. 4. *Two systems of conjugate points coincide or else separate each other, i. e., between two adjacent points of one system there is one and only one point of the second system.*

2. THE GENERALIZED COMPARISON THEOREM.

It will be the purpose of this section to generalize the Sturm comparison theorem, described in the introduction, to systems of the type (1. 3) by means of the notion of a conjugate system of points as worked out in § 1.

Let $\Omega(x, y, y')$ be defined by (1. 1), hypothesis (1. 2) being retained. Besides the system

$$J_i(y) = (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0, \tag{2.1}$$

we set up the integral

$$I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\,dx. \tag{2.2}$$

LEMMA 2. 1.† *If there is on x_1x_2 no point conjugate to x_1 with respect to the system* (2. 1), *then we always have*

$$I(y) > 0, \tag{2.3}$$

for every set of functions $y_i = y_i(x)$ of class D' ‡ *on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$, and which does not vanish identically on x_1x_2.*

In order to prove this lemma we shall prove first the existence of a conjugate system of solutions of (2. 1) with its determinant $\neq 0$ on x_1x_2. To show this, we extend the definition of P_{ik}, Q_{ik}, R_{ik}, on a slight extension $x_1 - \delta \leqq x \leqq x_1$ of x_1x_2. We do this in a way which preserves the continuity properties of the coefficients of $\Omega(x, y, y')$ and also (1. 2). We suppose this δ sufficiently small in order to have $U^*(x_1, x) \neq 0$ on $x_1 - \delta \leqq x \leqq x_1 + \delta$, $U^*(x_1, x)$, with $U^*(x_1, x_1) = 1$, being the determinant of the second conjugate system of solutions introduced in lemma 1. 3. The first determinant of lemma 1. 3, $U(x_1, x)$, is a continuous function of the initial point x_1 and therefore it follows from $U(x_1, x) \neq 0$ on $x_1 + \delta \leqq x \leqq x_2$, that also $U(x_0, x) \neq 0$ for $x_1 + \delta \leqq x \leqq x_2$ and some x_0 on the interval $x_1 - \delta \cdots x_1$ and sufficiently close to x_1. From Lemma 1. 1 and our assumption on $U^*(x_1, x)$, one gets also $U(x_0, x) \neq 0$ for $x_1 \leqq x \leqq x_1 + \delta$ and x_0 as described above. Hence $U(x_0, x) \neq 0$ on $x_1 \leqq x \leqq x_2$.

† Compare J. Hadamard, *Calcul des variations* I, Paris, 1910, § 288.

‡ A function is of class D' on x_1x_2, if it is continuous, and if the interval x_1x_2 can be divided into a finite number of subintervals, such that in each subinterval its derivative is continuous.

Let $y_i = y_i(x)$ be a set of admissible functions. With our special conjugate system just defined and (1.5), formula (1.8) holds. One notices that $\Omega_{u_i'} = Q_{ki}u_k + R_{ik}u_k'$, with $u_k = u_{kj}a_j$ and $u_k' = u'_{kj}a_j$, is continuous, even at the possible corners of $y_i = y_i(x)$. Therefore integrating (1.8) we get †

$$(2.4) \qquad I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\, dx = \int_{x_1}^{x_2} R_{ik}v_iv_k\, dx.$$

The interval x_1x_2 can be subdivided in a finite set of subintervals on each of which the v_i are continuous and not all identically zero. Our last formula (2.4) and (1.2) imply (2.3).

LEMMA 2.2. *If the points x_1 and x_2 are conjugate with respect to* (2.1), *then always*

$$(2.5) \qquad I(y) \geqq 0,$$

for every set $y_i = y_i(x)$ of class D' on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$.

Let $y_i = y_i(x)$ be a set of class D' on x_1x_2, with $y_i(x_1) = y_i(x_2) = 0$, and defining an arc C_{12} in (x, y_i)-space. Let 3 and 4 be the points in (x, y_i)-space whose coördinates are $(x_3, y_i(x_3))$ and $(x_4, 0)$ respectively $(x_1 < x_3 < x_4 < x_2)$. Lemma 2.1 applied to the interval x_1x_4 shows that $I(C_{13} + L_{34} + L_{42}) = I(C_{13} + L_{34}) \geqq 0$, L_{34} and L_{42} being segments of straight lines in (x, y_i)-space. Keeping 3 fixed and moving 4 towards the point 2, we get at the limit $I(C_{13} + L_{32}) \geqq 0$. Certainly $I(C_{13} + L_{32})$ has the limit $I(C_{12}) = I(y)$, if 3 tends to the point 2. The last inequality insures therefore (2.5) and our lemma is proved.

We shall prove now the following

COMPARISON THEOREM 2.1. *Besides the system*

$$(2.6) \qquad J_i(y) \equiv (d/dx)\Omega_{y_i'} - \Omega_{y_i} = 0,$$

we consider a second system of the same kind

$$(2.7) \qquad J^*_i(y) \equiv (d/dx)\Omega^*_{y_i'} - \Omega^*_{y_i} = 0,$$

$2\Omega^*(x, y, y') = P^*_{ik}y_iy_k + 2Q^*_{ik}y_iy_k' + R^*_{ik}y_i'y_k'$ *having like* Ω, *its coefficients of class C' on x_1x_2.*

† For another method of deriving this so called Clebsch transformation of the second variation see G. A. Bliss, "The Transformation of Clebsch in the Calculus of Variations," *Proceedings of the International Mathematical Congress*, Toronto, 1924, Vol. 1, pp. 589 603.

We suppose

$$\Omega(x, u, v) \leqq \Omega^*(x, u, v), \tag{2.8}$$

for $x_1 \leqq x \leqq x_2$, *and arbitrary values of* u_i *and* v_i. *The inequalities* (2.8) *and* (1.2) *imply*

$$R^*_{ik}(x) v_i v_k > 0 \text{ for } v_i v_i > 0,\ x_1 \leqq x \leqq x_2. \tag{2.9}$$

Then a conjugate system of points for (2.6) *has its points closer together than a conjugate system of points for* (2.7). *More precisely: Suppose that*

$$\xi^*_{-l}, \cdots, \xi^*_{-1},\ \xi,\ \xi^*_1, \cdots, \xi^*_k,$$

is the conjugate system for (2.7) *defined by the point* ξ. *The point* ξ *will then define for* (2.6) *a conjugate system which has at least* $k + l + 1$ *consecutive points*

$$\xi_{-l}, \cdots, \xi_{-1}, \xi, \xi_1, \cdots, \xi_k,$$

with

$$\xi_\alpha \leqq \xi^*_\alpha\ (\alpha = 1, 2, \cdots, k),\ \textit{and}\ \xi_{-\beta} \geqq \xi^*_{-\beta}\ (\beta = 1, 2, \cdots, l). \tag{2.10}$$

Restricted Comparison Theorem 2.2. *We replace the assumption* (2.8) *by the stronger one:*

$$\Omega(x, u, v) < \Omega^*(x, u, v), \tag{2.11}$$

for $x_1 \leqq x \leqq x_2$, $u_i u_i > 0$, *and arbitrary values* v_i. *The statement as in theorem* 2.1 *then holds after strengthening the inequalities* (2.10) *to*

$$\xi_\alpha < \xi^*_\alpha (\alpha = 1, 2, \cdots, k),\ \textit{and}\ \xi_{-\beta} > \xi^*_{-\beta}\ (\beta = 1, 2, \cdots, l). \tag{2.12}$$

Proof of theorem 2.1: We shall prove this theorem by induction and we start by proving the existence of ξ_1 with $\xi_1 \leqq \xi^*_1$. The points ξ and ξ^*_1 being conjugate and therefore also associated for (2.7), there is a solution of (2.7) $y_i = u_i(x)$ of class C' on $\xi\xi^*_1$, with $u_i(\xi) = u_i(\xi^*_1) = 0$ and not $\equiv 0$ on $\xi\xi^*_1$. An integration by parts gives

$$\begin{aligned} I^*(u) &= \int_\xi^{\xi^*_1} 2\Omega^*(x, u, u')\,dx = \int_\xi^{\xi^*_1} (u_i \Omega^*_{u_i} + u_i' \Omega^*_{u_i'})\,dx \\ &= \int_\xi^{\xi^*_1} u_i(\Omega^*_{u_i} - (d/dx)\Omega^*_{u_i'})\,dx = 0. \end{aligned} \tag{2.13}$$

From (2.8) and (2.13) we get

$$I(u) = \int_\xi^{\xi^*_1} 2\Omega(x, u, u')\,dx \leqq \int_\xi^{\xi^*_1} 2\Omega^*(x, u, u')\,dx = I^*(u) = 0.$$

This last result and lemma 2.1 show that there is a point ξ_1 conjugate to ξ with respect to (2.6), with $\xi_1 \leqq \xi^*_1$.

Suppose now that we know already the existence of $\xi_1, \xi_2, \cdots, \xi_{m-1}$, with $\xi_\alpha \leqq \xi^*_\alpha$, for $\alpha = 1, 2, \cdots, m-1$, $(m \leqq k)$. We want to show that there is a ξ_m with $\xi_m \leqq \xi^*_m$. The same argument as above applies again. The points ξ^*_{m-1} and ξ^*_m are conjugate with respect to (2.7) and $\xi_{m-1} \leqq \xi^*_{m-1}$ gives an admissible set $y_i = u_i(x)$ on the interval $\xi_{m-1}\xi^*_m$ with $I(u) \leqq I^*(u) = 0$. Lemma 2.1 insures again the existence of ξ_m conjugate to ξ_{m-1}, with $\xi_m \leqq \xi^*_m$.

The proof of theorem 2.1 is complete, because the same type of argument combined with the separation theorem 1.4 applies also in the case of the points $\xi_{-\beta}$ preceding ξ.

Proof of theorem 2.2: The inequality (2.11) implies (2.8) and the inequalities (2.10) are already established by theorem 2.1. All we have to do is to strengthen (2.10) to (2.12).

The set $y_i = u_i(x)$ used in (2.13) does not vanish identically on $\xi\xi^*_1$. The relations (2.13) and (2.11) therefore give $I(u) < I^*(u) = 0$. This result and lemma 2.2 make the equality $\xi_1 = \xi^*_1$ impossible. We therefore actually have $\xi_1 < \xi^*_1$. Suppose now $\xi_{m-1} < \xi^*_{m-1}$ to be true $(m \leqq k)$. A similar argument to that in the proof of theorem 2.1 provides an admissible set $y_i = u_i(x)$ on $\xi_{m-1}\xi^*_m$ with $I(u) < 0$. The equality $\xi_m = \xi^*_m$ would again contradict lemma 2.2. A similar argument proves the remaining inequalities (2.12).

3. The Generalized Oscillation Theorem.

With $\Omega(x, y, y')$ defined by (1.1) and (1.2) we set up the system

$$(d/dx)\Omega_{y_i'} - \Omega_{y_i} + A_{ik}(x, \lambda) y_k = 0, \quad (i = 1, 2, \cdots, n), \tag{3.1}$$

making on the coefficients $A_{ik}(x, \lambda)$ the following assumptions:

1. $A_{ik}(x, \lambda)$ $(i, k = 1, 2, \cdots, n)$ are supposed to be continuous functions of (x, λ) for $x_1 \leqq x \leqq x_2$, $\lambda \geqq \lambda_0$.

2. With the same λ_0 we suppose $A_{ik}(x, \lambda') u_i u_k < A_{ik}(x, \lambda'') u_i u_k$, for $x_1 \leqq x \leqq x_2$, $u_i u_i > 0$, and $\lambda_0 \leqq \lambda' < \lambda''$.

3. For every $M > 0$ arbitrarily large, there exists a number $\lambda(M)$, such that $A_{ik}(x, \lambda) u_i u_k > M$, for $x_1 \leqq x \leqq x_2$, $u_i u_i = 1$, and $\lambda > \lambda(M)$.

4. The interval $x_1 x_2$ contains no point conjugate to x_1 with respect to the system (3.1), taken for $\lambda = \lambda_0$.

We shall prove the following

OSCILLATION THEOREM 3.1. *The assumptions* 1, 2, 3, 4, *imply the existence of a unique sequence of values* $(\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots)$ *with*

$$\lim_{m\to\infty} \lambda_m = \infty, \tag{3.2}$$

and with the following property: For $\lambda = \lambda_m$ $(m = 1, 2, \cdots)$, *there is a conjugate system of points with respect to the system* (3.1), *containing exactly* $m+1$ *points and whose first and last point are* x_1 *and* x_2 *respectively.*

Of interest is the special case

$$A_{ik}(x, \lambda) = \lambda A_{ik}(x), \tag{3.3}$$

with $A_{ik}(x)$ *continuous and* $A_{ik}(x)u_iu_k$ *positive definite on* x_1x_2. *If* x_1x_2 *contains no point conjugate to* x_1 *with respect to the system* (1.3), *then our assumptions* 1, 2, 3, *and* 4, *are fulfilled with* $\lambda_0 = 0$. *The sequence* λ_m *has in this case the further property:*

$$\sum_{m=1}^{\infty} \lambda_m^{-\alpha} \text{ converges for } \alpha > 1/2, \text{ and diverges for } \alpha \leqq 1/2. \tag{3.4}$$

We define the sequence of functions

$$\xi_1(\lambda),\ \xi_2(\lambda), \cdots, \xi_m(\lambda), \cdots \quad \text{for } \lambda \geqq \lambda_0, \tag{3.5}$$

in the following way: For every $m = 1, 2, 3, \cdots$, $\xi_m(\lambda)$ shall be the m-th conjugate point to x_1 with respect to the system (3.1), or $\xi_m(\lambda) = x_2$ if there is no such point.

In order to prove the continuity and monotoneity of these functions we need some lemmas.

LEMMA 3.1. *Let* $x = \phi(\xi)$ *be the conjugate point to* ξ $(x_1 \leqq \xi \leqq x_2)$ *following* ξ, *conjugate with respect to* (1.3); *and let* $\phi(\xi) = x_2$ *if there is no such point. This function* $x = \phi(\xi)$ *is a continuous and non-decreasing function of* ξ *for* $x_1 \leqq \xi \leqq x_2$, *and also properly increasing on the interval defined by* $\phi(\xi) < x_2$, *which interval may have no points.*

The monotoneity property follows at once from the separation theorem 1.4. From the same theorem and theorem 1.3, one shows at once the impossibility of $\phi(\xi) < \phi(\xi + 0)$ or $\phi(\xi - 0) < \phi(\xi)$.

LEMMA 3.2. *Let* $2\Omega(x, y, y')$ *be defined by* (1.1), (1.2). *There are always two positive constants* b *and* c, *with*

$$(3.6) \qquad P_{ik} y_i y_k + 2Q_{ik} y_i y_k' + R_{ik} y_i' y_k' \geqq -b^2 y_i y_i + c^2 y_i' y_i'$$

for $x_1 \leqq x \leqq x_2$, and arbitrary values y_i and y_i', and consequently also

$$(3.7) \qquad P_{ik} y_i y_k + 2Q_{ik} y_i y_k' + R_{ik} y_i' y_k' \geqq -b^2 y_i y_i.$$

The quadratic form $R_{ik} y_i' y_k'$ being positive definite on $x_1 x_2$, we know that all the principal minors of $|R_{ik}|$ will be positive on $x_1 x_2$. For a particular $c > 0$ sufficiently small, the determinant $|R_{ik} - c^2 \delta_{ik}|$ will have the same property. The constant c being fixed in this way, we consider the determinant of order $2n$

$$(3.8) \qquad \begin{vmatrix} Q_{ik} & P_{ik} + b^2 \delta_{ik} \\ R_{ik} - c^2 \delta_{ik} & Q_{ik} \end{vmatrix}.$$

Its principal minors of order $1, 2, \cdots, n$, lying in the upper left hand corner of the matrix are positive on $x_1 x_2$. Its principal minors of order $n + 1$, $n + 2, \cdots, 2n$, similarly situated, are polynomials in b whose terms of highest degree are respectively $| R_{ik} - c^2 \delta_{ik} | b^2$, $| R_{ik} - c^2 \delta_{ik} | b^4, \cdots$, $| R_{ik} - c^2 \delta_{ik} | b^{2n}$. We certainly can choose b sufficiently large so that all these minors shall be positive for $x_1 \leqq x \leqq x_2$. This property of the determinant (3.8) insures the inequality (3.6), the quadratic form

$$2\Omega(x, y, y') + b^2 y_i y_i - c^2 y_i' y_i'$$

being positive definite for $x_1 \leqq x \leqq x_2$.

LEMMA 3.3. *With*

$$(3.9) \qquad I(y) = \int_{x_1}^{x_2} 2\Omega(x, y, y')\, dx,$$

we define Λ as the greatest lower bound of $I(y)$ in the class of sets $y_i = y_i(x)$ of class D' on $x_1 x_2$, with $y_i(x_1) = y_i(x_2) = 0$ and

$$(3.10) \qquad \int_{x_1}^{x_2} y_i y_i\, dx = 1.$$

Let ξ_1 be conjugate to x_1 with respect to (1.3). We shall prove that Λ is always finite † *and*

1) $\Lambda < 0$ if $\xi_1 < x_2$,

2) $\Lambda = 0$ if $\xi_1 = x_2$.

From (3. 7) and (3. 10) we get

$$I(y)=\int_{x_1}^{x_2}2\Omega dx \geqq -b^2\int_{x_1}^{x_2}y_iy_idx=-b^2,$$

and therefore Λ is finite and well defined. Our statement 1 is a consequence of the necessity of Jacobi's condition for a minimum of $I(y)$ without (3. 10). The second statement is equivalent to lemma 2. 2.

If there is no $\xi_1 \leqq x_2$, we know already that $\Lambda \geqq 0$ (lemma 2. 1) and the only difficulty is to prove that actually $\Lambda > 0$. Our assumptions are the same as in lemma 2. 1. Let (u_{ik}) be the conjugate system of solutions of (2. 1) introduced in the proof of lemma 2. 1. Its determinant never vanishes on x_1x_2: $U(x_0, x) \neq 0$ for $x_1 \leqq x \leqq x_2$. Let us replace in (2. 1) the quadratic form $2\Omega(x, y, y')$ by $2\Omega(x, y, y')-\delta y_iy_i$, with $\delta > 0$ very small. The point x_0 being fixed, $U(x_0, x)$ will become a continuous function of δ, say $U(x_0, x, \delta)$, which will also not vanish on x_1x_2 for some small $\delta > 0$. For such a δ the interval x_1x_2 certainly contains no conjugate points with respect to the altered system (2. 1) (lemma 1. 1), and (2. 3) gives

$$\int_{x_1}^{x_2}2\Omega(x, y, y')\,dx-\delta\int_{x_1}^{x_2}y_iy_idx \geqq 0,$$

for every admissible set $y_i=y_i(x)$, and hence $I(y) > \delta$ for all admissible sets with (3. 10). Hence $\Lambda \geqq \delta > 0$.

LEMMA 3. 4. *The first function $\xi_1(\lambda)$ of the sequence* (3. 5), *is a continuous and non-increasing function for $\lambda \geqq \lambda_0$, which is actually decreasing on the λ-interval on which $\xi_1(\lambda) < x_2$.*

The monotoneity properties follow from the assumption 2 of this section and the restricted comparison theorem 2. 2.

To prove the continuity of $\xi_1(\lambda)$, we consider the integral

$$I(y,\lambda)=\int_{x_1}^{x_2}[2\Omega(x, y, y')-A_{ik}(x,\lambda)y_iy_k]dx, \tag{3. 11}$$

the set $y_i=y_i(x)$ being an admissible set satisfying (3. 10). Let $\Lambda(\lambda)$ be the greatest lower bound of $I(y,\lambda)$ in this class of sets $y_i=y_i(x)$. We have seen that $\Lambda(\lambda)$ is well defined for $\lambda \geqq \lambda_0$ (lemma 3. 3). *We shall first prove that $\Lambda(\lambda)$ is continuous for $\lambda \geqq \lambda_0$*: For λ' fixed and λ variable (both $\geqq \lambda_0$) we have from (3. 11)

$$I(y,\lambda)-I(y,\lambda')=\int_{x_1}^{x_2}[A_{ik}(x,\lambda')-A_{ik}(x,\lambda)]y_iy_kdx.$$

Assumption 1 on the [illegible] the [illegible] $A_{ik}(x,\lambda)$ [illegible]

$$|A_{ik}(x,\lambda') - A_{ik}(x,\lambda)| < \epsilon \text{ for } |\lambda - \lambda'| < \delta,\ x_1 \leqq x \leqq x_2,$$
$$(i, k = 1, 2, \cdots, n),$$

and hence

$$|I(y,\lambda) - I(y,\lambda')| \leqq \epsilon \int_{x_1}^{x_2} \sum_{i,k=1}^{n} |y_i|\,|y_k|\,dx$$
$$= \epsilon \int_{x_1}^{x_2} (|y_1|^2 + \cdots + |y_n|^2)\,dx = \epsilon \text{ for } |\lambda - \lambda'| < \delta.$$

This implies also $|\Lambda(\lambda) - \Lambda(\lambda')| \leqq \epsilon$ for $|\lambda - \lambda'| < \delta$, and $\Lambda(\lambda)$ is therefore continuous for $\lambda \geqq \lambda_0$.

A discontinuity of the non-increasing function $\xi_1(\lambda)$ at $\lambda = \lambda'$ $(\geqq \lambda_0)$, would imply one at least of the inequalities

$$\xi_1(\lambda' + 0) < \xi_1(\lambda'), \tag{3.12}$$

$$\xi_1(\lambda') < \xi_1(\lambda' - 0). \tag{3.13}$$

We shall get a contradiction from each of these inequalities. To fix the ideas, we suppose (3.12) to hold. We choose a fixed x_3 with

$$\xi_1(\lambda' + 0) < x_3 < \xi_1(\lambda').$$

Let $\Lambda(\lambda)$ be the greatest lower bound of $I(y,\lambda)$ defined by (3.11) for the interval x_1x_3 instead of x_1x_2. For $\lambda > \lambda'$ we have

$$\xi_1(\lambda) < \xi_1(\lambda' + 0) < x_3$$

and hence by lemma 3.3 (case 1), applied to the integral (3.11) instead of (3.9), we get

$$\Lambda(\lambda) < 0, \text{ for } \lambda > \lambda'. \tag{3.14}$$

But $x_3 < \xi_1(\lambda')$, and the same lemma (case 3) gives

$$\Lambda(\lambda') > 0. \tag{3.15}$$

[illegible]

functions of the sequence (3.5) by induction, it will be convenient to have the following lemma:

LEMMA 3.5. *We define a function* $\phi(\xi, \lambda)$ *for* $x_1 \leqq \xi \leqq x_2$, $\lambda \geqq \lambda_0$, *as follows:* $x = \phi(\xi, \lambda)$ *is the conjugate point to* ξ, *following* ξ, *conjugate with respect to the system* (3.1), *or else* $\phi(\xi, \lambda) = x_2$ *if there is no such point. Let* $\xi = \xi(\lambda)$ *be a continuous and non-increasing function of* λ, *for* $\lambda \geqq \lambda_0$, *with* $x_1 \leqq \xi(\lambda) \leqq x_2$, *for* $\lambda \geqq \lambda_0$. *Then* $\psi(\lambda) = \phi(\xi(\lambda), \lambda)$ *is a continuous and non-increasing function of* λ, *for* $\lambda \geqq \lambda_0$, *and actually decreasing on the* λ-*interval defined by* $\phi(\xi(\lambda), \lambda) < x_2$, *which interval may have no points.*

The function $\phi(\xi, \lambda)$ is a function of two variables which is analogous to the function $\phi(\xi)$ of lemma 3.1, if one replaces the system (3.1) by (1.3). Lemma 3.1 insures therefore the monotoneity and continuity of $\phi(\xi, \lambda)$ as a function of ξ only when λ is fixed. Furthermore

$$\phi(x_1, \lambda) = \xi_1(\lambda),$$

and because the proof of lemma 3.4 works also for any ξ on x_1x_2, instead of $\xi = x_1$, this lemma 3.4 insures the monotoneity and continuity of $\phi(\xi, \lambda)$ as function of λ only when ξ is fixed. This, of course, does not imply the continuity of $\phi(\xi, \lambda)$ as function of both variables. But the more restricted statement of our lemma follows easily as indicated in the following paragraphs.

We prove first

a) *The values* (ξ', λ') *being fixed, then for every* $\epsilon > 0$ *there is a* $\delta > 0$, *such that*

$$(3.16) \qquad 0 \leqq \xi' - \xi < \delta,\ 0 \leqq \lambda - \lambda' < \delta \text{ imply } 0 \leqq \phi(\xi', \lambda') - \phi(\xi, \lambda) < \epsilon.$$

This statement is obvious for $\xi' = x_1$. In the proof we suppose $\xi' > x_1$. We have

$$(3.17) \quad \phi(\xi', \lambda') - \phi(\xi_0, \lambda^0) = \phi(\xi', \lambda') - \phi(\xi', \lambda^0) + \phi(\xi', \lambda^0) - \phi(\xi_0, \lambda^0).$$

The function $\phi(\xi', \lambda)$ being continuous in λ at $\lambda = \lambda'$, there is a $\lambda^0 > \lambda'$ with $0 \leqq \phi(\xi', \lambda') - \phi(\xi', \lambda^0) < \epsilon/2$. Furthermore $\phi(\xi, \lambda^0)$ is continuous at $\xi = \xi'$ and one can choose a $\xi_0 < \xi'$ with $0 \leqq \phi(\xi', \lambda^0) - \phi(\xi_0, \lambda^0) < \epsilon/2$. From (3.17) it follows $0 \leqq \phi(\xi', \lambda') - \phi(\xi_0, \lambda^0) < \epsilon$. This last result and the monotoneity of $\phi(\xi, \lambda)$ in ξ and λ certainly imply (3.16) for every $\delta < \xi' - \xi_0$ and $< \lambda^0 - \lambda'$.

Now we prove

b) *The values* (ξ', λ') *being fixed, then for every* $\epsilon > 0$ *there is a* $\delta > 0$, *such that*

$$(3.18) \quad 0 \leqq \xi - \xi' < \delta,\ 0 \leqq \lambda' - \lambda < \delta \textit{ imply } 0 \leqq \phi(\xi, \lambda) - \phi(\xi', \lambda') < \epsilon.$$

This statement is obvious if one or both of the equalities $\xi' = x_2$, $\lambda' = \lambda_0$ hold. In the proof we suppose $\xi' < x_2$, $\lambda' > \lambda_0$. The argument is the same as above. We have

$$(3.19) \quad \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda') = \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda^0) + \phi(\xi', \lambda^0) - \phi(\xi', \lambda').$$

There is a $\lambda^0 < \lambda'$ with $0 \leqq \phi(\xi', \lambda^0) - \phi(\xi', \lambda') < \epsilon/2$, and a $\xi_0 > \xi'$ with $0 \leqq \phi(\xi_0, \lambda^0) - \phi(\xi', \lambda^0) < \epsilon/2$, and therefore (3. 18) holds for every $\delta < \xi_0 - \xi'$ and $< \lambda' - \lambda^0$.

The proof for our lemma follows at once. The function

$$\psi(\lambda) = \phi(\xi(\lambda), \lambda)$$

is certainly non-increasing for $\lambda \geqq \lambda_0$, and is actually decreasing on the λ-interval for which $\psi(\lambda) < x_2$ and which may have no point (for instance if $\xi(\lambda) \equiv x_2$). For λ fixed, our statement a) insures the inequality

$$0 \leqq \phi(\xi(\lambda), \lambda) - \phi(\xi(\lambda + \eta), \lambda + \eta) < \epsilon$$

for $0 \leqq \eta < \delta'$ and hence $\psi(\lambda) = \psi(\lambda + 0)$. Similarly statement b) insures $\psi(\lambda) = \psi(\lambda - 0)$, and our lemma 3. 5 is proved.

LEMMA 3. 6. *The functions* $\xi_m(\lambda)$, $(m = 1, 2, 3, \cdots)$, *of the sequence* (3. 5) *are continuous and non-increasing functions of* λ *for* $\lambda \geqq \lambda_0$, *and actually decreasing on the* λ*-intervals for which* $\xi_m(\lambda) < x_2$.

For $m = 1$, this is the statement of lemma 3. 4. The proof goes on by induction. Suppose the theorem holds for $1, 2, \cdots, m - 1$. We have

$$\xi_m(\lambda) = \phi(\xi_{m-1}(\lambda), \lambda),$$

$\phi(\xi, \lambda)$ being the function defined by lemma 3. 5. By hypothesis $\xi_{m-1}(\lambda)$ is a continuous non-increasing function of λ for $\lambda \geqq \lambda_0$. Lemma 3. 5 says that $\xi_m(\lambda)$ is also a continuous non-increasing function of λ for $\lambda \geqq \lambda_0$, which is actually decreasing for $\xi_m(\lambda) < x_2$. This proves our lemma by induction.

The proof of the oscillation theorem 3. 1 now makes no difficulty. We have

$$(3.20) \quad \xi_1(\lambda) \geqq \xi_2(\lambda) \geqq \cdots \geqq \xi_m(\lambda) \geqq \cdots \text{ for } \lambda \geqq \lambda_0.$$

From assumption 4 of this section, we get $\xi_1(\lambda_0) = x_2$ and (3.20) gives

$$(3.21) \qquad \xi_m(\lambda_0) = x_2, \ (m = 1, 2, 3, \cdots).$$

We shall prove now

$$(3.22) \qquad \lim_{\lambda \to \infty} \xi_m(\lambda) = x_1, \ (m = 1, 2, 3, \cdots).$$

To prove this we determine the constant $a > 0$ satisfying

$$(3.23) \qquad P_{ik}(x) u_i u_k + 2Q_{ik}(x) u_i v_k + R_{ik}(x) v_i v_k \leqq a^2,$$

for $x_1 \leqq x \leqq x_2$, and $u_i u_i + v_i v_i = 1$. Inequality (3.23) and assumptio of this section give

$$(3.24) \qquad 2\Omega(x, y, y') - A_{ik}(x, \lambda) y_i y_k \leqq a^2(y_i y_i + y_i' y_i') - M y_i y_i,$$

for $x_1 \leqq x \leqq x_2$, $\lambda > \lambda(M)$, and arbitrary values y_i, y_i'.

We shall apply the comparison theorem 2.1 for

$$2\Omega^* = (a^2 - M) y_i y_i + a^2 y_i' y_i'$$

and the left hand side of (3.24) instead of 2Ω. The system (2.7) becon

$$(3.25) \qquad (d^2/dx^2) y_i + \alpha^2 y_i = 0,$$

with $\alpha = a^{-1}(M - a^2)^{1/2}$ (supposing $M > a^2$). In order to determine the c jugate system of points defined by $x = x_1$ with respect to the system (3.2 we remark that

$$u_{ik}(x) \equiv 0 \ \ (i \neq k), \quad u_{kk}(x) = \sin \alpha (x - x_1)$$

is a conjugate system of solutions of (3.25) and hence

$$U(x_1, x) = | u_{ik}(x) | = (\sin \alpha (x - x_1))^n.$$

The first zero of this determinant which follows x_1 is $\xi^*_1 = x_1 + \pi$

The m-th conjugate point of x_1, following x_1, will therefore be

$$\xi^*_m = x_1 + m\pi/\alpha.$$

The comparison theorem 2.1 gives

$$x_1 < \xi_m(\lambda) \leqq x_1 + m\pi a (M - a^2)^{-1/2}$$

and this implies (3.22), when we allow M to increase.

We define $\lambda_m > \lambda_0 (m = 1, 2, 3, \cdots)$ *as a value for which*

$$\xi_m(\lambda_m) = x_2, \quad \xi_m(\lambda_m + \eta) < x_2 \tag{3. 26}$$

for every $\eta > 0$. Its existence follows from (3. 21) and (3. 22). We shall prove first that x_2 *is the m-th conjugate point of* x_1 *with respect to the system* (3. 1) for $\lambda = \lambda_m$. We know that $\xi_m(\lambda_m + \eta) (< x_2)$ is the m-th conjugate point of x_1 with respect to (3.1) for $\lambda = \lambda_m + \eta$, for every $\eta > 0$. For $\lim \eta = 0$ we get the result that $\xi_m(\lambda_m) = x_2$ is the m-th conjugate point of x_1 with respect to (3. 1) for $\lambda = \lambda_m$. It follows also that $x_2 = \xi_{m+1}(\lambda_{m+1})$ is the $(m+1)$-st conjugate of x_1 with respect to (3. 1) for $\lambda = \lambda_{m+1}$, and hence $\xi_m(\lambda_{m+1}) < x_2 = \xi_m(\lambda_m)$. Therefore $\lambda_m < \lambda_{m+1}$ and

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots . \tag{3. 27}$$

We shall prove finally that

$$\lim_{m \to \infty} \lambda_m = \infty. \tag{3. 2}$$

Suppose (3. 2) does not hold and suppose that λ_m approaches Λ, as m approaches ∞. We take m sufficiently large so that $\xi_m(\Lambda) = x_2$. One therefore has $\xi_m(\Lambda) = x_2 = \xi_m(\lambda_m)$ and (3. 26) implies $\Lambda \leqq \lambda_m$, which result contradicts (3. 27) and the definition of Λ. Our oscillation theorem 3. 1 is therefore completely proved.

We consider now the special case (3. 3) and shall prove the statement (3. 4). Let k and K be two positive constants satisfying

$$k^2 \leqq A_{ik}(x) u_i u_k \leqq K^2, \quad \text{for} \quad x_1 \leqq x \leqq x_2, \quad u_i u_i = 1. \tag{3. 28}$$

The inequalities (3. 6), (3. 23), and (3. 28) give

$$\begin{aligned} c^2 y_i' y_i' - (K^2 \lambda + b^2) y_i y_i \leqq 2\Omega(x, y, y') \\ - \lambda A_{ik}(x) y_i y_k \leqq a^2 y_i' y_i' - (k^2 \lambda - a^2) y_i y_i, \end{aligned} \tag{3. 29}$$

for $x_1 \leqq x \leqq x_2$, $\lambda > 0$, and y_i, y_i' arbitrary. We shall apply the comparison theorem 2. 1 twice, comparing the three differential systems which correspond to the three quadratic forms of (3. 29). The system corresponding to the second form is our system (3. 1), while the systems corresponding to the first and the third form are of the type (3. 25). We have seen how to determine the conjugate systems of points for systems of the type (3. 25). The formulas there obtained and theorem 2. 1 give

$$x_1 + m\pi c(K^2\lambda_m + b^2)^{-1/2} \leqq \xi_m(\lambda_m) = x_2 \leqq x_1 + m\pi a(k^2\lambda_m - a^2)^{-1/2}$$

and hence

$$K^{-2}[m^2\pi^2c^2(x_2 - x_1)^{-2} - b^2] \leqq \lambda_m \leqq k^{-2}[m^2\pi^2a^2(x_2 - x_1)^{-2} + a^2].$$

These last inequalities show that $\sum_{m=1}^{\infty} \lambda_m^{-a}$ converges or diverges simultaneously with $\sum_{m=1}^{\infty} m^{-2a}$. Hence (3. 4) is proved.

We want to emphasize finally the difference between the theorem 3. 1 and Hickson's oscillation theorem (*loc. cit.*). Hickson established for the special case of Theorem 3. 1 the existence of an increasing sequence of characteristic numbers λ_n for each of which a certain boundary value problem has a finite number of linearly independent solutions η_i with elements vanishing at x_1 and x_2. In the preceding pages a similar sequence of characteristic numbers λ_n is found for each of which there exists a conjugate system in the sense defined above of precisely $n + 1$ points with x_1 and x_2 as its initial and final point.

UNIVERSITY OF CHICAGO,
CHICAGO, ILLINOIS.

ON IDENTITIES ARISING FROM SOLUTIONS OF q-DIFFERENCE EQUATIONS AND SOME INTERPRETATIONS IN NUMBER THEORY.

By G. W. STARCHER.

Introductory. The interpretation of certain kinds of identities in number theory has, since the time of Euler, stimulated numerous independent investigations for the purpose of obtaining identities involving infinite products and series of a simple character. Often identities were discovered empirically and later verified or proved, as was the case in some of the investigations of Euler and Ramanujan. Many of the proofs of known identities depend upon a purely algebraic rearrangement of products and series. Simple and often highly ingenious proofs of certain algebraic identities have been given by means of their interpretations in number theory.* Some of the most beautiful identities known, many of them having important applications to number theory, have been demonstrated by Jacobi † by means of the theory of elliptic functions. L. J. Rogers,‡ by quite general methods, has established identities of considerable interest and beauty in themselves.

This investigation is based on a study of two linear q-difference equations with linear coefficients taken in the forms (1. 1) and (4. 1) respectively. Each of these equations has a unique solution analytic in the neighborhood of the point zero and having there the value 1, and hereafter such a solution will be said to have the property A. Obtaining various expressions for such a solution, and equating them, we have identities.

While most of the identities found appear to be new they include many interesting and classic identities as special cases of the more general identities and thus afford new proofs for them. For this reason a number of such identities are stated and often new members are added. The method of exhibiting these identities serves to relate identities that have previously appeared to be quite unrelated.

1. *Solutions of a Non-homogeneous Equation.* Consider the equation

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* For example, see MacMahon, *Combinatory Analysis*, Vol. 2.

† "Theoria Evolutionis Functionum Ellipticarum," *Gesammelte Werke*, Vol. 1, pp. 141-239, and Ueber unendliche Reihen deren Exponenten zugleich in zwei verschiedenen quadratischen Formen enthalten sind, Vol. 2, pp. 218-288.

‡ "On the Expansion of Certain Infinite Products," *Proceedings of the London Mathematical Society*, Vol. 24 (1893), pp. 337-352; *ibid.*, Vol. 25 (1894), pp. 318-343.

$$(1.1) \qquad (1+\beta x)f(x)+(\gamma+\delta x)f(qx)+\lambda+\mu x=0, \qquad |q|<1.$$

If there is a solution of (1.1) having the property A it may be written in the form

$$(1.2) \qquad f_1(x)=1+\Sigma c_\nu x^\nu,$$

where here, as in the following pages, the index ν connotes the range 1, 2, 3, $\cdots$. In order that this series satisfy (1.1) and is other than the trivial solution $f_1(x)\equiv 1$ we must have

$$(1.3) \qquad 1+\gamma+\lambda=0, \qquad \epsilon=\beta+\delta+\mu\neq 0.$$

Then we have the relations

$$-c_1(1+\gamma q)=\epsilon; \qquad -c_\nu(1+\gamma q^\nu)=c_{\nu-1}(\beta+\delta q^{\nu-1}), \qquad (\nu>1),$$

which determine the c's uniquely. Hence we have the unique solution *

$$(1.4) \qquad f_1(x)=1+\Sigma\epsilon(-x)^\nu(\beta,\delta q)_{\nu-1}(\gamma q)'_\nu,$$

which is convergent if $|\beta x|<1$.

Let us consider the possibility of expressing such a solution in each of the forms

$$(1.5) \qquad f_2(x)=1+\Sigma c_\nu x^\nu(\beta x)'_\nu,$$

$$(1.6) \qquad f_3(x)=1+\Sigma c_\nu x(\beta x)'_\nu,$$

$$(1.7) \qquad f_4(x)=1+\Sigma c_\nu x(\gamma,\delta x)_{\nu-1}, \qquad (\beta=0),$$

$$(1.8) \qquad f_5(x)=1+\Sigma c_\nu x^{[(\nu+1)/2]}(\gamma,\delta x)_{[\nu/2]-1}, \qquad (\beta=0),$$

where $[k]$ denotes the least integer $\geq k$. Clearly each of these series, if it converges, defines a function having the property A.

Substituting (1.5) into (1.1), solving for the successive c's by equating coefficients of expressions of the form $x^\nu(\beta x)'_\nu$ to zero, we have the solution

$$(1.9) \qquad f_2(x)=1+\Sigma q^{\nu(\nu-1)/2}\epsilon(\delta,-\beta\gamma q)_{\nu-1}(\gamma q)'_\nu(\beta x)'_\nu(-x)^\nu.$$

For (1.6) we consider only the case $\delta=0$, to insure convergence of series obtained on substituting into (1.1), and obtain the solution

$$(1.10) \qquad f_3(x)=1+\Sigma(-)^\nu\epsilon(\gamma q)^{\nu-1}x(\beta x)'_\nu, \qquad (\delta=0).$$

Similarly when $\beta=0$ we get the solutions

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* For convenience we shall write $(a,x)_\nu$ for $(a+x)(a+qx)\cdots(a+q^{\nu-1}x)$, $(a,x)'_\nu$ for $1/(a,x)_\nu$; $(x)_\nu$ for $(1,x)_\nu$, and $(x)'_\nu$ for $1/(x)_\nu$, etc., and throughout this paper any symbol marked prime denotes the reciprocal of the expression denoted by the unprime symbol. In particular we make $(a,x)_0=1$ and $(x)=(x)_\infty=\Pi(1+q^{\nu-1}x)$.

Any values of the parameters or of x for which a series or product is undefined or is infinite will be known as exceptional values for that series or product. Such values we exclude by enclosing them in circles of arbitrarily small, but fixed, radius.

(1.11) $f_4(x) = 1 + \Sigma(-)^\nu \epsilon q^{\nu-1} x(\gamma, \delta x)_{\nu-1},$

(1.12) $f_5(x) = 1 + \Sigma(-x)^\nu \epsilon \delta^{\nu-1}$
$\times \{q^{3\nu(\nu-1)/2}(\gamma, \delta x)_{\nu-1}(\gamma q)'_{\nu-1} - q^{\nu(3\nu-1)/2}(\gamma, \delta x)_\nu(\gamma q)'_\nu\}.$

Clearly each of the last four series converge for all x.

2. *Case* $\lambda = \mu = 0$. *The Homogeneous Case.* By (1.3), (1.1) takes the simple form

(2.1) $$(1 + \beta x) f(x) = (1 - \delta x) f(qx).$$

We easily obtain as a solution having the property A

(2.2) $$f_6(x) = \Pi[(1 - \delta q^{\nu-1}x)/(1 + \beta q^{\nu-1}x)] = (-\delta x)(\beta x)'.$$

For the rest of this section let $\beta = 0$.

If in (2.1) we substitute

(2.3) $$f(x) = 1 + \Sigma c_\nu(-qx)_\nu,$$

we obtain (writing $c_0 = 1$)

$$1 + \Sigma c_\nu(1 - qx)(-q^2x)_{\nu-1} = 1 + \Sigma c_\nu(1 - q^{\nu+1}x)(-q^2x)_{\nu-1} - \Sigma c_{\nu-1}\delta x(-q^2x)_{\nu-1}.$$

On equating coefficients of $(-q^2x)_{\nu-1}$ we get

(2.4) $$c_\nu = \delta^\nu q^{-\nu}(-q)'_\nu.$$

If $|\delta| < |q|$, $f(x)$ is convergent for all x. By making $\beta = \lambda = \mu = 0$ and $x = -1$ in (1.4) and comparing with (2.2) we have from (2.3) when the c's have their values given by (2.4)

$$f(0) = 1 + \Sigma \delta^\nu q^{-\nu}(-q)'_\nu = 1/\Pi(1 - q^{\nu-2}\delta) = (-q^{-1}\delta)'.$$

Hence, if $\beta = 0$, $f_6(x)$ becomes

(2.5) $$f_7(x) = (-q^{-1}\delta)\{1 + \Sigma\delta^\nu q^{-\nu}(-q)'_\nu(-qx)_\nu\},$$

since $f_7(x)$ is also a solution with property A.

Replacing q by q^2 in (2.1), $\beta = 0$, we get

(2.6) $$f(x, q^2) = (1 - \delta x) f(q^2x, q^2).$$

If in (2.6) we substitute

(2.7) $$f(x, q^2) = 1 + \Sigma c_\nu(-\delta q^{-1}x)_\nu,$$

and equate the quantities which multiply the expressions $(-\delta x)_\nu$, $\nu = 1, 2, \cdots$, we have

$$c_1 = q/(1 - q), \qquad c_\nu = c_{\nu-1}q^\nu/(1 - q^\nu), \qquad (\nu > 1);$$

and hence the solution *

$$f(x, q^2) = 1 + \Sigma q^{\nu(\nu+1)/2}(-\delta q^{-1}x)_\nu(-q)'_\nu.$$

But $f(0, q^2) = (q)$ by comparison with (2.2) and (1.9). Then we have the solution with property A,

$$(2.8) \qquad f_7(x) = (q)'\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu(-\delta q^{-1}x)_\nu\},$$

which by (2.2) is the same as $\Pi(1 - \delta q^{2\nu-2}x)$.

3. *Applications and Interpretations.* We replace $f(x)$ by the symbol $f(\beta x, \gamma, \delta x, \mu x, q)$, so that f_3 becomes $f(\beta x, \gamma, 0, \mu x, q)$, f_4 and f_5 become $f(0, \gamma, \delta x, \mu x, q)$, f_6 becomes $f(\beta x, -1, \delta x, 0, q)$, f_7 becomes $f(0, -1, \delta x, 0, q)$, and $f_7(x, q^2)$ becomes $f(0, -1, \delta x, 0, q^2)$. From identities that arise from equating various expressions for the same function as obtained in the preceding sections we have †

$$(3.1) \quad f(\beta x, -1, \delta x, 0, q) = (-\delta x)(\beta x)' = 1 + \Sigma(\beta, \delta)_\nu(-q)'_\nu(-x)^\nu$$
$$= 1 + \Sigma q^{\nu(\nu-1)/2}(\delta, \beta)_\nu(-q)'_\nu(\beta x)'_\nu(-x)^\nu.$$

From (3.1) we obtain

$$(3.2) \quad f(qx, -1, qx, 0, q) = (-qx)(qx)' = 1 + 2\Sigma q^\nu(q)_{\nu-1}(-q)'_\nu(-x)^\nu$$
$$= 1 + 2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu(qx)'_\nu(-x)^\nu,$$

$$(3.3) \quad f(\pm qx, -1, \pm qx^2, 0, q) = (\mp qx^2)(\pm qx)' = 1 + \Sigma(\mp qx)^\nu(x)_\nu(-q)'_\nu$$
$$= 1 + \Sigma q^{\nu(\nu+1)/2}(x, 1)_\nu(-q)'_\nu(\pm qx)'_\nu(\mp x)^\nu,$$

$$(3.4) \quad f(q, -1, qx, 0, q) = (-qx)(q)' = 1 + \Sigma(-q)^\nu(x)_\nu(-q)'_\nu$$
$$= 1 + \Sigma(-)^\nu q^{\nu(\nu+1)/2}(x, 1)_\nu(-q)'_\nu(q)'_\nu,$$

$$(3.5) \quad f(qx, -1, -q, 0, q) = (q)(qx)' = 1 + \Sigma(-q)^\nu(x, -1)_\nu(-q)'_\nu$$
$$= 1 + \Sigma(-)^\nu q^{\nu(\nu+1)/2}(x, -1)_\nu(-q)'_\nu(qx)'_\nu.$$

These examples illustrate the variety of identities one may obtain in this way. Note the formulae similar to the last three given by the three corresponding expressions for $f(\pm qx^2, -1, \mp qx, 0, q)$, $f(-q, -1, qx, 0, q)$, and $f(qx, -1, q, 0, q)$ respectively. Note also the formulae obtained from those above by replacing x by $-x$, by ± 1, or by certain functions of q. For example

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* Since this was written my attention has been called to C. H. Ashton's dissertation, *Die Heineschen O-Functionen und ihre Anwendungen auf die Elliptische Functionen* (Munich, 1909). He has studied the same function in another form; also my $f(0, -1, qx, 0, q)$ is identical with his $0_{00}(z)$ with $x = e^{2\pi iz/w_1}$.

† Essentially the same formulae have been given by Cauchy, *Oeuvres* (1) VIII, pp. 42-50.

$$(3.6)\quad f(q,-1,q,0,q)=(-q)(q)'=1+2\Sigma(-q)^{\nu}(q)_{\nu-1}(-q)'_{\nu}$$
$$=1+2\Sigma(-)^{\nu}q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_{\nu}(q)'_{\nu},$$

and by the theory of elliptic functions the second member is ϑ_4. Similarly we find

$$\vartheta_2=2q^{1/4}\pi(1+q^{2\nu})^2(1-q^{2\nu})$$
$$=2q^{1/4}\Pi[(1-q^{4\nu})/(1-q^{4\nu-2})]=2q^{1/4}f(-q^2,-1,q^4,0,q^4).$$

Thus we have two new expressions for each of the Theta constants ϑ_2, ϑ_3, and ϑ_4. Another example of such identities in q is given by

$$(3.7)\quad f(-q,-1,-q,0,q)=(q)(-q)'$$
$$=1+2\Sigma q^{\nu}(q)_{\nu-1}(-q)'_{\nu}=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}[(-q)'_{\nu}]^2.$$

Again we have *

$$(3.8)\quad f(-qx,-1,q^{j+1}x,0,q)=1/\prod_{\nu=1}^{j}(1-q^{\nu}x)=1+\Sigma(qx)^{\nu}(-q^{j})_{\nu}(-q)'_{\nu}$$
$$=1+\sum_{\nu=1}^{j}q^{\nu^2}(-q^{j-\nu+1})_{\nu}(-q)'_{\nu}(-qx)'_{\nu}x^{\nu}=1+\Sigma_{\nu,\mu}P(\nu,\mu,\not> j)\,q^{\nu}x^{\mu},$$

where the symbol $P(\nu,\mu,\not> j)$ represents the number of partitions of ν into exactly μ summands none greater than j; and

$$(3.9)\quad f(q^{j+1}x,-1,qx,0,q)=\prod_{\nu=1}^{j}(1+q^{\nu}x)=1+\sum_{\nu=1}^{j}q^{\nu(\nu+1)/2}(-q^{j-\nu+1})_{\nu}(-q)'_{\nu}x^{\nu}$$
$$=1+\Sigma q^{\nu(\nu+1)/2}(-q^{j})_{\nu}(-q)'_{\nu}(q^{j+1}x)'_{\nu}x^{\nu}=1+\Sigma_{\nu,\mu}P^{i}(\nu,\mu,\not> j)\,q^{\nu}x^{\mu},$$

where the superscript i on the symbol P indicates that the parts are distinct. We may observe from the second members of (3.8) and (3.9) the theorem,

$$P(\nu,k,\not> j)=P^{i}(\nu+\tfrac{1}{2}k(k-1),k,\not> j+k-1).$$

From the expressions in § 2 we have

$$(3.10)\quad f(-qx,-1,0,0,q)=1/\Pi(1-q^{\nu}x)=1+\Sigma(qx)^{\nu}(-q)'_{\nu}$$
$$=1+\Sigma q^{\nu^2}(-q)'_{\nu}(-qx)'_{\nu}x^{\nu}=1+\Sigma q^{\nu}x(-qx)'_{\nu}$$
$$=1+\Sigma_{\nu,\mu}P(\nu,\mu,*)\,q^{\nu}x^{\mu}.$$

This is one of the fundamental identities in the theory of partitions. The symbol $P(\nu,\mu,*)$ represents the number of partitions of ν into μ positive summands. Replacing x by $q^{-1}x$ we have a similar identity involving the function which enumerates the partitions of ν into μ or fewer summands [illegible] By equating coefficients of $q^{\nu}x^{\mu}$ in these identities

* Cf. MacMahon *Combinatory Analysis*, Vol. 2, Ch. II. MacMahon demonstrates the identity (3.8) and the second and third members of (3.9).

we obtain identities in certain partition functions. When $x=1$ by the 2nd and 3rd members of (3.10) we have an identity due to Euler, and probably the most interesting identity in the theory of partitions. Generalizations are obtained by writing q^{m-n} for x and q^n for q.

Again we have *

$$(3.11)\quad f(0,-1,qx,0,q)=\Pi(1-q^{\nu}x)=1+\Sigma q^{\nu(\nu+1)/2}(-q)_{\nu}'(-x)^{\nu}$$
$$=1-\Sigma q^{\nu}x(-qx)_{\nu-1}$$
$$=1+\Sigma q^{\nu(3\nu-1)/2}(-x)^{\nu}(1-q^{2\nu}x)(-qx)_{\nu-1}(-q)'_{\nu},$$

which for $x=1$ gives

$$(3.12)\quad f(0,-1,q,0,q)=\Pi(1-q^{\nu})=1+\Sigma(-)^{\nu}q^{\nu(\nu+1)/2}(-q)'_{\nu}$$
$$=1-\Sigma q^{\nu}(-q)_{\nu-1}=1+\Sigma(-)^{\nu}q^{\nu(3\nu-1)/2}(1+q^{\nu})$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}.$$

The second and last members above constitute a celebrated identity due to Euler, of which many proofs have been given. If in (3.11) we replace x by $-x$ and equate coefficients of x^k we have the almost obvious theorem $P^{i}(\nu+\tfrac{1}{2}k(k-1),k,{}^*)=P(\nu,k,{}^*)$. Similar theorems come from (3.12).

Using the fact that $\Pi(1+q^{\nu})^{-1}=\Pi(1-q^{2\nu-1})$ we have †

$$(3.13)\quad (-q;q^2)f(-q,-1,qx,0,q^2)=f(0,-1,qx,0,q^2)=\Pi(1-q^{2\nu-1}x)$$
$$=(-q;q^2)\{1+\Sigma q^{\nu}(-x;q^2)_{\nu}(-q^2;q^2)'_{\nu}\}$$
$$=(-q;q^2)\{1+\Sigma(-)^{\nu}q^{\nu^2}(x^{-1};q^2)_{\nu}(-q)'_{2\nu}\}$$
$$=(-q;q^2)\{1+\Sigma q^{\nu(\nu+1)/2}(-x)_{\nu}(-q)'_{\nu}\},$$

which, by (3.11) with q replaced by q^2 and x by $q^{-1}x$,

$$=1+\Sigma q^{\nu^2}(-x)^{\nu}(-q^2;q^2)'_{\nu}=1-\Sigma q^{2\nu-1}x(-qx;q^2)_{\nu}$$
$$=1+\Sigma q^{3\nu^2-2\nu}(-x)^{\nu}(1-q^{4\nu-1}x)(-qx;q^2)_{\nu-1}(-q^2;q^2)'_{\nu}.$$

We can write

$$f(0,-1,qx,0,q^2)=\Pi[1-q^{\nu}x)/\Pi(1-q^{2\nu}x)]$$
$$=f(0,-1,qx,0,q)\,f(-q^2x,-1,0,0,q^2).$$

Each of these can be expressed as infinite series thus giving six additional, equally simple, expressions for the function in (3.13).

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* The identity of the 2nd and 3rd members is well known. That of the 2nd and last member has been verified by E. Netto. See *Lehrbuch der Combinatorik*, pp. 163-65.

† The symbol $(x;\,q^k)_{\nu}$ is written for $(1+x)(1+q^kx)(1+q^{2k}x)\cdots(1+q^{\nu k-k}x)$, accordingly we write $(x;\,q^k)$ for $\Pi(1+q^{k(\nu-1)}x)$. The third and sixth members of (3.13) constitute an identity due to Gauss and verified by Lebesgue. See *Liouville's Journal* (1), 5, p. 47.

From the first five members above we have

$$(3.14)\quad f(0,-1,q^2,0,q^2)/f(0,-1,q,0,q^2)=(-q^2;q^2)(-q;q^2)'$$
$$=f(-q,-1,q^2,0,q^2)=1+\Sigma q^{\nu}(-q;q^2)_{\nu}(-q^2)'_{\nu}$$
$$=1-\Sigma(-)^{\nu}q^{\nu^2+\nu-1}(-q^2;q^2)'_{\nu}(1-q)/(1-q^{2\nu-1})=1+\Sigma q^{\nu(\nu+1)/2}.$$

Similarly we see that

$$(3.15)\quad (-q^2;q^2)f(-q^2,-1,q^2x,0,q^2)=f(0,-1,q^2x,0,q^2)$$
$$=\Pi(1-q^{2\nu}x)=(-q^2;q^2)\{1+\Sigma q^{2\nu}(-x;q^2)_{\nu}(-q^2;q^2)'_{\nu}\}$$
$$=(-q^2;q^2)\{1+\Sigma(-)^{\nu}q^{\nu(\nu+1)}(x^{-1};q^2)_{\nu}(-q^4;q^4)'_{\nu}$$
$$=(-q;q^2)\{1+\Sigma q^{\nu(\nu+1)/2}(-qx)_{\nu}(-q)'_{\nu}\}.$$

By (3.11) we get other equally simple expressions for this function, and since

$$f(0,-1,q^2x,0,q^2)=f(0,-1,qx,0,q)\cdot f(-qx,-1,0,0,q^2)$$

we may, as above, add still other members to (3.15).

By previous results we get

$$(3.16)\quad f(x,0,\delta x,(\lambda-1)(1+\delta)x,q)=1+\Sigma(-x)^{\nu}\lambda(\delta)_{\nu+1}$$
$$=1+\Sigma q^{\nu(\nu-1)/2}\lambda\delta^{\nu-1}(1+\delta)(-x)^{\nu}(x)'_{\nu}.$$

In particular

$$(3.17)\quad f(-qx,0,-qx,0,q)=1+2\Sigma(qx)^{\nu}(q)_{\nu-1}=1+2\Sigma q^{\nu(\nu+1)/2}x^{\nu}(-qx)'_{\nu}.$$

These identities can be given interpretations in terms of the partition functions.

It is easy to verify that $f(-qx,0,-qx,-(x-2)qx,q)$ satisfies

$$(1-qx)g(x)-xg(qx)-1+(1+q)x-x^2=0,$$

which obviously has a solution with the property A expressible in the forms

$$(3.18)\quad g(x)=1+qx^2+q^2(1+q)x^3+q^3(1+q)(1+q^2)x^4+\cdots$$
$$=1+qx^2(-qx)'_2+q^6x^4(-qx)'_4+\cdots,$$

and this by previous results

$$=f(-qx,0,-qx,-(x-2)qx,q)=1+\Sigma q^{\nu(\nu+1)/2}(-qx)'_{\nu}x^{\nu+1}.$$

Similarly we may obtain other identities; e. g. those obtained by solving the equation satisfied by $h(x)=g(x)+x$.

By identities in § 2 we obtain

$$(3.19)\quad f(-qx,-\lambda,0,(1-\lambda)qx,q)=1+\Sigma\lambda q^{\nu}x^{\nu}(-\lambda q)'_{\nu}$$
$$=1+\Sigma(\lambda x)^{\nu}q^{\nu^2}(-\lambda q)'_{\nu}(-qx)'_{\nu}=1+\Sigma\lambda^{\nu}q^{\nu}x(-qx)'_{\nu}.$$

[illegible]

we have the theorem

$$P(a-c,b-1,\not> c)=P(a-b,c-1,\not> b),$$

where $P(0, 0, \not> \gamma) = 1$, otherwise $P(\alpha, \beta, \not> \gamma) = 0$ when α or β is 0. From this we have as corollaries

$$P(a, c, \not> c) = P(a-1, c-1, \not> c+1),$$
$$P(a, b, \not> c) = P(a+c-b-1, c-1, \not> b+1).$$

By (3.19) we have the interesting identities

$$(3.20)\quad f(-qx, -x, 0, (1-x)qx, q) = 1 + \Sigma q^{\nu} x^{\nu+1}(-qx)'_{\nu} = 1 + \Sigma q^{\nu^2} x^{2\nu}[(-qx)'_{\nu}]^2,$$

$$(3.21)\quad f(-qx, x, 0, (1+x)qx, q) = 1 - \Sigma q^{\nu} x^{\nu+1}(qx)'_{\nu} = 1 + \Sigma(-)^{\nu} q^{\nu^2} x^{2\nu}(-q^2x^2; q^2)'_{\nu} = 1 + \Sigma(-q)^{\nu} x^{\nu+1}(-qx)'_{\nu}.$$

When $|x| < 1$ we have

$$(3.22)\quad f(x, \gamma, 0, 0, q) = 1 + \Sigma(-x)^{\nu}(\gamma q)'_{\nu} = 1 + \Sigma(-)^{\nu}\gamma^{\nu-1} q^{\nu-1} x(x)'_{\nu} = 1 - \Sigma\gamma^{\nu-1} q^{\nu(\nu-1)} x^{\nu}(\gamma q)'_{\nu}(x)'_{\nu}.$$

This gives

$$(3.23)\quad f(-qx, -x, 0, 0, q) = 1 + \Sigma(qx)^{\nu}(-qx)'_{\nu} = 1 + \Sigma q^{\nu^2} x^{2\nu-1}[(-qx)'_{\nu}]^2.$$

Again from § 2 we have

$$(3.24)\quad f(\beta x, \gamma, \beta\gamma x, \mu x, q) = 1 + \Sigma\beta^{\nu-1}(\beta + \beta\gamma + \mu)(-x)^{\nu}/(1 + \gamma q^{\nu}) = 1 + \Sigma q^{\nu(\nu-1)/2}(\beta\gamma)^{\nu-1}(\beta + \beta\gamma + \mu)(-q)_{\nu-1}(\gamma q)'_{\nu}(\beta x)'_{\nu}(-x)^{\nu}.$$

From this we get

$$(3.25)\quad f(q^2, -q^{-1}. -q, -q^2 - 3q, q^2) = 1 + \frac{4q}{1-q} - \frac{4q^3}{1-q^3} + \frac{4q^5}{1-q^5} - \cdots$$
$$= 1 + \frac{4q}{(1-q)(1+q^2)} + \frac{4q^4(1-q^2)}{(1-q)(1-q^3)(1+q^2)(1+q^4)} + \cdots.$$

The second member, by the theory of elliptic functions, $= \vartheta_3{}^2$, and by a previous result $= \{f(-q, -1, -q, 0, -q)\}^2$. Similarly we have

$$(3.26)\quad f(q, -q^{-1}, -1, 1 - q - 4q^{1/2}, q^2) = \frac{4q^{1/2}}{1-q} f(q, -q, -q^2, 0, q^2)$$
$$= \frac{4q^{1/2}}{1-q} - \frac{4q^{3/2}}{1-q^3} + \frac{4q^{5/2}}{1-q^5} \cdots = \frac{4q^{1/2}}{1-q^2} + \frac{4q^{5/2}(1-q^2)}{(1-q^2)(1-q^6)} + \frac{4q^{13/2}(1-q^2)(1-q^4)}{(1-q^2)(1-q^6)(1-q^{10})} + \cdots$$
$$= \frac{4q^{1/2}}{1-q} - 4\Sigma q^{\nu^2+\nu-1/2}(-q)_{2\nu-1}(-q^2; q^4)'_{\nu}\,(-q^3; q^2)'_{\nu} = \vartheta_2{}^2 = 4q^{1/2}\{f(-q^2, -1, q^4, 0, q^4)\}^2.$$

Also we have an identity involving the series of Clausen *

$$(3.27)\qquad f(-q,-1,q,-q,q)=1+\Sigma q^{\nu}/(1-q^{\nu})$$
$$=1+\Sigma(-)^{\nu+1}q^{\nu(\nu+1)/2}(-q)'_{\nu}/(1-q^{\nu}).$$

Many other examples of identities involving $f(\beta x,\gamma,\delta x,\mu x,q)$ for particular values of the parameters might be given. Often such identities imply identities of interest having many terms involving only simple series and products; e. g. comparing (3.10), (3.11), and (3.17) we have

$$(3.28)\qquad \{f(q,-1,0,0,q)\}^{-1}=f(0,-1,-q,0,q)=f(-q,0,-q,q,q)$$
$$=f(-q,-1,0,0,q^2)=\{f(0,-1,q,0,q^2)\}^{-1},$$

which implies

$$(3.29)\qquad \{f(q,-1,0,0,q)\}^{-1}=\Pi(1+q^{\nu})=\Pi\,(1-q^{2\nu-1})=1+\Sigma q^{\nu}(-q^2;q^2)'_{\nu}$$
$$=1+\Sigma q^{2\nu^2-\nu}(-q)'_{2\nu}=1+\Sigma q^{2\nu-1}(-q;q^2)'_{\nu}=1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}$$
$$=1+\Sigma q^{\nu(3\nu-1)/2}(1+q^{2\nu})(q)_{\nu-1}(-q)'_{\nu}=1+\Sigma q^{\nu}(q)_{\nu-1}$$
$$=\{1+\Sigma q^{\nu(\nu+1)/2}\}/\{1+\Sigma(-)^{\nu}q^{3\nu^2-\nu}(1+q^{2\nu})\}=\text{etc.}$$

4. *The Homogeneous Linear Second Order Equation.* Consider the equation

$$(4.1)\qquad \phi(x)+(\alpha+\beta x)\phi(qx)+(\gamma+\delta x)\phi(q^2x)=0,$$

where $1+\alpha+\gamma=0$ and $\beta+\delta\neq 0$. A unique power series solution with property A is found to be

$$(4.2)\qquad \phi(x)=1+\Sigma q^{\nu(\nu-1)/2}(\beta,\delta)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)_{\nu},$$

where we have written

$$1+\alpha q^{\nu}+\gamma q^{2\nu}=1-q^{\nu}-\gamma q^{\nu}+\gamma q^{2\nu}=(1-q^{\nu})(1-\gamma q^{\nu}).$$

The series is convergent for all x.

When $\beta\neq 0$ let

$$(4.3)\qquad \phi(x)=\Pi(1-\beta q^{\nu-1}x)\Phi(x).$$

Then $\Phi(x)$ satisfies the q-difference equation

$$(4.4)\qquad \Phi(x)+\frac{\alpha+\beta x}{1-\beta x}\Phi(qx)+\frac{\gamma+\delta x}{(1-\beta x)(1-\beta qx)}\Phi(q^2x)=0.$$

If there is a solution of (4.4) expressible in the form

$$(4.5)\qquad \Phi(x)=1+\Sigma c_{\nu}(-\beta x)'_{\nu}x^{\nu},$$

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* *Journal für reine und angewandte Mathematik*, Vol. 3 (1828), p. 95. He states that the second member above

$$=q\left(\frac{1+q}{1-q}\right)+q^4\left(\frac{1+q^2}{1-q^2}\right)+q^9\left(\frac{1+q^3}{1-q^3}\right)+\cdots=\Sigma\tau(\nu)q^{\nu},$$

where $\tau(\nu)$ is the number of divisors of ν.

we find, on substituting into (4.4) and equating coefficients of the expressions $x^\nu(\beta x)'_\nu$ to zero,

$$c_1 = -(\delta + \beta\gamma q)/(1-q)(1-\gamma q),$$
$$c_\nu = -c_{\nu-1}q^{2\nu-2}(\delta+\beta\gamma q^\nu)/(1-q^\nu)(1-\gamma q^\nu).$$

Hence we have as a solution of (4.4)

$$(4.6)\quad \Phi(x) = 1 + \Sigma q^{\nu(\nu-1)}(\delta, \beta\gamma q)_\nu(-q)'_\nu(-\gamma q)'_\nu(-\beta x)'_\nu(-x)^\nu.$$

When $\beta = 0$ it can be shown that the equation satisfied by $\psi(x)$, where

$$(4.7)\quad \psi(x) = \Pi(1-\delta q^{\nu-1}x)^{-1}\phi(x),$$

does not have a solution of the form

$$(4.8)\quad \psi(x) = 1 + \Sigma c_\nu(-\delta x)'_\nu x^\nu,$$

in which the c's are simply determined, except when $\gamma = 0$. In this case $\psi(x)$ satisfies the equation

$$(4.9)\quad \psi(x) - \frac{1}{1-\delta x}\psi(qx) + \frac{\delta x}{(1-\delta x)(1-\delta qx)}\psi(q^2x) = 0,$$

having the solution

$$(4.10)\quad \psi(x) = 1 + \Sigma(-)^\nu q^{3\nu^2-2\nu}(\delta x)^{2\nu}(-q^2; q^2)'_\nu(-\delta x)'_{2\nu}.$$

When $\beta = \gamma = 0$ let

$$(4.11)\quad \Psi(x) = \Pi(1-q^\nu x)\phi(-qx\delta^{-1}).$$

Then $\Psi(x)$ satisfies the equation

$$(4.12)\quad \Psi(x) - (1-qx)\Psi(qx) - qx(1-qx)(1-q^2x)\Psi(q^2x) = 0.$$

If this equation has a solution expressible as a series with constant term 1, we find after substituting into (4.12), the expression multiplying the constant 1 is

$$1-(1-qx)-qx(1-qx)(1-q^2x) = 0\cdot x + q^2x^2 + q^3x^2(1-qx).$$

This suggests the series

$$\Psi(x) = 1 + \Sigma\{c_{2\nu-1}x^{2\nu-1}(-qx)_{\nu-1} + c_{2\nu}x^{2\nu}(-qx)_{\nu-1} + c'_{2\nu}x^{2\nu}(-qx)_\nu\}.$$

Substituting into (4.12) and computing the successive c's we have

$$c_1 = 0, \qquad c_2 = q^2, \qquad c'_2 = -q^3/(1-q), \cdots$$

and in general

$$(4.13)\quad c'_{2\nu} = c'_{2\nu}q^{2\nu} + c_{2\nu}q^{2\nu} - c'_{2\nu-2}q^{5\nu-2} + c_{2\nu-1}q^{4\nu-1},$$
$$(4.14)\quad c_{2\nu-1} = -c'_{2\nu-2}q^{3\nu-2} + c_{2\nu-1}q^{2\nu-1} + c'_{2\nu-2}q^{4\nu-3} - c_{2\nu-3}q^{5\nu-5} + c_{2\nu-2}q^{4\nu-3},$$
$$(4.15)\quad c_{2\nu} = -c_{2\nu-1}q^{3\nu-1} - c'_{2\nu-2}q^{5\nu-3} - c_{2\nu-2}q^{5\nu-3}.$$

Clearly we may solve (4.13) and (4.15) for $c_{2\nu}$ or $c'_{2\nu}$ in terms of c's with no subscript greater than $2\nu-1$ and hence for $c_{2\nu-2}$ and $c'_{2\nu-2}$ in terms of c's with subscripts not greater than $2\nu-3$. By repeated substitution for the c's with even subscripts in (4.14) we can ultimately show

$$c_{2\nu-1} = f(c_{2\nu-3}, c_{2\nu-5}, \cdots, c_1) + k(q).$$

Since $c_1 = 0$, $k(q) = 0$ and hence $c_3 = c_5 = \cdots = 0$. Then (4.13), (4.14), and (4.15) become

$$\begin{aligned} c'_{2\nu} &= c'_{2\nu}q^{2\nu} + c_{2\nu}q^{2\nu} - c'_{2\nu-2}q^{5\nu-2}, \\ 0 &= -c'_{2\nu-2}q^{3\nu-2} + c'_{2\nu-2}q^{4\nu-3} + c_{2\nu-2}q^{4\nu-3}, \\ c_{2\nu} &= -c'_{2\nu-2}q^{5\nu-3} - c_{2\nu-2}q^{5\nu-3}, \end{aligned}$$

from which we easily obtain

$$c_{2\nu} = (-1)^\nu q^{\nu(5\nu-1)/2}/\{(1-q)(1-q^2)\cdots(1-q^{\nu-1})\},$$

and

$$c'_{2\nu} = (-1)^\nu q^{\nu(5\nu-1)/2}/\{(1-q)\cdots(1-q^\nu)\}.$$

Hence we have * (if $\beta = \gamma = 0$)

(4.16) $\Psi(x) = \Pi(1-q^\nu x)\phi(-qx\delta^{-1})$

$$= 1 + \Sigma(-)^\nu\{q^{\nu(5\nu-1)/2}x^{2\nu}(-qx)_{\nu-1}(-q)'_{\nu-1} + q^{\nu(5\nu-1)/2}x^{2\nu}(-qx)_\nu(-q)'_\nu\}.$$

In particular we have

(4.17) $$\Psi(1) = 1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1+q^\nu) = \Pi(1-q^{5\nu})(1-q^{5\nu-3})(1-q^{5\nu-2}),$$

(4.18) $$\Psi(q) = 1 + \Sigma(-)^\nu q^{\nu(5\nu-3)/2}(1+q^{3\nu}) = \Pi(1-q^{5\nu})(1-q^{5\nu-4})(1-q^{5\nu-1})/(1-q).$$

The last members come from a well known identity due to Jacobi.

We return to (4.4) and seek a solution in the form

(4.19) $$\Phi(x) = 1 + \Sigma c_\nu(-\beta x)'_\nu,$$

where we make $|\delta/\beta| < |q|$ to insure convergence of series obtained after substituting in (4.4). Equating coefficients of $(-\beta x)'_\nu$ for $\nu = 1, 2, 3, \cdots$, we have finally the solution

(4.20) $$\Phi(x) = 1 + \Sigma(-\beta^{-1})^\nu(\delta, \beta\gamma q)_\nu(-q)'_\nu(-\beta x)'_\nu.$$

By (3.1)

$$\Phi(0) = 1 + \Sigma(-\beta^{-1})^\nu(\delta, \beta\gamma q)_\nu(-q)'_\nu = \Pi[1-\gamma q^\nu)/(1+\delta\beta^{-1}q^{\nu-1})].$$

* ... identity verified by Ramanujan (*Collected Papers* No. 26), and of which many ... have been given

Hence a solution of (4.4) with property A is

$$(4.21)\quad \Phi(x)=(\delta\beta^{-1})(-\gamma q)'\{1+\Sigma(-\beta^{-1})^{\nu}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\beta x)'_{\nu}\}$$

where $|\delta/\beta|<|q|$. But by analytic continuation we may extend the ran of validity of the series and require only that $|\delta/\beta|<1$, which is necessa for the convergence of the series.

If we write

$$\phi(q\delta x^2, q^2)=\Pi(1+\delta q^{\nu-1}x)\Phi_1(x),$$

and solve the equation satisfied by $\Phi_1(x)$, we obtain the solution

$$(4.22)\quad \Phi_1(x)=1+\Sigma(-\delta x)^{\nu}q^{\nu(\nu-1)/2}(-\gamma q;q^2)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(\delta x)'_{\nu}$$

provided $\beta=0$.

On writing $\phi(\beta x,\gamma,\delta x,q)$ for $\phi(x)$ we may summarize the main resul of this section in the following identities:

$$\begin{aligned}(4.23)\quad \phi(\beta x,\gamma,\delta x,q)&=1+\Sigma q^{\nu(\nu-1)/2}(\beta,\delta)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)^{\nu}\\ &=(-\beta x)\{1+\Sigma q^{\nu(\nu-1)}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(-\beta x)'_{\nu}(-x)\\ &=(-\beta x)(\delta\beta^{-1})(-\gamma q)'\{1+\Sigma(-\beta)^{\nu}(\delta,\beta\gamma q)_{\nu}(-q)'_{\nu}(-\beta x\end{aligned}$$

where the last member is present only when $|\delta/\beta|<1$,

$$\begin{aligned}(4.24)\quad &\phi(0,\gamma,q\delta^2x^2,q^2)\\ &\quad=(\delta x)\{1+\Sigma q^{\nu(\nu-1)/2}(-\delta x)^{\nu}(-\gamma q;q^2)_{\nu}(-q)'_{\nu}(-\gamma q)'_{\nu}(\delta x)'_{\nu}\\ (4.25)\quad &\phi(0,0,\delta x,q)\\ &\quad=(-\delta x)\{1+\Sigma q^{3\nu^2-2\nu}(-\delta^2x^2)^{\nu}(-q^2;q^2)'_{\nu}(-\delta x)'_{2\nu-1}\},\\ (4.26)\quad &\phi(0,0,-qx,q)\\ &\quad=(-qx)'\{1+\Sigma(-)^{\nu}q^{\nu(5\nu-1)/2}x^{2\nu}(1-q^{2\nu}x)(-qx)_{\nu-1}(-q)'_{\nu}\}\end{aligned}$$

5. *Applications.* From (4.23) we have

$$\begin{aligned}(5.1)\quad \phi(qx,\gamma,0,q)&=1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}(-\gamma q)'_{\nu}(-x)^{\nu}\\ &=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q)'_{\nu}(-\gamma q)'_{\nu}(-qx)'_{\nu}(-\gamma x)^{\nu}\}\\ &=(-qx)(-q\gamma)'\{1+\Sigma q^{\nu(\nu+1)/2}(-q)'_{\nu}(-qx)'_{\nu}(-\gamma)^{\nu}\},\end{aligned}$$

from which we easily obtain

$$\begin{aligned}(5.2)\quad &f(0,-1,-q^2x,0,q^2)\phi(-qx,-x,0,q^2)\\ &\quad=(q^2x;q^2)\{1+\Sigma q^{\nu^2}(-q^2;q^2)'_{\nu}(q^2x;q^2)'_{\nu}x^{\nu}\}\\ &\quad=(qx)\{1+\Sigma(-)^{\nu}q^{3\nu^2}(-q^2;q^2)'_{\nu}(qx)'_{2\nu}x^{2\nu}\}\\ &\quad=(qx;q^2)\{1+\Sigma q^{\nu^2+\nu}(-q^2;q^2)'_{\nu}(qx;q^2)'_{\nu}x^{\nu}\},\end{aligned}$$

$$\begin{aligned}(5.3)\quad \phi(qx,1,0,q)&=1+\Sigma q^{\nu(\nu+1)/2}(-x)^{\nu}[(-q)'_{\nu}]^2\\ &=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}[(-q)'_{\nu}]^2(-qx)'_{\nu}(-x)^{\nu}\}\\ &=(-qx)(-q)'\{1+\Sigma(-)^{\nu}q^{\nu(\nu+1)/2}(-q)'_{\nu}(-qx)'_{\nu}\}.\end{aligned}$$

Comparing (5.1) and (3.17) we have $\phi(-q, 0, 0, q) = f(-q, 0, -q, q, q)$.

By (4.23), (4.25), (4.26), and (5.2) we have *

$$
\begin{aligned}
(5.4) \quad \phi(0,0,-qx,q) &= 1 + \Sigma q^{\nu^2}(-q)'_\nu x^\nu \\
&= (qx)\{1 + \Sigma(-)^\nu q^{3\nu^2}(-q^2;q^2)'_\nu (qx)'_{2\nu} x^{2\nu}\} \\
&= (q^2x;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu (q^2x;q^2)'_\nu x^\nu\} \\
&= (qx;q^2)\{1 + \Sigma q^{\nu^2+\nu}(-q^2;q^2)'_\nu (qx;q^2)'_\nu x^\nu\} \\
&= (-qx)'\{1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1-q^{2\nu}x)(-qx)_{\nu-1}(-q)'_\nu x^{2\nu}\}.
\end{aligned}
$$

Again by (4.24), (5.1), and by symmetry in x we have

$$
\begin{aligned}
(5.5) \quad \phi(0,0,q^3x^2,q^2) &= (qx)\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu (qx)'_\nu (-x)^\nu\} \\
&= (-q^2x^2;q^2)\{1 + \Sigma q^{\nu(3\nu+1)/2}(-q)'_\nu (-q^2x^2;q^2)'_\nu x^{2\nu}\} \\
&= (-qx)\{1 + \Sigma q^{\nu(\nu+1)/2}(-q)'_\nu (-qx)'_\nu x^\nu\}.
\end{aligned}
$$

Similarly we find

$$
\begin{aligned}
(5.6) \quad \phi(0,0,x^2,q^4) &= (-qx;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu (-qx;q^2)'_\nu x^\nu\} \\
&= (-q^2x^2;q^4)\{1 + \Sigma q^{3\nu^2-\nu}(-q^2;q^2)'_\nu (-q^2x^2;q^4)'_\nu x^{2\nu}\} \\
&= (qx;q^2)\{1 + \Sigma q^{\nu^2}(-q^2;q^2)'_\nu (qx;q^2)'_\nu (-x)^\nu\}.
\end{aligned}
$$

By (5.4) and (4.17) we get

$$
\begin{aligned}
(5.7) \quad \phi(0,0,-q,q) &= 1 + \Sigma q^{\nu^2}(-q)'_\nu = (q)\{1 + \Sigma(-)^\nu q^{3\nu^2}(-q^2;q^2)'_\nu (q)'_{2\nu}\} \\
&= (q^2;q^2)\{1 + \Sigma q^{\nu^2}(-q^4;q^4)'_\nu\} = (q;q^2)\{1 + \Sigma q^{\nu^2+\nu}(-q^2;q^2)'_\nu (q;q^2)' \\
&= (-q)'\{1 + \Sigma(-)^\nu q^{\nu(5\nu-1)/2}(1+q^\nu)\} = 1/\Pi\,(1-q^{5\nu-4})(1-q^{5\nu-1}).
\end{aligned}
$$

Similarly we find

$$
\begin{aligned}
(5.8) \quad \phi(0,0,-q^4,q^4)/\Pi(1+q^{4\nu+2}) &= 1 + \Sigma(-)^\nu q^{3\nu^2+3\nu}(-q^2;q^2)'_\nu (q^6;q^4) \\
&= 1/\Pi(1-q^{20\nu-16})(1-q^{20\nu-4})(1+q^{4\nu+2}).
\end{aligned}
$$

By (5.6) we have two other expressions for the same function. Again by (5.5) and (4.17) we have

$$
\begin{aligned}
(5.9) \quad \phi(0,0,-q^2,q^2)/\Pi(1+q^{2\nu-1}) &= 1 + \Sigma(-)^\nu q^{\nu(3\nu-1)/2}(-q)'_\nu (q;q^2)'_\nu \\
&= 1/\Pi(1-q^{10\nu-8})(1-q^{10\nu-2})(1+q^{2\nu-1}),
\end{aligned}
$$

and by (5.5) two other members may be added. In all we can obtain fourteen different representations of $\phi(0,0,-q,q)$.

We could also write a similar set of identities for the function $\phi(0,0,-q^2x,q)$, which for $x=1$ is the function appearing in Ramanujan's second theorem.† For their beauty in themselves we state the following:

* Cf. L. J. Rogers, *Proceedings of the London Mathematical Society*, Vol. 25 (1894), p. 330.

† This theorem is stated by MacMahon, *loc. cit.*, p. 30.

$$(5.10)\quad \phi(0,0,q^3,q^2)=(q)\{1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}(-q^2;q^2)'_\nu\}$$
$$=(-q)\{1+\Sigma q^{\nu(\nu+1)/2}[(-q)'_\nu]^2\}$$
$$=(-q^2;q^2)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q)'_\nu(-q^2;q^2)'_\nu\},$$
$$(5.11)\quad \phi(0,0,1,q^4)=(-q;q^2)\{1+\Sigma q^{\nu^2}(-q)'_{2\nu}\}$$
$$=(q;q^2)\{1+\Sigma(-)^\nu q^{\nu^2}(-q^2;q^2)'_\nu(q;q^2)'_\nu\}$$
$$=(-q^2;q^4)\{1+\Sigma q^{3\nu^2-\nu}(-q^2;q^2)'_\nu(-q^2;q^4)'_\nu\}.$$

By (4.23) we get

$$(5.12)\quad \phi(\beta x,0,\delta x,q)=1+\Sigma q^{\nu(\nu-1)/2}(\beta,\delta)_\nu(-q)'_\nu(-x)^\nu$$
$$=(-\beta x)\{1+\Sigma q^{\nu^2-\nu}(-q)'_\nu(-\beta x)'_\nu(-\delta x)^\nu\}$$
$$=(-\beta x)(\delta\beta^{-1})\{1+\Sigma(-\delta/\beta)^\nu(-q)'_\nu(-\beta x)'_\nu\},$$
$$(5.13)\quad \phi(q,0,-q^2,q)=1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}=(-q)\{1+\Sigma q^{\nu^2+\nu}[(-q)'_\nu]^2\},$$
$$(5.14)\quad \phi(-q,0,q^2,q)=1+\Sigma q^{\nu(\nu+1)/2}=(q)\{1+\Sigma(-)^\nu q^{\nu^2+\nu}(-q^2;q^2)'_\nu\},$$

and this by (3.14) $=f(-q,-1,q^2,0,q^2)$. Thus we have a new expression for this function. From (5.12), (5.13), and (5.14) we get

$$(5.15)\quad \begin{aligned}\phi(q,0,-q^2,q)+\phi(-q,0,q^2,q)&=2\phi(-q^3,0,q^7,q^4),\\ \phi(-q,0,q^2,q)-\phi(q,0,-q^2,q)&=2q\phi(-q^5,0,q^9,q^4).\end{aligned}$$

Again we have by (5.12)

$$(5.16)\quad \phi(q,0,q^2,q)=1+\Sigma(-)^\nu q^{\nu(\nu+1)/2}(q)_\nu(-q)'_\nu$$
$$=(-q)\{1+\Sigma(-)^\nu q^{\nu^2+\nu}[(-q)'_\nu]^2\},$$
$$(5.17)\quad \phi(-q,0,q^3,q)=1+\Sigma q^{\nu(\nu+1)/2}(1-q^{\nu+1})/(1-q)=1/(1-q).$$

By (4.23)

$$(5.18)\quad \phi(qx,0,qx,q)=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu(-x)^\nu$$
$$=(-qx)\{1+\Sigma q^{\nu^2}(-q)'_\nu(-qx)'_\nu(-x)^\nu\},$$
$$(5.19)\quad \phi(-q,0,-q,q)=1+2\Sigma q^{\nu(\nu+1)/2}(q)_{\nu-1}(-q)'_\nu$$
$$=(q)\{1+\Sigma q^{\nu^2}(-q^2;q^2)'_\nu\},$$

which by (3.11)

$$=(q)f(0,-1,-q,0,q^2)=f(-q,-1,-q,0,q^2).$$

Note the two expressions for the latter as given by (3.2). Making use of expressions for $f(0,-1,-q,0,q^2)$ as given in § 3, we obtain several other members for (5.19). From (5.1) we have

$$(5.20)\quad \phi(qx,-1,0,q)=1+\Sigma q^{\nu(\nu+1)/2}(-q^2;q^2)'_\nu(-x)^\nu$$
$$=(-qx)\{1+\Sigma q^{\nu(3\nu+1)/2}(-q^2;q^2)'_\nu(-qx)'_\nu x^\nu\}$$
$$=(-qx)(q)'\{1+\Sigma q^{\nu(\nu+1)/2}(-q)'_\nu(-qx)'_\nu\}.$$

By comparison with (5.7), (3.1), and (5.3) we easily see that

$$(5.21)\quad \phi(-q,-1,0,q^2)=\phi(0,0,-q,q)/\Pi(1+q^{2\nu}) = 1/\Pi\,(1+q^{2\nu})(1-q^{5\nu-4})(1-q^{5\nu-1}),$$

$$(5.22)\quad \phi(q,-1,0,q)=f(q,-1,q,0,q)\phi(-q,1,0,q).$$

From (5. 20) we may show

$$(5.23)\quad \phi(-q,-1,0,q)+\phi(q,-1,0,q) = 2(q^3;q^4)(-q^2;q^4)'\phi(q^2,-q^{-1},0,q^4)=2\phi(-q^3,q^{-2},0,q^4),$$

and by comparing with (5. 1)

$$(5.24)\quad \phi(-q,-1,0,q)-\phi(q,-1,0,q) = 2q(q;q^4)(-q^2;q^4)'\phi(q^6,-q^{-3},0,q^4) = [2q/(1-q^2)]\phi(-q,q^2,0,q^4).$$

We point out two other identities of interest,

$$(5.25)\quad \phi(0,1,q^3x,q)=1+\Sigma q^{\nu(\nu+5)/2}[(-q)'_\nu]^2(-x)^\nu = (qx)\{1+\Sigma q^{\nu(\nu+1)/2}(-q;q^2)_\nu[(-q)'_\nu]^2(qx;q^2)'_\nu\}$$

$$(5.26)\quad \phi(x,\gamma,-qx,q)=1+\Sigma q^{\nu(\nu-1)/2}(-\gamma q)'_\nu(-x)^\nu = (-x)\{1+\Sigma q^{\nu^2}x^\nu(-q)'_\nu(-x)'_\nu(1-\gamma)/(1-\gamma q^\nu)\}.$$

By comparison with § 2 we have

$$(5.27)\quad \phi(-qx,\gamma,q^2x,q)=f(0,-\gamma,-qx,0,q),$$

and by comparison with (5. 1) we have

$$(5.28)\quad \phi(x,1,-qx,q)=(-x)\phi(0,q^{-1}x,0,q)=\phi(x,0,0,q).$$

Again comparing with § 2 (5. 1), and (3. 10),

$$(5.29)\quad \phi(x,-1,-qx,q)=f(0,1,x,0,q),$$

$$(5.30)\quad \phi(-1,-1,q,q)=2(q)\phi(q,0,0,q^2)=2\phi(-q,0,q,q) = 2(q)f(q,-1,0,0,q)=f(0,1,-1,0,q).$$

Another way of obtaining identities from those already found is illustrated by the following. On equating coefficients of γ^k in the last two members of (5. 26) and using (3. 8) we get

$$(5.31)\quad \Sigma q^{[\nu(\nu-1)/2]+k}(-q^\nu)_k(-q)'_k(-x)^\nu = -(-x)\{\Sigma q^{\nu^2+(k-1)\nu}(-q)'_{\nu-1}(-x)'_\nu x^\nu\}.$$

Replacing x by -1, dividing by q^k, and reducing, this becomes

$$(5.32)\quad 1+\Sigma q^{\nu(\nu+1)/2}(-q^{\nu+1})_k(-q)'_k=(q)\{1+\Sigma(-)^\nu q^{\nu^2+(k+1)\nu}(-q^2;q^2)_\nu\} = (q)\phi(q^{k+2},0,0,q^2)=\phi(-q,0,q^{k+2},q),$$

[illegible] values being obtained by comparison with (5. 1) and (5. 19)

Then we may write

$$(5.33)\quad \phi(-1,\gamma,q,q)=\phi(-1,0,q,q)+\Sigma_k\phi(-q,0,q^{k+2},q)q^{k^2}\gamma^k.$$

But by (5.26) and (5.12) we have

$$\phi(-1,0,q,q)=(q)\{1+\Sigma(-)^\nu q^{\nu^2}(-q^2;q^2)'_\nu+1+\Sigma(-)^\nu q^{\nu^2+\nu}(-q^2;q^2)'_\nu\}$$
$$=(q)\{(-q;q^2)+(-q^2;q^2)\}=1+(q)(-q^2;q^2)=1+(-q^2;q^2)(-q;q^2)'.$$

By (5.12) and (3.1)

$$\phi(-q,0,q^{k+2},q)=(-q)'_k f(-q^{k+1},-1,q^2,0,q^2)$$
$$=(-q^2;q^2)(-q^{k+1};q^2)'(-q)'_k.$$

Then (5.33) becomes

$$(5.34)\quad \begin{aligned}\phi(-1,\gamma,q,q)&=1+(-q^2;q^2)(-q;q^2)'\\&\quad+\Sigma_k(-q^2;q^2)(-q^{k+1};q^2)(-q)'_k(\gamma q)^k\\&=1+(-q^2;q^2)(-q;q^2)'\\&\quad\times\{1+\Sigma_k(-q^2;q^2)_k(\gamma q)^{2k}\}+\Sigma_k(-q;q^2)_k(\gamma q)^{2k-1}\\&=f(-q,-\gamma^2,0,q-\gamma q,q^2)\\&\quad+(-q^2;q^2)(-q;q^2)'\{1+\Sigma_k(-q^2;q^2)_k(\gamma q)^{2k}\}.\end{aligned}$$

Then by (3.10)

$$(5.35)\quad \begin{aligned}\phi(-1,\gamma,q,q)-f(-q,-\gamma^2,0,q-\gamma q,q^2)\\=(-q^2;q^2)(-q;q^2)'(-q^2\gamma^2;q^2)',\end{aligned}$$

which obviously implies an identity of several members.

By (4.23) we get

$$(5.36)\quad \begin{aligned}\phi(x,\gamma,-q^2x,q)&=1+\Sigma q^{\nu(\nu-1)/2}(-\gamma q)'_\nu(-x)^\nu(1-q^{\nu+1})/(1-q)\\&=(-x)\{1+\Sigma q^{\nu^2+\nu}(-q^{-1}\gamma)_\nu(-\gamma q)'_\nu(-x)'_\nu x^\nu\}\\&=(-x)(-q^2)(-\gamma q)'\{1+\Sigma q^{2\nu}(-\gamma q^{-1})_\nu(-q)'_\nu(-x)'_\nu\},\end{aligned}$$

which with (5.26) gives the identity

$$(5.37)\qquad \phi(q^3,q^2,-q^5,q)=(-q^3)\phi(q^5,q^4,-q^7,q^2);$$

and

$$(5.38)\quad \begin{aligned}\phi(qx,\gamma,q^2x,q)&=1+\Sigma q^{\nu(\nu+1)/2}(q)_\nu(-q)'_\nu(-\gamma q)'_\nu(-x)^\nu\\&=(-qx)\{1+\Sigma q^{\nu^2+\nu}(\gamma)_\nu(-q)'_\nu(-\gamma q)'_\nu(-qx)'_\nu(-x)^\nu\\&=(qx)(q)(-q\gamma)'\{1+\Sigma(-qx)^\nu(\gamma)_\nu(-q)'_\nu(-qx)'_\nu\}\end{aligned}$$

which with previous results gives

$$(5.39)\qquad \phi(x,-x,qx,q)=f(qx,-1,x,0,q).$$

Other relations of this kind might be found. From such relations we could evidently state identities of many members as we have already done for a number of identities involving known functions.

UNIVERSITY OF ILLINOIS.

SUMMATION FACTORS WHICH ARE POWERS OF A COMPLEX VARIABLE.

By Walter H. Durfee.

1. *Introduction.* By a well known theorem of Frobenius,* if the power series $y(z) = \sum_{n=0}^{\infty} a_n z^n$ has the unit circle as circle of convergence, and if $\sum_{n=0}^{\infty} a_n$ is summable by Cesàro's k-th mean to the value s, then $\lim y(z) = s$ as $z \to +1$ along any path lying wholly between two fixed chords of the unit circle intersecting at $z = +1$.

Frobenius stated the theorem for the case, $k = 1$. Extensions have been made by various writers, notably Hölder † and Fejér.‡ Hurwitz,§ also, has shown that if $\sum_{n=0}^{\infty} a_n$ is summable C_k to the value s, and if $g_n(z)$ is a function of z such that, 1) $\lim_{z \to 1} g_n(z) = 1$, 2) for each $|z| < 1$, $n^k g_n(z)$ is bounded for all n,¶ and 3) $\sum_{n=0}^{\infty} n^k |\Delta^{k+1} g_n(z)|$ converges for each $|z| < 1$ and is bounded for all $|z| < 1$; then $\sum_{n=0}^{\infty} a_n g_n(z)$ converges for $|z| < 1$, and $\lim_{z \to 1} \sum_{n=0}^{\infty} a_n g_n(z) = s$.

The theorems of this paper deal with functions which satisfy conditions equivalent to these, the series under discussion being of the form $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$, where a_n is real and z is a complex variable. The inquiry is directed toward determining the conditions on $f(n)$ under which the series approaches a limit as $z \to +1$.

For the purposes of this paper the functions $f(n)$ are restricted to the class of logarithmico-exponential functions,‖ so expressed that $f(0) = 0$; for brevity these will be designated as L-functions. Certain properties of these

* *Journal für die reine und angewandte Mathematik*, Vol. 89 (1880), p. 262.

† *Mathematische Annalen*, Vol. 20 (1882), p. 535.

‡ *Mathematische Annalen*, Vol. 58 (1904), p. 62.

§ Cf. Morse, "Certain Definitions of Summability," *American Journal of Mathematics*, Vol. 45 (1923), p. 263.

¶ The bound may depend upon z.

‖ Hardy, *Orders of Infinity*, III, 2.

functions, which will be useful in the subsequent pages, are: if $f(x)$ is an L-function, then $f(x)$ and all its derivatives are ultimately continuous, non-vanishing, monotonic, and of constant sign; as $x \to +\infty$, $f(x)$ approaches either infinity, or zero, or some other definite limit; $f'(x)$ is also an L-function, and $f''(x) = o[f'(x)]$ entails $f'(x) = o[f(x)]$.

Fejér, in his paper on Fourier Series,* proved that Frobenius' theorem is valid for series of the form $y(z) = \sum_{n=0}^{\infty} a_n z^{n^p}$, where p may be any integer greater than zero. It has been shown, however, by Hardy, in a paper "On Certain Oscillating Series," † that if $f(n)$ has too rapid a rate of increase the series $\sum_{n=0}^{\infty} a_n z^{f(n)}$ may not approach any limit as $z \to +1$. A discussion of certain series of this type will be found on a later page.

2. *Statement of Theorems.* The discussion will be facilitated by the definition of a region within the unit circle, which we shall call R, to which the variable z is confined. As in Frobenius' theorem, let there be two chords intersecting at the point $z = +1$. They may, without loss of generality, be considered as making equal angles with the real axis. We now define R as the open region, in a neighborhood of $z = +1$, bounded by these chords and interior to them.

THEOREM I. *If the series* $\sum_{n=0}^{\infty} a_n$ *is summable* C_k *to the value* s, *and if* a) $f(x)$ *is an* L-*function, and* $f(0) = 0$, b) $\log x = o[f(x)]$, c) $f(x) > 0$, $f'(x) > 0$, and d) $f''(x)/f'(x) = O[1/x]$, *then the series* $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ *converges for* $|z| < 1$, *and* $\lim y(z) = s$ *as* $z \to 1$ *along any path lying wholly in* R.‡

THEOREM II. *If* $\sum_{n=0}^{\infty} a_n$ *is summable* C_1 *to the value* s, *and if* a) $f(x)$ *is an* L-*function, and* $f(0) = 0$, b) $f(x) = O[\log x]$, c) $f(x) \to +\infty$, $f'(x) > 0$, and d) $a_n = O[\lambda^{f(n)} f'(n)]$, or $s_n = O[\lambda^{f(n)}]$, *for every* $\lambda > 1$, *then the series* $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ *converges for* $|z| < 1$, *and* $\lim y(z) = s$ *as* $z \to 1$ *along any path lying wholly in* R.

3. *Proof of Lemmas.* It will be convenient at this point to establish

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* *Mathematische Annalen*, Vol. 58 (1904), p. 62.

† *Quarterly Journal of Mathematics*, Vol. 38 (1906-7), p. 269.

‡ A somewhat similar theorem, due to H. L. Garabedian, has recently been published. See *Annals of Mathematics*, Vol. 32 (1931), p. 83.

the truth of a number of lemmas which will be of use in the proof of the above theorems.

LEMMA 1. *As* $z \to 1$ *in* R, $|\log z| = O(\log \rho)$, *where* $\rho = |z|$.

Let θ be the amplitude of z, $-\pi/2 < \theta < \pi/2$. Then $|\log z| \leqq |\log \rho| + |\theta|$. As $z \to 1$, $\theta \sim \sin\theta$, and since $\log \rho = \log[1-(1-\rho)] \sim -(1-\rho)$, $|\log \rho| \sim 1-\rho$. Now let $|z-1| = r$, and call the acute angle between $(z-1)$ and the real axis β. Then $\sin\theta = r/\rho \sin\beta$; and $\rho^2 = r^2 + 1 - 2r\cos\beta$. Noting that as $z \to 1$, $(1+\rho)/\rho \to 2$, and $2\cos\beta - r \to 2\cos\beta$, we have

$$\frac{|\theta|}{|\log \rho|} \sim \frac{\sin\theta}{1-\rho} = \frac{(1+\rho)\sin\theta}{1-\rho^2}$$

$$= \frac{[(1+\rho)r/\rho]\sin\beta}{r(2\cos\beta - r)} = \frac{(1+\rho)\sin\beta}{\rho(2\cos\beta - r)} \sim \frac{2\sin\beta}{2\cos\beta} = \tan\beta.$$

Since z lies in R, $\tan\beta$ is less than some constant, A, determined by the chords bounding R, whence

$$|\log z| \leqq |\log \rho| + A|\log \rho| = O(\log \rho).$$

LEMMA 2. *If $f(x)$ is an L-function, and if $f(x)$ and its first k derivatives exist and are continuous, non-vanishing, and of constant sign for $x \geqq a$, then for such values of x they either all bear the same sign, or all derivatives up to a certain one bear the same sign and thereafter they alternate.*

To prove this, suppose that for $x \geqq a$, $g(x) \equiv f^{(i)}(x)$ and $g'(x)$ are of opposite sign. By Taylor's Theorem, for $x \geqq a$,

$$g(x) = g(a) + (x-a)g'(a) + [(x-a)^2/2]\, g''\{a + \theta(x-a)\} \qquad 0 < \theta < 1.$$

Then, for $x > a$ we may write,

$$g''\{a + \theta(x-a)\} = [2/(x-a)^2]\{g(x) - (x-a)g'(a) - g(a)\}.$$

If $g(a) > 0$, and $g'(a) < 0$, then by hypothesis $g(x) > 0$ for $x > a$, and the first two terms in the braces on the right-hand side of the above expression are, signs considered, positive; while the last, which is negative, is a constant. The second term increases indefinitely with x; hence for sufficiently great values of x, $g''\{a + \theta(x-a)\} > 0$. But for $x > a$, $a + \theta(x-a) > a$, that is, there are values of $x > a$ for which $g''(x) > 0$. Since by hypothesis $g''(x)$ is of constant sign for $x \geqq a$, it follows that for all $x > a$, [illegible]

If on the other hand, $g(a) < 0$, and $g'(a) > 0$, then for $x > a$, $g(x) < 0$.

and for sufficiently great values of x, $g''\{a+\theta(x-a)\}<0$. It follows that there are values of $x>a$ for which $g''(x)<0$, and hence that $g''(x)<0$ for all $x \geqq a$.

Thus if $g(x)$ and $g'(x)$ are of opposite sign for $x \geqq a$, then $g(x)$ and $g''(x)$ are of like sign for $x \geqq a$; which proves the lemma.

LEMMA 3. *If $f''(x)/f'(x)=O(1/x)$, then for any $k \geqq 1$, $f^{(k)}(x)/f'(x)=O(1/x^{k-1})$.*

The statement is obviously true for $k=1$. Suppose it true for some $k>1$. Differentiating, we have

$$\frac{f'f^{(k+1)}-f^{(k)}f''}{(f')^2}=\frac{f^{(k+1)}}{f'}-\frac{f^{(k)}}{f'}\cdot\frac{f''}{f'}$$
$$=\frac{f^{(k+1)}}{f'}-O\left(\frac{1}{x^k}\right)=O\left[\frac{d}{dx}\left(\frac{1}{x^{k-1}}\right)\right]=O\left[\frac{1}{x^k}\right],$$

or $f^{(k+1)}/f'=O[1/x^k]$. Thus, if the statement is true for $k=2$, as by hypothesis it is, then it is true for all $k \geqq 2$, and the proof is complete.

LEMMA 4. *If $f(x)=O(\log x)$, and if $f(x)\rightarrow+\infty$ and $f'(x)$ is positive, then $f''(x)$ is negative for sufficiently great values of x; and further, $f''(x)/f'(x)=O(1/x)$.*

If $f(x)=O(\log x)$, then $f'(x)=O(1/x)$*; that is, for sufficiently large values of x, $|f'(x)|<M/x$, where M is some positive constant. Hence $f'(x)\rightarrow 0$ as $x\rightarrow+\infty$. Since $f(x)$ is an L-function, so is $f'(x)$; and since $f'(x)\rightarrow 0$, ultimately monotonically, $f''(x)$ must be ultimately negative. The truth of the first statement in the lemma is thus established. The function $xf''(x)/f'(x)$ is an L-function, and therefore approaches either zero, or $-\mu$, or $-\infty$, where μ is some positive constant. Suppose that, for $x \geqq X$, $xf''(x)/f'(x)<-\alpha$, where $\alpha>1$. Then $f''(x)/f'(x)+\alpha/x<0$. Integrating from X to x, we have

$$\log f'(x)+\alpha\log x-\log f'(X)-\alpha\log X<0, \quad \text{or} \quad x^\alpha f'(x)<q(X),$$

where
$$q(X)=X^\alpha f'(X).$$

Putting this in the form $f'(x)-q(X)/x^\alpha<0$, and integrating once more from X to x,

$$f(x)-q(X)/(1-\alpha)x^{\alpha-1}-f(X)+q(X)/(1-\alpha)X^{\alpha-1}<0,$$
or
$$f(x)<f(X)+q(X)/(\alpha-1)X^{\alpha-1}-q(X)/(\alpha-1)x^{\alpha-1}.$$

* Hardy, *Orders of Infinity.*

Then we have $\limsup_{x\to+\infty} f(x) \leqq f(X) + Xf'(X)/(\alpha - 1)$, which contradicts the hypothesis $f(x) \to +\infty$. It follows that $\lim_{x\to+\infty} xf''(x)/f'(x) = -\mu$ exists, and $-1 \leqq -\mu \leqq 0$, whence also $f''(x)/f'(x) = O(1/x)$.

LEMMA 5. *The terms in the k-th derivative, with respect to n, of $z^{f(n)}$ can be set in one-to-one correspondence with the partitions of k, the components of any partition being the orders of the derivatives of $f(n)$ in the corresponding term, so that the number of occurrences of a component in a partition indicates the exponent of that derivative. The number of components of a partition is the same as the exponent of* $\log z$ *in the corresponding term. The numerical coefficients of the terms are all positive.*

If we set $\phi(n) = z^{f(n)}$, the first three derivatives are

$$\phi'(n) \;= z^{f(n)}f'(n)\log z,$$
$$\phi''(n) \;= z^{f(n)}\{(f'(n)\log z)^2 + f''(n)\log z\},$$
$$\phi'''(n) = z^{f(n)}\{(f'(n)\log z)^3 + 3f'(n)f''(n)(\log z)^2 + f'''(n)\log z\}.$$

Inspection shows that the statements in the lemma are valid for $k \leqq 3$. Suppose now that they are true for some fixed k. Then each term of $\phi^{(k)}$ is of the form $T_\phi = Az^f(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}(\log z)^h$, where $h = \sum_{i=1}^{k}\alpha_i$, any, or all but one, of the α_i may be zero, and $k = \sum_{i=1}^{k} i\alpha_i$.

From any partition of k partitions of $(k+1)$ may be formed, either by adding a new component, 1, or by increasing any single component by unity; and if each partition of k be so treated in every possible way all the partitions of $(k+1)$ will be obtained.

Now, $\phi^{(k+1)} = d\phi^{(k)}/dn$, and by differentiating T_ϕ we obtain

$$T'_\phi = dT_\phi/dn = Az^f\{(f')^{\alpha_1+1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}(\log z)^{h+1} + (f')^{\alpha_1}\cdots(f^{(k)})^{\alpha_k}\sum_{i=1}^{k}\alpha_i[f^{(i+1)}/f^{(i)}](\log z)^h\}.$$

These terms correspond to the partitions of $(k+1)$ obtainable from that partition of k which corresponds to T_ϕ. Further, in each term of T'_ϕ the sum of the exponents is equal to the exponent of $\log z$, and $\sum i\alpha_i = k+1$. Finally, the numerical coefficients are either A or $A\alpha_i$, and in either case they bear the same sign as A.

Thus if the statements of the lemma are true for some particular k, by induction they are also true for $(k+1)$. Being verified for $k = 3$, they hold for every positive integral value of k.

4. *Proof of Theorems.* We proceed now to the consideration of the first Theorem. In showing that, under the conditions of this theorem $y(z)$ converges within the unit circle it will be advantageous to go further and prove that for $|z| < 1$, $\sum_{n=0}^{\infty} | a_n z^{f(n)} |$ is convergent. We note that since $\sum_{n=0}^{\infty} a_n$ is C_k-summable, $| a_n | < Gn^k$, where G is a positive constant; it will therefore suffice to prove the convergence of $\sum_{n=0}^{\infty} n^k \rho^{f(n)}$, $0 \leqq \rho < 1$, $\log n = o[f(n)]$. Since $f(n)/\log n \to +\infty$, for a fixed ρ, and $n > N$, we have $[f(n)/\log n] \log \rho < -k-2$. That is, $f(n) \log \rho < -(k+2) \log n$, or $\rho^{f(n)} < 1/n^{k+2}$. Then $n^k \rho^{f(n)} < 1/n^2$ for $n > N$. The convergence of $\sum_{n=0}^{\infty} n^k \rho^{f(n)}$ thus follows from that of $\sum_{n=0}^{\infty} 1/n^2$, and the absolute convergence of $y(z)$ within the unit circle is thus established.

The series $\sum_{n=0}^{\infty} a_n$ being summable C_k, we have $\lim_{n\to\infty} C_n^{(k)} = s$, where

$$C_n^{(k)} = S_n^{(k)} / \binom{n+k}{k} = [S_0^{(k-1)} + S_1^{(k-1)} + \cdots + S_n^{(k-1)}] / \binom{n+k}{k},$$

and $S_n^{(0)} = s_n = a_0 + a_1 + \cdots + a_n$. Thus, if we define $C_{-r}^{(k)} = 0$ for $r > 0$, we have

$$S_n^{(k-1)} = S_n^{(k)} - S_{n-1}^{(k)} = \binom{n+k}{k} C_n^{(k)} - \binom{n+k-1}{k} C_{n-1}^{(k)}$$
$$S_n^{(k-2)} = S_n^{(k-1)} - S_{n-1}^{(k-1)} = \binom{n+k}{k} C_n^{(k)} - 2\binom{n+k-1}{k} C_{n-1}^{(k)} + \binom{n+k-2}{k} C_{n-2}^{(k)}.$$

and so on. This process ultimately yields

$$S_n^{(0)} = s_n = \binom{n+k}{k} C_n^{(k)} - \binom{k}{1}\binom{n+k-1}{k} C_{n-1}^{(k)}$$
$$+ \binom{k}{2}\binom{n+k-2}{k} C_{n-2}^{(k)} - + \cdots + (-1)^k \binom{k}{k}\binom{n}{k} C_{n-k}^{(k)},$$
$$a_n = s_n - s_{n-1} = \binom{n+k}{k} C_n^{(k)} - \binom{k+1}{1}\binom{n+k-1}{k} C_{n-1}^{(k)}$$
$$+ \cdots + (-1)^k \binom{k+1}{k}\binom{n}{k} C_{n-k}^{(k)} + (-1)^{k+1}\binom{k+1}{k+1}\binom{n-1}{k} C_{n-k-1}^{(k)}.$$

This may be written,

$$a_n = \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p}\binom{n+k-p}{k} C_{n-p}^{(k)}$$

and accordingly we have

$$y(z) = \sum_{n=0}^{\infty} \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p}\binom{n+k-p}{k} C_{n-p}^{(k)} z^{f(n)}.$$

If now we set up the expression.

$$\begin{aligned}
&\binom{k+1}{0}\left\{\binom{k}{k}C_0^{(k)} + \binom{1+k}{k}C_1^{(k)}z^{f(1)} + \binom{2+k}{k}C_2^{(k)}z^{f(2)} + \cdots\right\}\\
&-\binom{k+1}{1}\left\{\binom{k-1}{k}C_{-1}^{(k)} + \binom{k}{k}C_0^{(k)}z^{f(1)} + \binom{1+k}{k}C_1^{(k)}z^{f(2)} + \cdots\right\}\\
&+\binom{k+1}{2}\left\{\binom{k-2}{k}C_{-2}^{(k)} + \binom{k-1}{k}C_{-1}^{(k)}z^{f(1)} + \binom{k}{k}C_0^{(k)}z^{f(2)} + \cdots\right\}\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+(-1)^{k+1}\binom{k+1}{k+1}\left\{\binom{k-k-1}{k}C_{-k+1}^{(k)} + \binom{k-k}{k}C_{-k}^{(k)}z^{f(1)} + \binom{k-k+1}{k}C_{-k+1}^{(k)}z^{f(2)} + \cdots\right.
\end{aligned}$$

each row, in view of the definition $C_{-r}^{(k)} = 0,\ r > 0$, is seen to be a series of the form $\sum_{n=0}^{\infty}\binom{k+1}{q}\binom{n+k}{k}C_n^{(k)}z^{f(n+q)}$. The terms of this series are not greater in absolute value than the corresponding terms of $\sum_{n=0}^{\infty}\binom{k+1}{q}(n+1)^kC_n^{(k)}z^{f(n)}$, for $f(n+q) \geqq f(n)$, and

$$\binom{n+k}{k} = \frac{(n+k)(n+k-1)\cdots(n+2)(n+1)}{k(k-1)\cdots 2\cdot 1} \leqq (n+1)^k.$$

Since also $C_n^{(k)}$ approaches a definite limit as $n \to \infty$, this latter series converges with $\sum_{n=0}^{\infty} n^kz^{f(n)}$, the convergence of which has already been established. It follows that each of the row-series in the above expression is convergent, $|z| < 1$. Since the number of rows, $(k+2)$, is finite, the series formed of the sums of columns also converges and represents the same function as the sum of the row-series. But the series of column-sums is exactly the function,

$$y(z) = \sum_{n=0}^{\infty}\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}\binom{n+k-p}{k}C_{n-p}^{(k)}z^{f(n)}.$$

It thus appears that we can write

$$\begin{aligned}
y(z) = \binom{k+1}{0}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n)} - \binom{k+1}{1}\sum_{n=0}^{\infty}\binom{n+k-1}{k}C_{n-1}^{(k)}z^{f(n)}&\\
+ - \cdots + (-1)^{k+1}\binom{k+1}{k+1}\sum_{n=0}^{\infty}\binom{n-1}{k}C_{n-k-1}^{(k)}z^{f(n)}&.
\end{aligned}$$

Rewriting this in such manner as to omit terms containing $C_{-r}^{(k)},\ r > 0$, we have

$$\begin{aligned}
y(z) = \binom{k+1}{0}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n)} - \binom{k+1}{1}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n+1)}&\\
+ - \cdots + (-1)^{k+1}\binom{k+1}{k+1}\sum_{n=0}^{\infty}\binom{n+k}{k}C_n^{(k)}z^{f(n+k+1)}&,
\end{aligned}$$

or

$$y(z) = \sum_{n=0}^{\infty}\binom{n+k}{k}\left\{\sum_{p=0}^{k+1}(-1)^p\binom{k+1}{p}z^{f(n+p)}\right\}C_n^{(k)} = \sum_{n=0}^{\infty} h_n(z)C_n^{(k)}.$$

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where $$b_n(z) = \binom{n+k}{k} \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} z^{f(n+p)}.$$

It must now be shown that the transformation whose coefficients are $b_n(z)$ is regular; that is, that the limit of the sum of the series $\sum_{n=0}^{\infty} b_n(z) C_n^{(k)}$ as $z \to 1$ in R is the same as the limit, s, of the sequence $C_n^{(k)}$. The necessary and sufficient conditions for regularity in this case are four,* namely,

I. For each n, $\lim_{z\to 1} b_n(z) = 0$.

II. For each z in R, $\sum_{n=0}^{\infty} | b_n(z) |$ converges.

III. For all z in R, $\sum_{n=0}^{\infty} | b_n(z) |$ is bounded.

IV. $\lim_{z\to 1} \sum_{n=0}^{\infty} b_n(z) = 1$.

Since
$$\lim_{z\to 1} b_n(z) = \binom{n+k}{k} \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} = \binom{n+k}{k} (1-1)^{k+1} = 0,$$
for each value of n, condition I is evidently satisfied.

Bearing in mind that $\binom{n+k}{k} \leqq (n+1)^k$, and that $| z^{f(n+p)} | \leqq \rho^{f(n)}$ for $p \geqq 0$, we have
$$| b_n(z) | \leqq \binom{n+k}{k} \sum_{p=0}^{k+1} \binom{k+1}{p} \rho^{f(n)} \leqq 2^{k+1}(n+1)^k \rho^{f(n)}.$$
Thus
$$\sum_{n=0}^{\infty} | b_n(z) | \leqq 2^{k+1} \sum_{n=0}^{\infty} (n+1)^k \rho^{f(n)},$$
which converges with $\sum_{n=0}^{\infty} n^k \rho^{f(n)}$. Therefore the series $\sum_{n=0}^{\infty} | b_n(z) |$ is convergent for each z in R, and condition II is satisfied.

It is equally simple to deal with condition IV. We have
$$C_n^{(k)} = \left[1/\binom{n+k}{k}\right] \left\{ \binom{n+k}{k} a_0 + \binom{n+k-1}{k} a_1 + \cdots + \binom{k}{k} a_n \right\}.$$
In particular, if $a_0 = 1$ and $a_i = 0$, $i > 0$, then for every n, $C_n^{(k)} = a_0 = 1$. In this case, therefore,
$$y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)} = a_0 z^{f(0)} = 1,$$
and likewise
$$y(z) = \sum_{n=0}^{\infty} b_n(z) C_n^{(k)} = \sum_{n=0}^{\infty} b_n(z).$$

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* *Bulletin of the American Mathematical Society*, Vol. 28 (1922), p. 17.

Equating these expressions [illegible], [illegible] the $\sum_{n=0}^{\infty} a_n(z) = 1$ and accordingly condition IV is satisfied.

There remains condition III. We note, in the first place, that since $f(x)$ is an L-function there is a value of x, say n_0, such that for $x > n_0$, $f(x)$ and its first $(k+1)$ derivatives exist, and are continuous, monotone, and of constant sign. If, therefore, we set $\phi(n) = z^{f(n)}$, we have, for $n > n_0$, the first difference

$$\Delta\phi(n) = \phi(n) - \phi(n+1) = \int_{n+1}^{n} \phi'(r_1)\, dr_1,$$

the second difference

$$\Delta^2\phi(n) = \Delta\phi(n) - \Delta\phi(n+1) = \int_{n+1}^{n}\int_{r_1+1}^{r_1} \phi''(r_2)\, dr_2 dr_1,$$

and the $(k+1)$-st difference

$$\Delta^{k+1}\phi(n) = \Delta^k\phi(n) - \Delta^k\phi(n+1) = \int_{n+1}^{n}\int_{r_1+1}^{r_1}\cdots\int_{r_k+1}^{r_k} \phi^{(k+1)}(r_{k+1})\, dr_{k+1}\cdots dr_2 dr_1.$$

Now, in this last expression, set $r_1 = n + s_1$, $r_2 = r_1 + s_2$, and in general $r_i = r_{i-1} + s_i$.

$$\Delta^{k+1}\phi(n) = \int_1^0\int_1^0\cdots\int_1^0 \phi^{(k+1)}(\mu)\, ds_{k+1}\cdots ds_2 ds_1,$$

where $\mu = n + s_1 + s_2 + \cdots + s_{k+1}$. Observing that

$$|b_n(z)| = \binom{n+k}{k} \left| \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} z^{f(n+p)} \right| \leqq (n+1)^k |\Delta^{k+1}\phi(n)|,$$

and that the above expression for $\Delta^{k+1}\phi(n)$ is valid for $n > n_0$, we may now write

$$\begin{aligned}\sum_{n=0}^{\infty} |b_n(z)| &\leqq \sum_{n=0}^{n_0} |b_n(z)| \\ &\qquad + \sum_{n=n_0+1}^{\infty} (n+1)^k \left| \int_1^0\int_1^0\cdots\int_1^0 \phi^{(k+1)}(\mu)\, ds_{k+1}\cdots ds_2 ds_1 \right| \\ &\leqq \sum_{n=0}^{n_0} |b_n(z)| \\ &\qquad + \sum_{n=n_0+1}^{\infty} (n+1)^k \int_0^1\int_0^1\cdots\int_0^1 |\phi^{(k+1)}(\mu)|\, ds_{k+1}\cdots ds_2 ds_1\end{aligned}$$

$$\sum_{n=n_0+1}^{\infty} (n+1)^k \int_0^1 \int_0^1 \cdots \int_0^1 | \phi^{k+1}(\mu) | \, ds_{k+1} \cdots ds_2 ds_1$$

is bounded for all z in R.

It is evident that the statements made in Lemma 5 regarding $z^{f(n)}$ are equally valid for $\rho^{f(n)}$, where $\rho = | z |$. If we set $\psi(n) = \rho^{f(n)}$, and indicate by $T\psi$ a term of $\psi^{(k)}(n)$,

$$T\psi = A\rho^f (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k} (\log \rho)^h, \qquad h = \sum_{i=1}^{k} \alpha_i,$$

we note that $T'\psi = T\psi \{ f' \log \rho + \sum_{i=1}^{k} \alpha_i f^{(i+1)} / f^{(i)} \}$.

Suppose now that the alternation of signs of the $f^{(i)}$ commences with f''. Then, for $n > n_0$, $f^{(i)}$ and $f^{(i+1)}$ have opposite signs, and since $f' \log \rho < 0$, it follows that $T\psi$ and $T'\psi$ bear opposite signs, and hence also $\psi^{(k)}$ and $\psi^{(k+1)}$. On observing that $\psi'(n) = \rho^{f(n)} f'(n) \log \rho < 0$, we conclude that

$$(-1)^{k+1} \psi^{(k+1)}(n) > 0, \qquad n > n_0.$$

By Lemma 1, and the conclusion just stated,

$$| \phi^{(k+1)}(n) | \leqq B \, | \psi^{(k+1)}(n) | = (-1)^{k+1} B \psi^{(k+1)}(n),$$

where z is in R and B is some positive constant. Since $\mu \geqq n$, we can write

$$\sum_{n_0+1}^{\infty} (n+1)^k \int_0^1 \int_0^1 \cdots \int_0^1 | \phi^{(k+1)}(\mu) | \, ds_{k+1} \cdots ds_2 ds_1$$
$$\leqq B \sum_{n_0+1}^{\infty} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^{k+1} (\mu+1)^k \psi^{(k+1)}(\mu) \, ds_{k+1} \cdots ds_2 ds_1.$$

A first integration gives

$$B \sum_{n_0+1}^{\infty} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^{k+1} [(\mu+1)^k \psi^{(k)}(\mu) - k(\mu+1)^{k-1} \psi^{(k-1)}(\mu)$$
$$+ k(k-1)(\mu+1)^{k-2} \psi^{(k-2)}(\mu) - + \cdots + (-1)^k k! \, \psi(\mu)]_{s_{k+1}=0}^{s_{k+1}=1} \, ds_k \cdots ds_2 ds_1.$$

This may be expressed as $B \sum_{n_0+1}^{\infty} \{ g(n) - g(n+1) \}$, where

$$g(n) = \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^k \{ (\mu_1+1)^k \psi^{(k)}(\mu_1)$$
$$- k(\mu_1+1)^{k-1} \psi^{(k-1)}(\mu_1) + - \cdots + (-1)^k k! \, \psi(\mu_1) \} \, ds_k \cdots ds_2 ds_1,$$

in which $\mu_1 = n + s_1 + s_2 + \cdots + s_k$.

The terms in the integrand on the right-hand side of this expression are of the form $(-1)^m (k!/m!)(\mu_1+1)^m \psi^{(m)}(\mu_1)$ [illegible]

$$U_m = \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^m (k!/m!)(\mu_1+1)^m \psi^{(m)}(\mu_1)\, ds_k \cdots ds_2 ds_1$$

we can write $g(n) = \sum_{m=0}^{k} U_m$. Noting that $\mu_1 \leqq n + k$, we see that

$$U_m \leqq \frac{k!\,(n+k+1)^m}{m!} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^m \psi^{(m)}(\mu_1)\, ds_k \cdots ds_2 ds_1$$

$$= \frac{k!\,(n+k+1)^m}{m!} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^m [\psi^{(m-1)}(\mu_1)]_{s_k=0}^{s_k=1}\, ds_{k-1} \cdots ds_2 ds_1 .$$

Now if m is even, $\psi^{(m-1)}(\mu_1)$ is negative and increasing, if m is odd, $\psi^{(m-1)}(\mu_1)$ is positive and decreasing; in either case, therefore, $(-1)^m \psi^{(m-1)}(\mu_1)$ is negative and increasing, and hence

$$U_m < \frac{k!\,(n+k+1)^m}{m!} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^{m-1} \psi^{(m-1)}(\mu_2)\, ds_{k-1} \cdots ds_2 ds_1,$$

where $\mu_2 = n + s_1 + s_2 + \cdots + s_{k-1}$.

After m such integrations we find

$$U_m < \frac{k!\,(n+k+1)^m}{m!} \int_0^1 \int_0^1 \cdots \int_0^1 \psi(\mu_{m+1})\, ds_{k-m} \cdots ds_2 ds_1,$$

where $\mu_{m+1} = n + s_1 + \cdots + s_{k-m}$. Since $\psi(\mu_{m+1}) = \rho^{f(\mu_{m+1})}$ is a positive, monotonically decreasing, function of μ_{m+1}, it follows that

$$0 \leqq U_m < [k!\,(n+k+1)^m/m!]\rho^{f(n)},$$

and hence that

$$\limsup_{n\to\infty} U_m \leqq \limsup_{n\to\infty} [k!\,(n+k+1)^m/m!]\rho^{f(n)}.$$

But this limit is [illegible] provided condition (b) of Theorem I [illegible]

Thus $\lim_{n\to\infty} U_m = 0$, and as there are but a finite number, $(k+1)$, of such terms in $g(n)$, we have $\lim_{n\to\infty} g(n) = 0$. As a result,

$$B\sum_{n_0+1}^{\infty}\{g(n)-g(n+1)\} = Bg(n_0+1)$$
$$< B\sum_{m=0}^{k}[k!\,(n_0+k+2)^m/m!\,]\rho^{f(n_0+1)} \leqq Bk!\sum_{m=0}^{k}[n_0+k+2)^m/m!\,],$$

which is bounded.

Collecting these results, we have

$$\sum_{n-0}^{\infty} |\, b_n(z)\,| < 2^{k+1}[1+2^k+\cdots+(n_0+1)^k] + Bk!\sum_{m=0}^{k}(n_0+k+2)^m/m!\,.$$

It is thus proved that providing the alternation of signs of the derivatives of $f(x)$ commences with $f''(x)$ condition III is satisfied, $\sum_{n-0}^{\infty} |\, b_n(z)\,|$ is bounded for all z in R.

On the other hand, if $f''(x) > 0$, then for this, and possibly other derivatives, $\operatorname{sgn} f^{(i)} = (-1)^i$. These may enter certain terms of $\phi^{(k+1)}(n)$, for example $T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_q}$, in such manner as to make the signs of these terms different from that of the leading term, $z^{f(n)}\{f'(n)\log z\}^{k+1}$. In this case we have

$$|\,\phi^{(k+1)}(n)\,| \leqq (-1)^{k+1}B\psi^{(k+1)}(n) + (-1)^k 2B(T\psi_1 + T\psi_2 + \cdots + T\psi_q).$$

The series $\sum_{n_0+1}^{\infty}(n+1)^k\,|\,\Delta^{k+1}\phi(n)\,|$ is bounded if $\sum_{n_0+1}^{\infty} n^k\,|\,\Delta^{k+1}\phi(n)\,|$ is bounded; for the sake of simplicity the proof will be given for the latter. We have

$$\sum_{n_0+1}^{\infty} n^k\,|\,\Delta^{k+1}\phi(n)\,| \leqq B\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{k+1}\mu^k\psi^{(k+1)}(\mu)\,ds_{k+1}\cdots ds_2 ds_1$$
$$+\,2B\sum_{n_0+1}^{\infty} n^k\int_0^1\int_0^1\cdots\int_0^1 (-1)^k(T\psi_1 + T\psi_2 + \cdots + T\psi_q)\,ds_{k+1}\cdots ds_2 ds_1.$$

The first of these summations, after a first integration, gives as before, $B\sum_{n_0+1}^{\infty}\{g(n)-g(n+1)\}$, where now

$$g(n) = \int_0^1\int_0^1\cdots\int_0^1 (-1)^k\{\mu_1{}^k\psi^{(k)}(\mu_1) - k\mu_1{}^{k-1}\psi^{(k-1)}(\mu_1)$$
$$+-\cdots+(-1)^k k!\,\psi(\mu_1)\}\,ds_k\cdots ds_2 ds_1$$

and $\mu_1 = n + s_1 + s_2 + \cdots + s_k$. Here again we may put $g(n) = \sum_{m=0}^{k} U_m$,

where,

$$U_m = \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^m (k!/m!) \mu_1{}^m \psi^{(m)}(\mu_1)\, ds_k \cdots ds_2 ds_1.$$

In this case, however, we cannot be sure that $(-1)^m \psi^{(m-1)}(\mu_1)$ is negative and increasing; so the former procedure cannot be followed beyond this point.

But the terms of $\psi^{(m)}(\mu_1)$ are all of the form

$$T\psi = A\rho^f (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(m)})^{\alpha_m} (\log \rho)^h, \qquad h = \sum \alpha_i, \quad m = \sum i\alpha_i.$$

By Lemma 3, and condition (d) of the theorem, there is a positive constant, M_i, such that, for n sufficiently great,

$$| f^{(i)}(\mu_1) |^{\alpha_i} < M_i{}^{\alpha_i} [f'(\mu_1)]^{\alpha_i} / \mu_1{}^{\alpha_i(i-1)}.$$

Supposing n_0 so chosen that this inequality holds for all values of i occurring in any of the $T\psi$, and setting $\prod_{i=1}^{m} M_i{}^{\alpha_i} = M$, we have

$$| T\psi | < A\rho^{f(\mu_1)} M [f'(\mu_1)]^h \, | \log \rho |^h / \mu_1{}^{m-h}.$$

Thus the absolute values of the terms in the integrand,

$$(-1)^m (k!/m!) \mu_1{}^m \psi^{(m)}(\mu_1)$$

are less than

$$(AMk!/m!) \rho^{f(\mu_1)} \mu_1{}^h \, [f'(\mu_1)]^h,$$

provided $\rho > 1/e$. In this expression A, M, k, m, are constants $\mu_1 \leqq n + k$, $f'(\mu_1) \leqq f'(n+k)$, and $\rho^{f(\mu_1)} \leqq \rho^{f(n)}$; hence

$$\limsup_{n\to\infty} \rho^{f(\mu_1)} \mu_1{}^h [f'(\mu_1)]^h$$
$$\leqq \limsup_{n\to\infty} \exp\{f(n) \log \rho + h[\log(n+k) + \log f'(n+k)]\}.$$

But $\log n = o[f(n)]$, and since $f'' = o(f')$ entails $f' = o(f)$, it follows that the limit on the right-hand side of the above expression is zero.

There being a finite number of such terms in the integrand of the expression for U_m, it is clear that $\lim_{n\to\infty} U_m = 0$, and hence that $\lim_{n\to\infty} g(n) = 0$. We thus have

$$B \sum_{n_0+1}^{\infty} \{g(n) - g(n+1)\} = Bg(n_0 + 1).$$

But

$$| g(n_0 + 1) | \leqq \sum^{k} | U_m(n_0 + 1) |$$
$$\leqq \sum_{m=0}^{k} (k!/m!) \int_0^1 \int_0^1 \cdots \int_0^1 \sum_{T\psi} \mu_1{}^m \, | T\psi | \, ds_k \cdots ds_2 ds_1,$$

o

where the summation in the integrand extends over all the terms in $\psi^{(m)}$, and $\mu_1 = n_0 + 1 + s_1 + \cdots + s_k$. It has also been shown that

$$| T\psi | < A\rho^{f(\mu_1)} M[f'(\mu_1)]^h \, | \log \rho \, |^h / \mu_1^{m-h} < AM[f'(\mu_1)]^h / \mu_1^{m-h} \quad \text{if} \quad \rho > 1/e.$$

Hence

$$| g(n_0 + 1) | < \sum_{m=0}^{k} AM(k!/m!) \sum_h \int_0^1 \int_0^1 \cdots \int_0^1 \mu_1^h [f'(\mu_1)]^h ds_k \cdots ds_2 ds_1;$$

and since $\mu_1 \leqq n_0 + k + 1$ and $f'(\mu_1)$ is a positive, monotonically increasing function of μ_1,

$$| g(n_0 + 1) | < \sum_{m=0}^{k} AM(k!/m!) \sum_h (n_0 + k + 1)^h [f'(n_0 + k + 1)]^h,$$

where h takes on all values corresponding to the terms $T\psi$ in $\psi^{(m)}$.

Thus $g(n_0 + 1)$ is bounded for all z in R, and so also is the first summation in the expression for $\sum_{n_0+1}^{\infty} n^k \, | \Delta^{k+1}\phi(n) \, |$.

With regard to the second of these summations, we note that any of the $T\psi_i$ therein is of the form

$$T\psi_i = A_i \rho^f (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k+1)})^{\alpha_{k+1}} (\log \rho)^h, \qquad h = \sum_{i=1}^{k+1} \alpha_i.$$

Suppose now that the derivatives for which $\operatorname{sgn} f^{(i)} = (-1)^i$ are $f^{(i_1)}, f^{(i_2)}, \cdots, f^{(i_r)}$, and for these let

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r} = \alpha, \qquad i_1\alpha_{i_1} + i_2\alpha_{i_2} + \cdots + i_r\alpha_{i_r} = \beta.$$

By Lemma 3, and condition (d), we may set

$$| f^{(i)}(\mu) \, |^{\alpha_i} < M_i^{\alpha_i} [f'(\mu)]^{\alpha_i} / \mu^{\alpha_i(i-1)},$$

and accordingly

$$| T\psi_i | < A_i M \rho^f (f')^{\alpha_1 + \alpha} \cdots (f^{(k+1)})^{\alpha_{k+1}} \, | \log \rho \, |^h / \mu^{\beta - \alpha},$$

where now the $f^{(i_1)}, \cdots, f^{(i_r)}$, no longer appear, and $\sum \alpha_i = h$, but $\sum i\alpha_i = k - \beta + \alpha + 1 = j + 1 < k + 1$. Then we have

$$\sum_{n_0+1}^{\infty} n^k \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^k T\psi_i ds_{k+1} \cdots ds_2 ds_1$$

$$< A_i M \sum_{n_0+1}^{\infty} \int_0^1 \int_0^1 \cdots \int_0^1 (-1)^l \mu^j \rho^f (f')^{\alpha_1 + \alpha} \cdots (f^{(k+1)})^{\alpha_{k+1}} (\log \rho)^h ds_{k+1} \cdots ds_1$$

But the integrand in the latter expression is, except for a constant factor,

one of the terms of $(-1)^{j+1}\mu^j\psi^{(j+1)}(\mu)$. If the terms in $\psi^{(j+1)}$ are all of like sign,

$$\sum_{n_0+1}^{\infty} n^k \int_0^1\int_0^1\cdots\int_0^1 (-1)^k T\psi_i\, ds_{k+1}\cdots ds_2 ds_1$$

$$< G\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 (-1)^{j+1}\mu^j\psi^{(j+1)}(\mu)\, ds_{k+1}\cdots ds_2 ds_1,$$

which can be shown bounded as was done for $\psi^{(k+1)}$ in like circumstances. But if the terms are not all of like sign the above process is repeated with $\psi^{(j+1)}$ instead of $\psi^{(k+1)}$. As the order of the derivative of ψ is lowered at each repetition we are certain ultimately to reach the form

$$\sum_{n_0+1}^{\infty}\int_0^1\int_0^1\cdots\int_0^1 -\psi'(\mu)\, ds_{k+1}\cdots ds_2 ds_1,$$

which can readily be proved bounded.

The second summation being thus proved bounded, it is established that $\sum_{n=0}^{\infty} |\, b_n(z)\,|$ is bounded for all z in R; condition III is satisfied. This completes the proof of the regularity of the transformation whose coefficients are $b_n(z)$. It follows that, as $z \to 1$ in R,

$$\lim_{z\to 1} y(z) = \lim_{z\to 1}\sum_{n=0}^{\infty} b_n(z) C_n^{(k)} = \lim_{n\to\infty} C_n^{(k)} = s,$$

and the proof of Theorem I is complete.

Owing to the less general nature of Theorem II, its proof is considerably simpler. We set up the series

$$w(z) = \sum_{n=0}^{\infty}(n+1)\{z^{f(n)} - 2z^{f(n+1)} + z^{f(n+2)}\}C_n^{(1)} = \sum_{n=0}^{\infty} b_n(z) C_n^{(1)};$$

and the proof consists in showing that, under the conditions of the theorem, $w(z) \to s$ as $z \to 1$ in R, and that for all z in R, $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)}$ converges to the same value as $w(z)$.

As before, since $\lim_{n\to\infty} C_n^{(1)} = s$, we must establish the regularity of the transformation whose coefficients are $b_n(z)$. The conditions for regularity are those given in the proof of Theorem I.

It is evident that for each n, $\lim_{z\to 1} b_n(z) = (n+1)(1-2+1) = 0$, so that condition I is satisfied.

Since $f(x) = O(\log x)$, by Lemma 4 the alternation of signs of the derivatives begins with $f''(x)$. Accordingly, if we again set $\phi(n) = z^{f(n)}$, $\psi(n) = \rho^{f(n)}$, and recall Lemma 1, we can find an n_0 such that for $n > n_0$, $|\phi^{(k)}(n)| \leqq B|\psi^{(k)}(n)| = (-1)^k B\psi^{(k)}(n)$, where z is in R and B is some positive constant. We also have, as before, but this time with $k = 1$,

$$|b_n(z)| = (n+1)\,|z^{f(n)} - 2z^{f(n+1)} + z^{f(n+2)}| = (n+1)\,|\Delta^2\phi(n)|.$$

Thus

$$\sum_{n=0}^{\infty} |b_n(z)| = \sum_{n=0}^{n_0} |b_n(z)| + \sum_{n=n_0+1}^{\infty} (n+1)\,|\Delta^2\phi(n)|.$$

Since the absolute value of $b_n(z)$ is not greater than $4(n+1)$, the first summation is less than $4\{1+2+\cdots+(n_0+1)\} = 2(n_0+1)(n_0+2)$. With regard to the second summation it will suffice to consider $\sum_{n=n_0+1}^{\infty} n\,|\Delta^2\phi(n)|$.

We have $\Delta^2\phi(n) = \int_1^0\int_1^0 \phi''(\mu)\,ds_2 ds_1$, where $\mu = n + s_1 + s_2$, and so

$$\sum_{n=n_0+1}^{\infty} n\,|\Delta^2\phi(n)| = \sum_{n=n_0+1}^{\infty} n \left| \int_1^0\int_1^0 \phi''(\mu)\,ds_2 ds_1 \right|$$

$$\leqq \sum_{n=n_0+1}^{\infty} n \int_0^1\int_0^1 |\phi''(\mu)|\,ds_2 ds_1 \leqq B \sum_{n_0+1}^{\infty} n \int_0^1\int_0^1 \psi''(\mu)\,ds_2 ds_1.$$

Now, $\psi''(\mu) = \rho^{f(\mu)}\{[f'(\mu)\log\rho]^2 + f''(\mu)\log\rho\}$, and by Lemma 4 there can be found constants, $M_1 > 0$ and $M_2 > 0$, such that for $n > n_0$, $f'(x) < M_1/x$, $|f''(x)| < M_2 f'(x)/x$. Finally, for $\rho > 1/e$, $|\log\rho| > [\log\rho]^2$; whence

$$\psi''(\mu) < -\rho^{f(\mu)}(\log\rho)\{M_1 f'(\mu)/\mu + M_2 f'(\mu)/\mu\} = -M\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho),$$

where $M = M_1 + M_2$. Then, since $n \leqq \mu$

$$\sum_{n_0+1}^{\infty} n\,|\Delta^2\phi(n)| < BM \sum_{n_0+1}^{\infty} n \int_0^1\int_0^1 -\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho)\,ds_2 ds_1$$

$$\leqq BM \sum_{n_0+1}^{\infty} \int_0^1\int_0^1 -\rho^{f(\mu)} f'(\mu)(\log\rho)\,ds_2 ds_1$$

$$= BM \sum_{n_0+1}^{\infty} \int_0^1 \left[-\rho^{f(\mu)}\right]_{s_2=0}^{s_2=1} ds_1 = BM \sum_{n_0+1}^{\infty} \{g(n) - g(n+1)\},$$

where $g(n) = \int_0^1 \rho^{f(n+s_1)} ds_1 < \rho^{f(n)}$. Therefore $\limsup_{n\to\infty} g(n) \leqq \lim_{n\to\infty} \rho^{f(n)} = 0$, that is to say, $\lim_{n\to\infty} g(n) = 0$. On the other hand, $g(n_0+1) < \rho^{f(n_0+1)} < 1$, whence $\sum_{n_0+1}^{\infty} n\,|\,\Delta^2\phi(n)\,| < BM$, which is bounded.

We have thus proved that for all z in R, $\sum_{n=0}^{\infty} |\,b_n(z)\,|$ is bounded; since the series is one of positive terms it is also convergent for each z in R. Conditions II and III are thus satisfied.

To test the series for IV, $\lim_{z\to 1} \sum_{n=0}^{\infty} b_n(z) = 1$, we suppose as before that $a_0 = 1$, $a_i = 0$, $i > 0$, whence for each n, $C_n^{(1)} = a_0 = 1$. Thus, in this case, $w(z) = \sum_{n=0}^{\infty} b_n(z)$, and $y(z) = \sum_{n=0}^{\infty} a_n z^{f(n)} = a_0 z^{f(0)} = 1$. If now it can be shown that, for all z in R, $w(z) = y(z)$, then IV will certainly be satisfied.

To this end we note that $a_n = (n+1)C_n^{(1)} - 2nC_{n-1}^{(1)} + (n-1)C_{n-2}^{(1)}$, so that

$$y(z) = \sum_{n=0}^{\infty} \{(n+1)C_n^{(1)} - 2nC_{n-1}^{(1)} + (n-1)C_{n-2}^{(1)}\}\, z^{f(n)}.$$

If we indicate by y_m and w_m the sum of the first $(m+1)$ terms of $y(z)$ and $w(z)$ respectively, we have

$$\begin{aligned} y_n - w_{n-2} &= \{(n+1)C_n^{(1)} - 2nC_{n-1}^{(1)}\}\, z^{f(n)} + nC_{n-1}^{(1)} z^{f(n-1)} \\ &= \{(n+1)C_n^{(1)} - nC_{n-1}^{(1)}\}\, z^{f(n)} + \{z^{f(n-1)} - z^{f(n)}\}\, nC_{n-1}^{(1)}. \end{aligned}$$

The coefficient in the first term of this expression is simply s_n, and by condition (d) of the theorem for a given $\lambda > 1$ there is a constant H such that for all n sufficiently great $|\,s_n\,| \leqq H\lambda^{f(n)}$. If, therefore, for a fixed value of z, we choose λ so that $1/\lambda > |\,z\,| = \rho$, then for n sufficiently great $|\,s_n z^{f(n)}\,| = |\,s_n\,|\,\rho^{f(n)} \leqq H(\lambda\rho)^{f(n)}$. But the value of this quantity approaches zero as n increases, wherefore the first term in the above expression for $(y_n - w_{n-2})$ also approaches zero as n increases.

The second term is $n[\Delta z^{f(n-1)}]C_{n-1}^{(1)}$, and the absolute value of this is less than $Dn\,|\,C_{n-1}^{(1)}\,| \int_0^1 -\rho^{f(\mu)} f'(\mu)(\log\rho)\, ds$, where $\mu = n-1+s$, and D is a positive constant. Since $[f'(\mu)] < M_1/\mu \leqq M_1/(n-1)$, and $|\log\rho\,| < 1$ if $\rho > 1/e$, and $\int_0^1 \rho^{f(\mu)} ds < \rho^{f(n-1)}$, we can say that

$$n\,|\,\Delta z^{f(n-1)}\,|\,|\,C_{n-1}^{(1)}\,| < DM_1 n/(n-1)\,|\,C_{n-1}^{(1)}\,|\,\rho^{f(n-1)};$$

and this expression certainly approaches zero as n increases. As a result we

have $\lim_{n\to\infty} (y_n - w_{n-2}) = 0$, or $y(z) = w(z)$ for all z in R. Condition IV is accordingly satisfied, and the transformation is regular; hence as $z \to 1$ in R, $\lim_{z\to 1} y(z) = \lim_{z\to 1} w(z) = \lim_{n\to\infty} C_n{}^{(1)} = s$.

It remains only to show that $y(z)$ is convergent everywhere within the unit circle. Its convergence within the region R has been established, and it is obvious that for any fixed z, $|z| < 1$, we may so choose the chords defining R that z will lie within the angle formed by them. We now define R' as that part of the annular region between circles of radius $\epsilon > 0$ and $(1-\epsilon)$, with centers at the origin, which lies within this same angle, ϵ being so chosen that z lies in R'.

Recalling the proof for the case where z was confined to R, we have in the present case also for n greater than some n_0, $|\phi^{(k)}(n)| \leqq (-1)^k B\psi^{(k)}(n)$, and $\sum_{n=0}^{\infty} |b_n(z)| = \sum_{n=0}^{n_0} |b_n(z)| + \sum_{n_0+1}^{\infty} (n+1) |\Delta^2\phi(n)|$. The first summation is bounded, and the second likewise provided that $B\sum_{n_0+1}^{\infty} n \int_0^1 \int_0^1 \psi''(\mu)\, ds_2 ds_1$ is bounded. Since $|\log\rho| < |\log\epsilon|$ we now have

$$\psi''(\mu) < -M\,|\log\epsilon|\,\rho^{f(\mu)}[f'(\mu)/\mu](\log\rho)$$

and hence $B\sum_{n_0+1}^{\infty} n \int_0^1 \int_0^1 \psi''(\mu)\, ds_2 ds_1 < BM\,|\log\epsilon|$. It follows that for z in R', $\sum_{n=0}^{\infty} |b_n(z)|$, and hence $w(z)$, is bounded and convergent.

Again,

$$y_n - w_{n-2} = \{(n+1)C_n{}^{(1)} - nC_{n-1}^{(1)}\}z^{f(n)} + \{z^{f(n-1)} - z^{f(n)}\}nC_{n-1}^{(1)}.$$

The first term approaches zero as n increases, and for the second

$$\begin{aligned} n\,|\{z^{f(n-1)} - z^{f(n)}\}C_{n-1}^{(1)}| &= n\,|[\Delta z^{f(n-1)}]C_{n-1}^{(1)}| \\ &< DM_1 n/(n-1)\,|C_{n-1}^{(1)} \log\epsilon)\,|\rho^{f(n-1)}, \end{aligned}$$

which also approaches zero as n increases. It thus appears that $y(z) = w(z)$, and $y(z)$ consequently converges, for all z in R'.

It is clear that for any fixed z, $0 < |z| < 1$, ϵ may be so chosen that z will lie within R'. It is obvious also that $y(z)$ is convergent for $z = 0$. It thus follows that $y(z)$ is convergent everywhere within the unit circle; and this completes the proof of Theorem II.

The proof as given makes use of the second form of condition (d),

$s_n = O[\lambda^{f(n)}]$ for every $\lambda > 1$. The condition $a_n = O[\lambda^{f(n)} f'(n)]$ for every $\lambda > 1$, is stronger than $s_n = O[\lambda^{f(n)}]$; it is sufficient to assure the fulfillment of the latter. For, if the former holds, then there is a constant H, and an N, such that for $n > N$, $a_n < H\lambda^{f(n)} f'(n)(\log \lambda)$. Thus for $n > N$,

$$| s_n | \leqq | s_N | + | a_{N+1} | + \cdots + | a_n | < | s_N | + H(\log \lambda)\{\lambda^{f(N+1)} f'(N+1) \cdots + \lambda^{f(n)} f'(n)\}.$$

Now λ can be so chosen that $\lambda^{f(n)} f'(n)$ is ultimately a decreasing function; for since $f(n) = O(\log n)$ and $f'(n) = O(1/n)$, an N_1 can be found such that for $n > N_1$, $f(n) < M_3(\log n)$ and $f'(n) < M_1/n$; hence for such values of n, $\lambda^{f(n)} f'(n) < \lambda^{M_3(\log n)} \cdot M_1/n = M_1 e^{(M_3 \log \lambda - 1)\log n}$. Since M_3 does not depend upon the choice of λ the latter may be chosen so small that $(M_3 \log \lambda - 1) < 0$, whence $M_1 e^{(M_3 \log \lambda - 1)\log n} \to 0$. Accordingly, for $n > N$, N_1, we can write

$$| s_n | < | s_N | + H \int_N^n \lambda^{f(x)} f'(x)(\log \lambda)\, dx = | s_N | + H\lambda^{f(n)} - H\lambda^{f(N)}.$$

Then $$\frac{| s_n |}{\lambda^{f(n)}} < \frac{| s_N | - H\lambda^{f(N)}}{\lambda^{f(n)}} + H, \quad \text{and} \quad \lim_{n\to\infty} \sup \frac{| s_n |}{\lambda^{f(n)}} \leqq H.$$

But it follows that an N_2 can be found such that for $n > N_2$, $| s_n | \leqq H\lambda^{f(n)}$, or, what is the same thing, $s_n = O[\lambda^{f(n)}]$.

Either form of condition (d) is sufficient. On the other hand, $a_n = o[\lambda^{f(n)}]$ is necessary. For if $\sum_{n=0}^{\infty} a_n x^{f(n)}$, $0 < x < 1$, is convergent, then $\sum_{n=0}^{\infty} a_n/\lambda^{f(n)}$ is also convergent for all $\lambda > 1$. It follows that $\lim_{n\to\infty} a_n/\lambda^{f(n)} = 0$, whence $a_n = o[\lambda^{f(n)}]$.

5. *Oscillating Series.* It was mentioned in the introductory paragraphs that if the exponents of z in the series $\sum_{n=0}^{\infty} a_n z^{f(n)}$ have too rapid a rate of increase the series does not necessarily approach any limit as $z \to 1$. Hardy * cites as particular instances the series $\sum_{n=0}^{\infty} (-1)^n x^{p^n}$, $p \geqq 2$ integer, and $\sum_{n=0}^{\infty} (-1)^n x^{n!}$; he shows that these both oscillate about the value ½, the Cesàro mean of their coefficients, as $x \to 1^-$ along the real axis.

Belinfante † has proved that the power-series $y(x) = \sum_{n=0}^{\infty} (-1)^n x^{f(n)}$,

* *Quarterly Journal of Mathematics*, Vol. 38 (1906-7), p. 269.

† *Koninklijke Akademie van Wetenschappen te Amsterdam*, Verslagen 32 (1923), p. 472.

$f(n)$ integer, approaches no limit as $x \to 1^-$, provided two constants, $a > 1$ and b, can be found such that for all sufficiently great n, $1 + (a^2 - 1)/2a < b \leqq f(n+1)/f(n) \leqq a$. The function $f(n) = p^n$, $p > 1$ integer, falls into this category, for then we may choose $a = b = p$, and $1 + (p^2 - 1)/2p < p = p^{n+1}/p^n$.

This theorem may be extended to the case where $f(n)$ is not an integer as follows. Let $[f(n)]$ be the greatest integer in $f(n)$, so that $[f(n)] \leqq f(n) < [f(n) + 1]$, and

$$\frac{[f(n+1)]}{[f(n)+1]} < \frac{f(n+1)}{f(n)} < \frac{[f(n+1)+1]}{[f(n)]},$$

whence

$$\frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} < 1 < \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]}.$$

Then, for all sufficiently great n.

$$1 + \frac{a^2-1}{2a} < b \leqq \frac{f(n+1)}{f(n)} < \frac{f(n+1)}{f(n)} \cdot \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]}$$
$$\leqq \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]}\, a,$$

and likewise,

$$\frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]}\, b \leqq \frac{f(n+1)}{f(n)} \cdot \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]}$$
$$< \frac{f(n+1)}{f(n)} \leqq a.$$

Now choose $a_1 > a$, and $b_1 < b$, such that $1 + (a_1{}^2 - 1)/2a_1 < b_1 < b$.*

Then, since

$$\lim_{n\to\infty} \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]} = \lim_{n\to\infty} \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]} = 1,$$

we have for all sufficiently great n,

$$1 + \frac{a_1{}^2-1}{2a_1} < b_1 < \frac{f(n)[f(n+1)]}{f(n+1)[f(n)+1]}\, b \leqq \frac{[f(n+1)]}{[f(n)+1]} < \frac{f(n+1)}{f(n)}$$
$$< \frac{[f(n+1)+1]}{[f(n)]} \leqq \frac{f(n)[f(n+1)+1]}{f(n+1)[f(n)]}\, a < a_1.$$

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* That such a choice can be made is apparent from the fact that $1 + (a^2 - 1)/2a$ is an increasing function of a. Since $1 + (a^2 - 1)/2a < b$ an a_1 and b_1 can always be found such that $1 + (a^2 - 1)/2a < 1 + (a_1{}^2 - 1)/2a_1 < b_1 < b$. But then certainly $a_1 < a$.

It now follows that the series with integral exponents,

$$g(x) = x - x^{[f(1)]} + x^{[f(2)+1]} - x^{[f(3)]} + x^{[f(4)+1]} - + \cdots,$$
$$h(x) = x - x^{[f(1)+1]} + x^{[f(2)]} - x^{[f(3)+1]} + x^{[f(4)]} - + \cdots,$$

satisfy the conditions of Belinfante's theorem, and hence approach no limit as $x \to 1^-$. But $[f(2n+1)] \leqq f(2n+1) < [f(2n+1)+1]$, and $f[(2n)+1] > f(2n) \geqq [f(2n)]$, whence $g(x) < y(x) < h(x)$. Now

$$h(x) - g(x) = (1-x)\{x^{[f(1)]} + x^{[f(2)]} + x^{[f(3)]} + \cdots\}.$$

Since $1 < b \leqq f(n+1)/f(n)$, ultimately $f(n) \geqq Cb^n$ where C is a positive constant; and hence for any positive constant M, ultimately $[f(n)] > Mn$. Thus there is an n_0 such that for $n \geqq n_0$, $x^{[f(n)]} < x^{Mn}$. Then

$$h(x) - g(x) < (1-x)\{n_0 - 1 + \sum_{n=n_0}^{\infty} x^{[f(n)]}\} < (1-x)\{n_0 + \sum_{n=n_0}^{\infty} x^{Mn}\}$$
$$= (1-x)n_0 + (1-x)\frac{x^{Mn_0}}{1-x^M} = (1-x)n_0 + \frac{x^{Mn_0}}{1+x+\cdots+x^{M-1}}.$$

Hence

$$\limsup_{x\to 1^-} \{h(x) - g(x)\} \leqq \lim_{x\to 1^-} \left\{ (1-x)n_0 + \frac{x^{Mn_0}}{1+x+\cdots+x^{M-1}} \right\} = \frac{1}{M}.$$

But M was any positive number, whence $\lim\limits_{x\to 1^-} \{h(x) - g(x)\} = 0$. It follows that $\lim\limits_{x\to 1^-} \{h(x) - y(x)\} = 0$, and $y(x)$ accordingly oscillates between the same limits as $h(x)$.

6. *Limits of Oscillation.* It is interesting to see what can be done toward determining these limits. Hardy has shown that for the series $\sum\limits_{n=0}^{\infty} (-1)^n x^{n!}$ they are actually 0 and 1, and the same is true for any series whose exponents increase at a more rapid rate than $n!$. For the series $y(z) = \sum\limits_{n=0}^{\infty} (-1)^n x^{p^n}$, since $x > y(x) > x - x^p$, it is evident that 0 and 1 constitute lower and upper bounds; but a somewhat closer approximation may be obtained, as follows. Consider the ratio $m(x) = (x^p - x^{p^2})/(x - x^p)$. Its first derivative, with respect to x, is $[(p-1)x^p/(x-x^p)^2]\psi(x^{p-1})$, where $\psi(t) = 1 - (p+1)t^p + pt^{p+1}$. Now $\psi'(t) - p(p+1)t^{p-1}(t-1)$ is negative if $t < 1$, and furthermore, $\psi(1) - \psi(t) = (1-t)\psi'(\eta) < 0$, where $t < \eta < 1$. Thus for $t < 1$, $\psi(t) > \psi(1) = 1 - (p+1) + p = 0$, and since $x^{p-1} < 1$, it follows that

$dm(x)/dx > 0$, and $m(x)$ is a function increasing with x. Then $m(x^p)$ increases with x^p, and so also with x, and in general $m(x^{p^k})$ increases with x.

Now suppose that the expression $(x^{p^{2k}} - x^{p^{2k+1}})/(x - x^p)$ increases with x. If so, the product

$$\frac{x^{p^{2k}} - x^{p^{2k+1}}}{x - x^p} \cdot \frac{x^{p^{2k+1}} - x^{p^{2k+2}}}{x^{p^{2k}} - x^{p^{2k+1}}} \cdot \frac{x^{p^{2k+2}} - x^{p^{2k+3}}}{x^{p^{2k+1}} - x^{p^{2k+2}}} = \frac{x^{p^{2k+2}} - x^{p^{2k+3}}}{x - x^p}$$

must also increase with x. But the supposition is true for $k = 1$, since $(x^{p^2} - x^{p^3})/(x - x^p) = m(x) \cdot m(x^p)$ certainly increases with x; hence it is true by induction for any positive integer k. From this we conclude, since

$$\frac{y(x)}{x - x^p} = 1 + \sum_{k=1}^{\infty} \frac{x^{p^{2k}} - x^{p^{2k+1}}}{x - x^p},$$

that $y(x)/(x - x^p)$ is a function increasing with x, and hence that $y(x)/(x - x^p) > y(x^p)/(x^p - x^{p^2})$. Observing now that $y(x^p) = x - y(x)$, we have, on the one hand, $\{(x^p - x^{p^2})/(x - x^p) + 1\}y(x) > x$, or $y(x) > x[(x - x^p)/(x - x^{p^2})]$, and on the other hand, $[(x^p - x^{p^2})/(x - x^p)] \times \{x - y(x^p)\} > y(x^p)$, or $y(x^p) < x\,[(x^p - x^{p^2})/(x - x^{p^2})]$. Taking derivatives of numerator and denominator in these expressions, we find,

$$\lim_{x \to 1^-} \left\{ x \, \frac{x - x^p}{x - x^{p^2}} \right\} = \lim_{x \to 1^-} \left\{ \frac{1 - px^{p-1}}{1 - p^2 x^{p^2-1}} \right\} = \frac{1}{1+p},$$

and

$$\lim_{x \to 1^-} \left\{ x \, \frac{x^p - x^{p^2}}{x - x^{p^2}} \right\} = \lim_{x \to 1^-} \left\{ \frac{px^{p-1} - p^2 x^{p^2-1}}{1 - p^2 x^{p^2-1}} \right\} = \frac{p}{1+p}.$$

Thus, $\liminf\limits_{x \to 1^-} y(x) \geqq 1/(1+p)$, and $\limsup\limits_{x \to 1^-} y(x) \leqq p/(1+p)$.

It will be seen that as $p \to 1$ these limits both approach $\frac{1}{2}$. It should be remarked, however, that they do not seem to be very close approximations to the actual limits of oscillation, as is evidenced by numerical calculation for certain particular values of p. The series $y(x)$ converges so slowly for values of x near 1 that this calculation would be very tedious but for a device ascribed by Hardy to Wedderburn. We set up the companion series,

$$g(x) = \frac{1}{2} + \frac{\log x}{1+p} + \frac{1}{2!}\,\frac{(\log x)^2}{1+p^2} + \frac{1}{3!}\,\frac{(\log x)^3}{1+p^3} + \cdots,$$

which is convergent for $0 < x$. If we then set $F(x) = y(x) - g(x)$, we have, $F(x) + F(x^p) = y(x) - g(x) + y(x^p) - g(x^p)$

$$= x - \left\{ 1 + \frac{(1+p)\log x}{1+p} + \frac{1}{2!}\,\frac{(1+p^2)(\log x)^2}{1+p^2} + \cdots \right\} = x - e^{\log x} = 0.$$

Now, [illegible] $g(x)$ [illegible] zero [illegible] as

$x \to 1$, $F(x)$ will oscillate about zero with the same amplitude. Further, since $F(x) + F(x^p) = 0$, we have $F(x) = F(x^{p^2}) = -F(x^{p^3})$ for any positive integer k. While $y(x)$ converges slowly for values of x near unity, $g(x)$ converges there very rapidly. A proper choice of k, however, yields [illegible] $y(x^{p^k})$ and $g(x^{p^k})$, both of which converge fairly quickly. Thus the relation

$$y(x) = F(x) + g(x) = -F(x^{p^k}) + g(x) = y(x^{p^k}) - g(x^{p^k}) + g(x)$$

renders the calculation of $y(x)$ comparatively simple for values of x near unity. In the figure are shown graphs of $F(x)$ for three values of p, namely $p = 2.0$, 1.75, and 1.5. It will be noted that the amplitude of the oscillation for $p = 2.0$ is of the order of five times that for $p = 1.75$, and one hundred times that for $p = 1.5$. In the latter case then, the actual range of oscillation of $y(x)$ at $x = 1$ is approximately from 0.499978 to 0.500022, whereas the bounds given by the expressions above are 0.4 and 0.6.

7. *Another Type of Oscillating Series.* In the article mentioned above Hardy also discusses the series

$$M(a) = 1/(1+a) - 1/(1+pa) + 1/(1+p^2a) - + \cdots, \; p > 1.$$

He shows that this series likewise oscillates in the limit as a approaches zero along positive real values. The behavior of $M(a)$ near $a = 0$ is closely analogous to that of $y(x)$ near $x = 1$. In this connection it is interesting to note that the series $M(a)$ may be obtained from $y(x)$ by a rather simple transformation. The series

$$tx^{t-1}y(x) = t\{x^t - x^{t-1+p} + x^{t-1+p^2} - + \cdots\}, \qquad t > 0 \text{ fixed},$$

is uniformly convergent in the closed interval $0 \leq x \leq \xi < 1$. Therefore we can write

$$t\int_0^{\xi} x^{t-1}y(x)\,dx = t\int_0^{\xi} x^t\,dx - t\int_0^{\xi} x^{t-1+p}\,dx + t\int_0^{\xi} x^{t-1+p^2}\,dx - + \cdots$$
$$= \frac{t\xi^{t+1}}{t+1} - \frac{t\xi^{t+p}}{t+p} + \frac{t\xi^{t+p^2}}{t+p^2} - + \cdots, \qquad 0 < \xi < 1.$$

[illegible]

whence it appears that our series is uniformly convergent for $0 \leqq \xi \leqq 1$. We may therefore pass to the limit, and write

$$t \int_0^1 x^{t-1} y(x)\, dx = t/(t+1) - t/(t+p) + t/(t+p^2) - + \cdots.$$

If we now set $1/t = a$, we have the desired expression,

$$M(a) = 1/(1+a) - 1/(1+pa) + 1(1+p^2 a) - + \cdots.$$

We proceed to develop some of the more interesting properties of this series. In the first place,

$$M(a/p^2) = 1/[1+(a/p^2)] - 1/[1+(a/p)] + M(a) > M(a),$$

and therefore the sequence, $M(a)$, $M(a/p^2)$, $M(a/p^4)$, $\cdots$, is monotone increasing. Since $M(a/p^{2n}) < 1$ for every n, it is also bounded above, and so possesses a limit. We set, $\lim_{n\to\infty} M(a/p^{2n}) = \mu(a)$. This limit-function plays a part similar to that of $F(x)$. It is evident that $\mu(a) = \mu(p^2 a)$; and also, since

$$M(a/p^{2n}) = 1/[1+(a/p^{2n})] - M(a/p^{2n-1}) = 1/[1+(a/p^{2n})] - M(pa/p^{2n}),$$

we obtain, when n is allowed to increase indefinitely, the relation $\mu(a) = 1 - \mu(pa)$.

Again, since

$$\mu(a) = M(a) + \left[\frac{1}{1+a/p^2} - \frac{1}{1+a/p}\right] + \left[\frac{1}{1+a/p^4} - \frac{1}{1+a/p^3}\right] + \cdots,$$

and since

$$\frac{1}{1+a/p^{2n}} - \frac{1}{1+a/p^{2n-1}} = \left[1 - \frac{1}{1+a/p^{2n-1}}\right] - \left[1 - \frac{1}{1+a/p^{2n}}\right]$$
$$= \frac{a/p^{2n-1}}{1+a/p^{2n-1}} - \frac{a/p^{2n}}{1+a/p^{2n}} = \frac{1}{1+p^{2n-1}/a} - \frac{1}{1+p^{2n}/a},$$

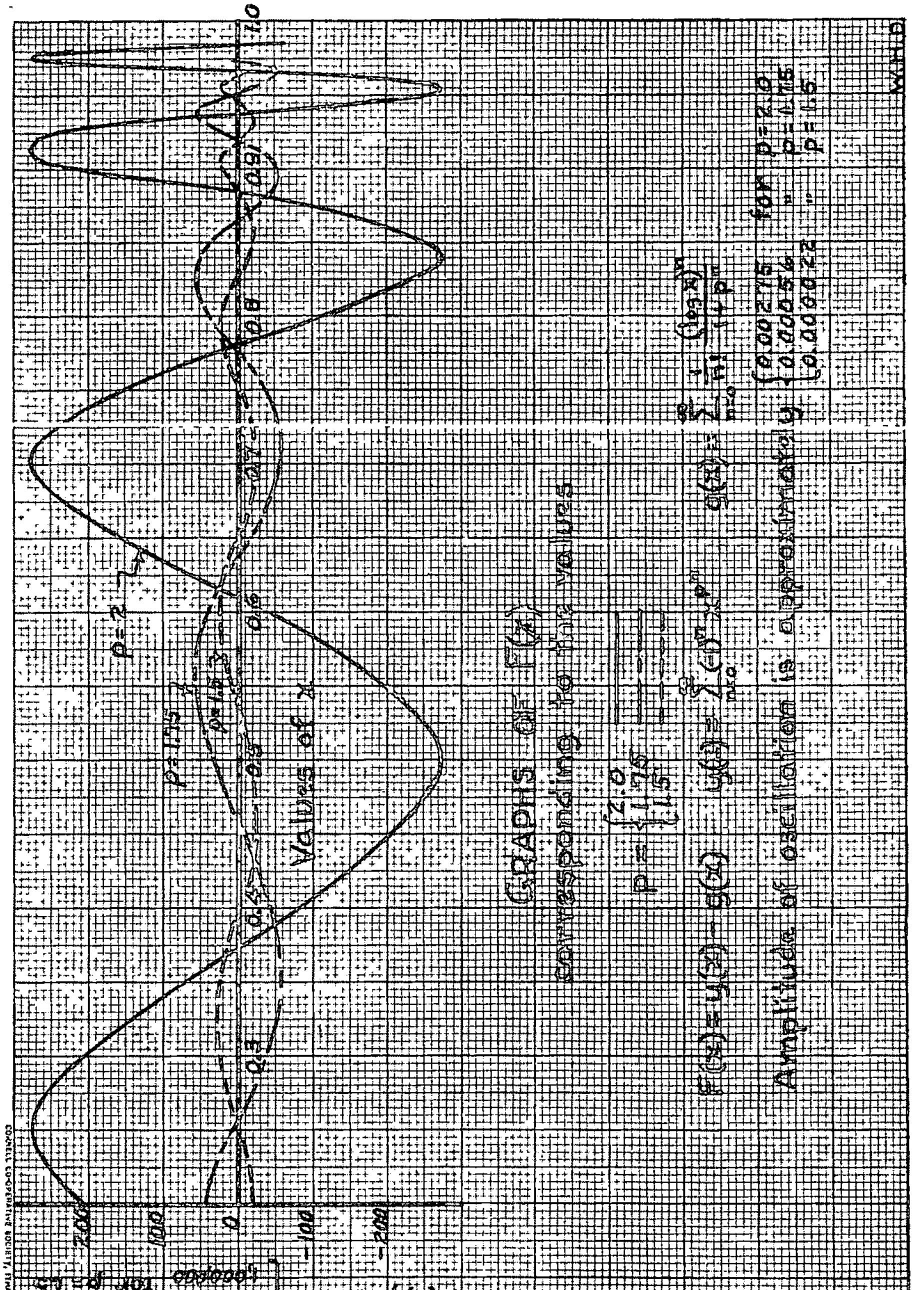
we have

$$\mu(a) = M(a) + \left[\frac{1}{1+p/a} - \frac{1}{1+p^2/a}\right] + \left[\frac{1}{1+p^3/a} - \frac{1}{1+p^4/a}\right] + \cdots,$$

or $\mu(a) = M(a) + M(p/a)$. From this relation we deduce

$$\mu(a) = M(a) + M(p/a) = M(p/a) + M[p/(p/a)] = \mu(p/a).$$

By means of these equations we can readily find the value of $\mu(a)$ when


GRAPHS OF F(x)
corresponding to the values
P = {2.0, 1.75, 1.5
F(x) = y(x) − g(x)
y(x) = Σ (−1)^n x^(p^n)
g(x) = Σ (1/n!) (log x)^n / (1 + p^n)
Amplitude of oscillation is approximately {0.00275 for p = 2.0; 0.00056 " p = 1.75; 0.000022 " p = 1.5
Values of x
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
p = 2
p = 1.75
p = 1.5
F(x), Unit is {1/100,000 for p = 2, 1.75; 1/1,000,000 for p = 1.5
200
100
0
−100
−200
W.H.D.

A NEW CALCULUS OF NUMERICAL FUNCTIONS.

By D. H. Lehmer.*

1. *Introduction.* In the theory of numerical functions there are two outstanding types of finite integration, namely

$$(1) \qquad \sum_{\nu=0}^{n} f(\nu) \qquad \text{and} \qquad (2) \qquad \sum_{\delta \mid n} f(\delta)$$

where δ runs over the divisors of n. The first of these processes leads to the calculus of finite differences, while the second one gives rise to a calculus of numerical functions which has been extensively studied.† If we make the sums (1) and (2) homogeneous as follows:

$$(3) \qquad \sum_{\nu=0}^{n} f(\nu) g(n-\nu) \qquad \text{and} \qquad (4) \qquad \sum_{\delta \mid n} f(\delta) g(n/\delta)$$

we obtain formulas for the n-th coefficient of the product of two power or Dirichlet series respectively. Both these sums are of the form $\sum f(a) g(b)$, where (a, b) are solutions of $\psi(a, b) = n$, for $\psi(a, b) = a + b$ and ab respectively.‡

It is the purpose of the present investigation to develop the fundamentals of a new calculus of numerical functions for which $\psi(a, b) = [a, b]$, the least common multiple of a and b. That is, we shall discuss sums of the form $\sum_n f(i) g(j)$, where it will be understood that for the subscript n, the sum extends over the solutions of $[i, j] = n$. Thus for $n = 6$

$$\begin{aligned} \sum_6 f(i) g(j) = f(1) g(6) &+ f(2) g(3) + f(2) g(6) \\ &+ f(3) g(2) + f(3) g(6) + f(6) g(1) \\ &+ f(6) g(2) + f(6) g(3) + f(6) g(6). \end{aligned}$$

2. *The function $d(i, n)$.* In what follows we suppose that $n = \prod_{\nu=1}^{t} p_\nu^{a_\nu}$ where p_ν are distinct primes. The function $d(i, n)$ may be defined as follows: $d(i, n) = 0$, if i is not a divisor of n. Otherwise $d(i, n)$ is the largest divisor d of i for which n/d is prime to d. To obtain a formula for $d(i, n)$ suppose

* National Research Fellow.

† Dickson, *History of the Theory of Numbers*, Vol. 1, Ch. [illegible]

‡ For another type of $\psi(x, y)$, see N. V. Bervy, *Matematicheskii Sbornik*, Vol. 18, pp. 519-[illegible].

that $i = \prod_{\nu=1}^{t} p_\nu^{\beta_\nu}$ where $0 \leqq \beta_\nu \leqq \alpha_\nu$, and that ν_k $(k = 1, 2, 3, \cdots)$ are those values of ν for which $\beta_\nu = \alpha_\nu$. Then $d(i, n) = \prod_{\nu=\nu_k} p_\nu^{\alpha_\nu}$. If $\alpha_\nu = 1$, $(\nu = 1, 2, \cdots t)$ then n is called a *simple number*. If n is simple, $d(i, n) = i$. If n is a power of a prime, then $d(i, n) = 1$ or n according as $i < n$, or $i = n$.

THEOREM 1. *If i_0 is a fixed divisor of n, then the solutions j of $[i_0, j] = n$ are $j = n/\delta$, where δ are the divisors of $d(i_0, n)$.*

Proof. If $i_0 = 1$, the theorem is obvious since $d(1, n) = 1$. For $i_0 > 1$, let us order the prime factors p_ν of $n = \prod p_\nu^{\alpha_\nu}$ in such a way that

$$i_0 = \prod_{\nu=1}^{h} p_\nu^{\alpha_\nu} \prod_{\nu=h+1}^{t} p_\nu^{\beta_\nu},$$

where $\beta_\nu < \alpha_\nu$. Any solution j of $[i_0, j] = n$, must be a divisor of n. Let $j = \prod_{\nu=1}^{t} p_\nu^{\gamma_\nu}$, then it follows from the definition of the L. C. M. that γ_ν must satisfy the following requirements

$$\text{(I)} \qquad \max.\,(\alpha_\nu, \gamma_\nu) = \alpha_\nu \qquad (\nu = 1, 2, \cdots, h)$$

$$\text{(II)} \qquad \max.\,(\beta_\nu, \gamma_\nu) = \alpha_\nu \qquad (\nu = h+1, \cdots, t)$$

(I) is satisfied for any γ_ν such that $0 \leqq \gamma_\nu \leqq \alpha_\nu$, while (II) requires $\gamma_\nu = \alpha_\nu$. Hence $j = \prod_{\nu=1}^{h} p_\nu^{\gamma_\nu} \prod_{\nu=h+1}^{t} p_\nu^{\alpha_\nu} = n/\delta$, where δ divides $\prod_{\nu=1}^{h} p_\nu^{\alpha_\nu} = d(i_0, n)$. Hence the theorem.

COROLLARY 1. *If $n = p^\alpha$, then*

$$(5) \qquad \sum_{p^\alpha} f(i)g(j) = f(p^\alpha) \sum_{s=0}^{\alpha-1} g(p^s) + g(p^\alpha) \sum_{s=0}^{\alpha-1} f(p^s) + f(p^\alpha)g(p^\alpha)$$

Proof. Since in this case $d(i, n) = 1$, or n according as $i < n$, or $i = n$, the corollary follows from the theorem by collecting the coefficients of $f(p^s)$.

3. *Factorable Functions.* A numerical function $f(n)$ is said to be factorable if $f(1) = 1$, and

$$f(n)f(m) = f(mn)$$

for every pair of coprime integers m, n. In particular

$$f(n) = \prod_{\nu=1}^{t} f(p_\nu^{\alpha_\nu}).$$

THEOREM 2. *If $f(n)$ and $g(n)$ are factorable functions then $F(n) = \sum_n f(i)g(j)$ is also factorable.*

Proof. Let m, n be any pair of integers for which $(m, n) = 1$, then

$$F(m)F(n) = \sum_m f(i_1)g(j_1) \sum_n f(i_2)g(j_2).$$

Since $(i_1, i_2) = (j_1, j_2) = 1$, it follows that

$$\text{(S)} \qquad F(m)F(n) = \sum f(i_1 i_2)g(j_1 j_2).$$

Since also $(i_1, j_2) = (i_2, j_1) = 1$, we have

$$[i_1 i_2, j_1 j_2] = [i_1, j_1] \cdot [i_2, j_2] = mn.$$

Therefore every term of the above double sum (S) appears in

$$\text{(S}') \qquad F(mn) = \sum_{mn} f(i)g(j).$$

Conversely if $[i, j] = mn$, then $[(m, i), (m, j)] = m$ and $[(n, i), (n, j)] = n$. Hence if $i_1 = (m, i)$, $i_2 = (n, i)$, $j_1 = (m, j)$, $j_2 = (n, j)$, then $[i_1, j_1] = m$ and $[i_2, j_2] = n$, so that every term in (S$'$) is in (S). Therefore

$$F(m)F(n) = \sum_{mn} f(i)g(j) = F(mn)$$

which is the theorem.

THEOREM 3. *If $f(n)$ and $g(n)$ are factorable functions, then*

$$(6) \qquad \sum_n f(i)g(j) = \prod_{\nu=1}^{t} \{f(p_\nu^{a_\nu}) \sum_{s=0}^{a_\nu - 1} g(p_\nu^s) + g(p_\nu^{a_\nu}) \sum_{s=0}^{a_\nu} f(p_\nu^s)\}.$$

This follows immediately from Theorem 2, and Corollary 1.

4. *Integration.* The following lemma which properly belongs to section 8 is interpolated here to facilitate the proofs of Theorems 5, 9, 11, and 12.

LEMMA. *If $f(n)$, $g(n)$ and $h(n)$ are any numerical functions, then*

$$\begin{aligned}\sum_n f(i) \sum_j g(j_1)h(j_2) &= \sum_n g(i) \sum_j f(j_1)h(j_2) \\ &= \sum_n h(i) \sum f(j_1)g(j_2)\end{aligned}$$

where $[j_1, j_2] = j$.

Proof. Consider all one rowed matrices (i_1, i_2, i_3) for which $[i_1, i_2, i_3] = n$. Then it is seen that each of the sums of the theorem is equal to

$$\sum f(i_1)g(i_2)h(i_3) \qquad [i_1, i_2, i_3] = n.$$

Let us now consider in detail the case in which $g(n) = 1$, or in other words sums of the form

$$(7) \qquad \sum_n f(i) = F(n).$$

The operation which carries f into F is a type of finite integration, analogous

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to the well known types (1) and (2), and may be referred to as L. C. M. integration, or simply integration when no confusion can arise.

The sum of the type (2) may be reduced to the type (1) as follows

$$\sum_{\delta \mid n} f(\delta) = \sum_{\nu=1}^{n} f(\nu)\epsilon(n/\nu)$$

where $\epsilon(x) = 1$, or 0 according as x is or is not an integer, but it is seldom advantageous to discuss a sum over divisors in this form. In a similar way a sum of the type (7) may be written

$$\Sigma_n f(i) = \sum_{\delta \mid n} f(\delta)\mu_1(d(\delta, n)) \tag{8}$$

where $\mu_1(n)$ is the number of divisors of n. Although this is a sum over divisors it is not always easily dealt with as such.

5. *The Inversion Function.* In the present theory the following function takes the place of Möbius' μ function.

THEOREM 4. *Let* $M(n) = \prod_{\nu=1}^{t} \{(\alpha_\nu + 1)^{-1} - \alpha_\nu^{-1}\}$, $M(1) = 1$, *then* $\Sigma_n M(i) = [1/n]$.

Proof. Obviously M is factorable and the theorem is true for $n = 1$. If $n > 1$, by (6)

$$\begin{aligned}\Sigma_n M(i) &= \prod_{\nu=1}^{t} \{M(1) + M(p_\nu) + \cdots + M(p_\nu^{\alpha_\nu - 1}) + (\alpha_\nu + 1)M(p_\nu^{\alpha_\nu})\} \\ &= \prod_{\nu=1}^{t} \{1 + \tfrac{1}{2} - 1 + \tfrac{1}{3} - \tfrac{1}{2} + \cdots \\ &\qquad + \alpha_\nu^{-1} - (\alpha_\nu - 1)^{-1} + (\alpha_\nu + 1)((\alpha_\nu + 1)^{-1} - \alpha_\nu^{-1})\} = 0;\end{aligned}$$

hence the theorem.

THEOREM 5. *If* $F(n) = \Sigma_n f(i)$, *then* $f(n) = \Sigma_n F(i) M(j)$.

Proof.
$$\begin{aligned}\Sigma_n F(i) M(j) &= \Sigma_n M(j) \Sigma_i f(i_1) \\ &= \Sigma_n f(j) \Sigma_i M(i_1).\end{aligned}$$

This follows from the Lemma with i and j interchanged and one of the functions identically 1. But $\Sigma_i M(i_1) = [1/i]$. Hence the theorem.

This theorem is an analogue of Dedekind's inversion formula, and gives an explicit formula for the derivative of F.

6. *Successive Integrals.* Any numerical function determines a sequence

$$\cdots f_{-3}(n),\quad f_{-2}(n),\quad f_{-1}(n),\quad f_0(n),\quad f_1(n),\quad f_2(n), \cdots \tag{9}$$

in which $f_{r+1}(n) = \sum_n f_r(i)$ or, what is the same, $f_{r-1}(n) = \sum_n f_n(i) M(j)$.

THEOREM 6. *If any f_r is factorable, so is every function of the sequence* (9).

This follows at once from Theorem 2.

THEOREM 7.* *Let f and g be two factorable functions and let*

$$f(p^\alpha) = \rho(p^{\alpha+1}) - \rho(p^\alpha)$$
$$g(p^\alpha) = \theta(p^{\alpha+1}) - \theta(p^\alpha),$$

$\rho(1) = \theta(1) = 0$. *Finally if $F(n) = \sum f(i) g(j)$ then*

$$F(p^\alpha) = \rho(p^{\alpha+1})\theta(p^{\alpha+1}) - \rho(p^\alpha)\theta(p^\alpha).$$

Proof. By Theorem 3,

$$\begin{aligned} F(p^\alpha) &= \{\rho(p^{\alpha+1}) - \rho(p^\alpha)\}\{1 + \sum_{s=1}^{\alpha-1} (\theta(p^{s+1}) - \theta(p^s))\} \\ &\quad + \{\theta(p^{\alpha+1}) - \theta(p^\alpha)\}\{1 + \sum_{s=1}^{\alpha} (\rho(p^{s+1}) - \rho(p^s))\} \\ &= \{\rho(p^{\alpha+1}) - \rho(p^\alpha)\}\,\theta(p^\alpha) + \{\theta(p^{\alpha+1}) - \theta(p^\alpha)\}\,\rho(p^{\alpha+1}) \\ &= \rho(p^{\alpha+1})\theta(p^{\alpha+1}) - \rho(p^\alpha)\theta(p^\alpha), \end{aligned}$$

which is the theorem.

To study the properties of the general sequences (9) it is first necessary to consider the special case in which $f_0(n) = [1/n]$. The r-th member of this special sequence we denote by $M_r(n)$.

THEOREM 8. *If $M_0(n) = [1/n]$, then* $\prod_{\nu=1}^{t} \{(\alpha_\nu + 1)^r - \alpha_\nu^r\} = M_r(n)$.

Proof. In Theorem 7 put $\theta(p^\alpha) = \alpha$. Then $g(n) = 1$. Assume the above expression for $M_r(n)$. Then if we set $\rho(p^\alpha) = \alpha^r$ we have $f(n) = M_r(n)$ and by Theorem 7, $M_{r+1}(p^\alpha) = \sum_{p^\alpha} M_r(i) = (\alpha+1)^{r+1} - \alpha^{r+1}$. If we put $\theta(p^\alpha) = \alpha^{-1}$, then $g(n) = M(n)$ and for the same choice of $\rho(p^\alpha)$, $M_{r-1}(p^\alpha) = \sum_{p^\alpha} M_r(i) M(j) = (\alpha+1)^{r-1} - \alpha^{r-1}$. But the theorem is true for $r = 0$. Hence the induction is complete. It is to be observed that

$$M(n) = M_{-1}(n), \qquad M_r(1) = 1.$$

The formula for $M_r(n)$ must not be used for $n = 1$, $r < 1$.

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* An equivalent statement of this theorem: $\Sigma F(\delta) = \Sigma f(\delta) \Sigma g(\delta)$, $\delta \mid n$, has been [illegible] by von Sterneck, *Monatshefte für Mathematik und Physik*, Vol. 5 (1894), [illegible] for non-factorable functions as well. Von Sterneck's proof is not adequate however. A rigorous proof of his general theorem will appear in a forthcoming paper.

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THEOREM 9. *If $f_0(n)$ is any numerical function, then*

$$f_r(n) = \sum_n f_0(i) M_r(j).$$

Proof. The theorem is trivial for $r = 0$. Assuming it true for $f_r(n)$, we may write, (using the lemma with $g(n) = 1$)

$$\begin{aligned} f_{r+1}(n) &= \sum_n f_r(j) = \sum_n \sum_j f_0(j_1) M_r(j_2) \\ &= \sum_n f_0(i) \sum_j M_r(j_1) = \sum_n f_0(i) M_{r+1}(j). \end{aligned}$$

For $f_{r-1}(n)$ we have

$$\begin{aligned} f_{r-1}(n) &= \sum_n M_{-1}(i) f_r(j) = \sum_n M_{-1}(i) \sum_j f_0(j_1) M_r(j_2) \\ &= \sum_n f_0(i) \sum_j M_{-1}(j_1) M_r(j_2) = \sum_n f_0(i) M_{r-1}(j). \end{aligned}$$

This proves the theorem.

Since $f_r(n)$ defines the same sequence (9) as does $f_0(n)$ we may put the preceding theorem into the following form.

THEOREM 10. *If $r = r_1 + s_1 = r_2 + s_2$ be any two partitions of r, then*

$$f_r(n) = \sum_n f_{r_1}(i) M_{s_1}(j) = \sum_n f_{r_2}(i) M_{s_2}(j).$$

More generally we have

THEOREM 11. *If $r = r_1 + s_1 = r_2 + s_2$, then for any pair of numerical functions $f_0(n)$ and $g_0(n)$,*

$$\sum_n f_{r_1}(i) g_{s_1}(j) = \sum_n f_{r_2}(i) g_{s_2}(j).$$

Proof.

$$\begin{aligned} \sum_n f_{r_1}(i) g_{s_1}(j) &= \sum_n f_{r_1}(i) \sum_j g_0(j_1) M_{s_1}(j_2) \\ &= \sum_n g_0(i) \sum_j f_{r_1}(j_1) M_{s_1}(j_2) = \sum_n g_0(i) f_r(j). \end{aligned}$$

The last two equalities follow from the Lemma and from Theorem 9. Since the last result does not depend on r_1 and s_1, but only on their sum, the theorem follows.

The preceding theorem justifies the following notation

$$F_{r+s}(n) = \sum_n f_r(i) g_s(j).$$

As $r + s$ varies we obtain a sequence

$$(10) \quad \ldots F_{-3}(n), F_{-2}(n), F_{-1}(n), F_0(n), F_1(n), F_2(n), F_3(n), \cdots.$$

The relation between any two elements of the sequence (10) is the same as that between corresponding elements of (9). To show this it is sufficient to prove the following:

THEOREM 12. *If $F_{r+s}(n) = \sum_n f_r(i) g_s(j)$ where f_0 and g_0 are two arbitrary functions, then*

$$F_{r+1}(n) = \sum_n F_r(i).$$

Proof. By definition

$$\begin{aligned} F_{r+1}(n) &= \sum_n f_0(i) g_{r+1}(j) = \sum_n f_0(i) \sum_j g_r(j_1) \\ &= \sum_n \sum_j f_0(j_1) g_r(j_2) = \sum_n F_r(j), \end{aligned}$$

which is the theorem.

7. *Multiple Integrals.* It is possible to give another interpretation to the sequence (9) for $r > 0$ by considering all the one rowed matrices $(i_0, i_1, i_2, \cdots, i_r)$ of positive integers whose L. C. M. $[i_0, i_1, i_2, \cdots, i_r] = n$.

THEOREM 13. *If $f_r(n)$, $r > 0$, is a member of the sequence (9), then*

$$f_r(n) = \sum f_0(i_0), \qquad [i_0, i_1, \cdots, i_r] = n.$$

Proof. The theorem is obvious for $r = 1$ as it reduces to the definition of $f_1(n)$. Assuming the theorem true for $f_r(n)$ we have

$$\begin{aligned} f_{r+1}(n) &= \sum_n f_r(i) \\ &= \sum_n \sum f_0(i_0) \qquad [i_0, i_1, \cdots, i_r] = i \qquad [i, j] = n. \end{aligned}$$

If we write $j = i_{r+1}$, we have

$$n = [i, i_{r+1}] = [[i_0, i_1, \cdots, i_r], i_{r+1}] = [i_0, i_1, \cdots, i_{r+1}].$$

Then
$$f_{r+1}(n) = \sum f_0(i_0) \qquad [i_0, i_1, \cdots, i_{r+1}] = n.$$

This completes the induction.

If we set $f_0(n) = 1 = M_1(n)$, we find by applying Theorem 13 that the number of solutions of

$$[i_0, i_1, i_2, \cdots, i_r] = n$$

is $M_{r+1}(n)$. These matrices correspond to the r-th divisors* of n in the calculus of divisors.

8. *General r-th order Inversion.* From Theorems 9 and 13 we obtain the following generalization of Theorem 5.

THEOREM 14. *If*

$$F(n) = \sum f(i_0) \qquad [i_0, i_1, \cdots, i_r] = n$$

then

$$f(n) = \sum F(i_0) M(i_1) M(i_2) \cdots M(i_r)$$

* *American Journal of Mathematics*, Vol. 52 (1930), pp. 293-304.

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Proof. If we write f_0 for f, then $F = f_r$ by Theorem 13.

Since $[i_0, i_1, \cdots, i_{r-1}, i_r] = [i_0, [i_1, [i_2, [\cdots, [i_{r-1}, i_r]] \cdots]$, we may replace $[i_{k+1}, i_{k+2}, \cdots, i_r]$ by j_k, $k = 0, 1, \cdots, r-2$, and have $[i_k, j_k] = j_{k-1}$. In particular $[i_0, j_0] = n$. The sum in (12) may be written

$$\sum_n f_r(i_0) \sum_{j_0} M(i_1) M(i_2) \cdots M(i_r) =$$

$$\sum_n f_r(i_0) \sum_{j_0} M(i_1) \sum_{j_1} M(i_2) \cdots \sum_{j_{r-3}} M(i_{r-2}) \sum_{j_{r-2}} M(i_{r-1}) M(i_r). \tag{13}$$

But the very last sum has the value $M_{-2}(j_{r-2})$. The last two sums therefore may be replaced by

$$\sum_{j_{r-3}} M(i_{r-2}) M_{-2}(j_{r-2}) = M_{-3}(j_{r-3}).$$

Continuing this process the whole sum (13) is seen to collapse to

$$\sum_n f_r(i_0) M_{-r}(j_0) = f_0(n)$$

by Theorem 9. This proves Theorem 14.

9. *L. C. M. Multiplication.* The sums (11) and (12) are the simplest examples of the general type

$$F(n) = \sum f_1(i_1) f_2(i_2) f_3(i_3) \cdots f_r(i_r) \tag{14}$$

where $f_\nu(n)$ are any numerical functions (not necessarily belonging to a sequence (9)). The function F might be called the "L. C. M. product"* of the f's, and we might write $F = f_1 \cdot f_2 \cdot f_3 \cdots f_r$. Such a symbolic product enjoys all the properties of ordinary multiplication and with this operation as basis, it is possible to construct a theory of numerical functions of which the present paper is but a small part.

10. *Examples.* The following examples illustrate the nature of the functions in the sequence (9), which arise from a number of familiar factorable functions.

	$f_0(n)$	$f_r(p^\alpha)$
I	n^k	$p^{k\alpha}\alpha^r + M_r(p^\alpha)\sigma_k(p^\alpha)$
II	$\lambda(n)$	$\begin{cases} -\alpha^r & \alpha \text{ odd} \\ (\alpha+1)^r & \alpha \text{ even} \end{cases}$
III	$\mu(n)$	$\mu(p^\alpha)$
IV	$\phi^{(k)}(n)$	$p^{\alpha k}(\alpha+1)^r - p^{(\alpha-1)k}\alpha^r$
V	$2^{t(n)}$	$2\alpha^r + M_r(p^\alpha)M_2(p^\alpha)$
VI	$2^{t(n)}\lambda(n)$	$\lambda(p^\alpha)M_{2r}(p^\alpha)/M_r(p^\alpha)$.

* This is the analogue of what E. T. Bell designates ideal product in the divisor theory. Compare: *University of Washington Publications*, Vol. 1, No. 1, pp. 9, 10.

To illustrate L. C. M. products we give the following examples:

VII $\quad \sum_n 2^{t(i)}\mu_1(j) = \mu_1(n)\mu_1(n^3)$

VIII $\quad \sum_n 2^{t(i)}\lambda(i)\lambda(j) = \lambda(n)$

IX $\quad \sum_n 2^{t(i)}\lambda(i)2^{t(j)} = \lambda(n)\prod_{\nu=1}^{t(n)} 4\alpha_\nu$

X $\quad \sum_n f(i)\mu^2(j) = \prod_{\nu=1}^{t} \{2f(p_\nu^{\alpha_\nu}) + \mu^2(p_\nu^{\alpha_\nu})\},$

where f is any factorable function. The next 5 examples illustrate the L. C. M. powers of certain functions.

XI $\quad \sum_n \mu_1(i)\mu_1(j) = \mu_1{}^3(n)$

XII $\quad \sum_n 2^{t(i)}2^{t(j)} = 2^{t(n)}\mu_1(n^2)$

XIII $\quad \sum_n 2^{t(i)}\lambda(i)2^{t(j)}\lambda(j) = [1/n]$

XIV $\quad \sum\{\mu(i_0)\mu(i_1)\cdots\mu(i_r)\}^2 = \mu^2(n)M_{r+1}(n)$

XV $\quad \sum\phi(i_0)\phi(i_1)\cdots\phi(i_r) = \phi^{(r+1)}(n).$

In the last two formulas $[i_0, i_1, \cdots, i_r] = n$, $(r > 0)$. In the above table $\lambda(n) = \prod_{\nu=1}^{t(n)} (-1)^{\alpha_\nu}$ is Liouville's function, $\sigma_k(n)$ is the sum of k-th powers of the divisors of n, and $\phi^{(k)}(n) = n^k \prod_{\nu=1}^{t} (1 - p_\nu^{-k})$ is Jordan's generalized totient function.

11. *Periodic Sequences.* Thus far we have developed features of our calculus which have precise analogues in the divisor calculus. It would be misleading not to consider some of the essential dissimilarities of the two theories. The first of these has to do with periodicities in the sequence (9), and its divisor theory analogue,* namely the sequence obtained from $f_0(n)$ by repeated application of the operation (2). One may show that no sequence of this latter type is periodic except for the trivial case of $f_0(n) \equiv 0$. For the sequence (9) however we see from example III that the sequence defined by $f_0(n) = \mu(n)$ is periodic with the period 1. To show that this is essentially the only periodic sequence, we use

THEOREM 15. *If $f(n)$ satisfies the following conditions.*

$$f(n) = \sum_n f(i)M_r(j) \qquad (r \neq 0) \tag{15}$$

$$f(1) = 0 \tag{16}$$

then $f(n)$ vanishes identically.

* For a detailed account of this sequence see *American Journal of Mathematics,* Vol. 52 (1930), pp. 296-299.

Proof. Suppose that $f(n) \not\equiv 0$. Then an integer L exists such that

$$f(\nu) = 0, \quad \nu < L, \quad f(L) \neq 0, \quad L > 1. \tag{17}$$

In (15) set $n = L$, then in view of (17)

$$f(L) = \sum_L f(i) M_r(j) = f(L) \sum_{\delta | L} M_r(\delta). \tag{18}$$

But

$$\sum_{\delta | L} M_r(\delta) = \prod_{\nu=1}^{t} \sum_{s=0}^{\alpha_\nu} M_r(p^s) = \prod_{\nu=1}^{t} (\alpha_\nu + 1)^r = \mu_1^r(L).$$

Since $r \neq 0$, (18) implies that $\mu_1(L) = 1$, which contradicts the hypothesis $L > 1$. Hence the theorem.

THEOREM 16. *If* $f(n) = \sum_n f(i) M_r(j)$ $r \neq 0$ *then* $f(n) = f(1)\mu(n)$.

Proof. Let $f(n) - f(1)\mu(n) = \chi(n)$ *then*

$$\begin{aligned}\sum_n \chi(i) M_r(j) &= \sum_n f(i) M_r(j) - f(1) \sum_n \mu(i) M_r(j) \\ &= f(n) - f(1)\mu(n) = \chi(n)\end{aligned}$$

by hypothesis and example III. Moreover $\chi(1) = 0$ and therefore by Theorem 15, $\chi(n)$ vanishes identically. Hence the theorem follows.

Thus if any two members of a sequence (9) are equal, all members are equal and are proportional to the μ function. This property of μ is only a special case of

THEOREM 17. *Let* $g(n)$ *be any numerical function for which* $g(1) = 1$, *then* $\sum_n g(i)\mu(j) = \mu(n)$.

Proof. By Theorem 1

$$\sum_n g(i)\mu(j) = \sum_{i | n} g(i) \sum_{\delta | d(i,n)} \mu(n/\delta).$$

Since μ is factorable $\mu(n/\delta) = \mu(n/d(i,n))\mu(d(i,n)/\delta)$. Hence

$$\begin{aligned}\sum_n g(i)\mu(j) &= \sum_{i|n} g(i)\mu(n/d(i,n)) \sum_{\delta | d(i,n)} \mu(\delta) \\ &= \sum_{i|n} g(i)\mu(n/d(i,n))[1/d(i,n)].\end{aligned}$$

It follows that the coefficient of $g(i)$ is 0 or $\mu(n)$ according as $d(i,n) > 1$, or $= 1$. Hence, if n is not simple, $\mu(n) = 0$ and the whole sum vanishes proving the theorem. If n is simple we have seen that $d(i,n) = i$. In this case the only non-zero term is that for which $i = 1$. Hence $\sum_n g(i)\mu(j) = g(1)\mu(n) = \mu(n)$. This proves the theorem. By Theorem 16, μ is the only function which satisfies Theorem 17.

COROLLARY. *If $g(n)$ is any function at all, then*

$$\sum_n g(i)\mu(j) = g(1)\mu(n). \tag{19}$$

12. *The Equation $f\cdot g = h$.* In the preceding section we considered two special cases of the equation

$$\sum_n f(i)g(j) = h(n) \tag{20}$$

in which $h(n) = f(n)$. The equation (20) has other special cases which should be compared with those of its analogue

$$\sum_{\delta|n} f(\delta)g(n/\delta) = h(n). \tag{21}$$

For brevity we shall write (20) in the form $f\cdot g = h$, adopting the notation of section 8. Likewise (21) will be written $fg = h$.

Case I. $f = g = h$. It may be shown that the only solutions of $ff = f$ are $f(n) \equiv 0$, $f(n) = [1/n]$. The equation $f\cdot f = f$ has however an infinity of solutions. To exhibit infinitely many factorable solutions it is sufficient to select any finite or infinite increasing sequence $\{\nu_k\}$ $(k = 1, 2, 3, \cdots)$ of non-negative integers. Define $f(n)$ as that factorable function for which

$$f(p^\alpha) = \begin{cases} (-1)^k, & \alpha = \nu_k \\ 0, & \text{otherwise.} \end{cases}$$

Then by (5)

$$\sum_{p^\alpha} f(i)f(j) = f^2(p^\alpha) + 2f(p^\alpha)\sum_{s=0}^{\alpha-1} f(p^s). \tag{22}$$

If α is not a member of the sequence $\{\nu_k\}$, the right hand member of (22) vanishes. Otherwise it becomes for $\alpha = \nu_k$

$$1 + 2(-1)^k \sum_{a=0}^{k-1} (-1)^a = 1 + 2(-1)^k(1-(-1)^k)/2 = (-1)^k = f(p^{\nu_k})$$

Therefore $f\cdot f = f$. Special examples of these functions are $[1/n]$, $\mu(n)$, $\lambda(n)$. There are also infinitely many non-factorable functions for which $f\cdot f = f$.

Case II. $f = g$, $h(n) = [1/n] = M_0(n)$. The equation $ff = M_0$ has only two solutions $f(n) = \pm M_0(n)$. But $f\cdot f = M_0$ has infinitely many other solutions. To exhibit infinitely many factorable solutions choose any function $\theta(n)$ for which $\theta(1) = 1$, $\theta(n) = \pm 1$ $(n > 1)$. Define $f(n)$ as that factorable function for which

$$f(p^\alpha) = \theta(\alpha + 1) \quad \theta(\alpha), \qquad \alpha > 0.$$

Then by Theorem 7

$$\sum_{p^a} f(i)f(j) = \theta^2(\alpha+1) - \theta^2(\alpha) = 0.$$

Obviously for $n = 1$, $f \cdot f = 1$. Hence for any such f, $f \cdot f = M_0$. The only never vanishing factorable solution of this equation is $f(n) = 2^{t(n)}\lambda(n)$, obtained from $\theta(n) = (-1)^{n+1}$ (Example XIII).

Case III. $h = 0$. The equation $fg = 0$ implies either $f(n) = 0$ or $g(n) = 0$, or both. However one may find infinitely many solutions of $f \cdot g = 0$ in which neither function is identically zero. In fact it follows from the corollary of Theorem 17 that if $f = \mu$, then $f \cdot g = 0$ is satisfied by every function g, which vanishes for $n = 1$.

The discussion of the infinite series aspect of the above theory will be included in a paper on infinite series in general.

ON CERTAIN IDENTITIES IN THETA FUNCTIONS.

By Russell Smith Park.

1. *Introduction.* The object of this paper is to obtain certain identities in Jacobi's theta functions, by a method essentially the same as that used by Latimer in his paper, "On Certain Identities in Theta Functions." * In that paper sets of integral elements in generalized quaternion algebras were used in which all the parameters α, β, γ were odd. It is proposed in this paper to obtain similar identities when $\gamma = 2\sigma$, where σ is an odd integer. The sets of integral elements, to be used in obtaining the first identities, were obtained by Miss Darkow † for a generalized quaternion algebra A. The elements to be used in obtaining the last two identities are algebraic numbers.

2. *Change of notation and definition of sets.* Consider the rational algebra A_1 with the basal numbers

$$1, \qquad I = (\beta\gamma)^{1/2} i, \qquad J = (\gamma\alpha)^{1/2} j, \qquad K = (\alpha\beta)^{1/2} k,$$

where $\gamma = 2\sigma$ and α, β, σ are positive odd integers such that $\alpha\beta\sigma$ contains no square factor > 1, and i, j, k are the usual quaternion units.

In Miss Darkow's algebra A the multiplication table involved two even parameters α and β which were factored into

$$\alpha = 2\mu\delta, \qquad \beta = 2\nu\delta,$$

where μ, ν, δ were odd integers and 2δ was the positive g. c. d. of α and β. If we assume that $\alpha < 0$, $\beta > 0$, and in the multiplication table of A replace μ, ν, δ, e_1, e_2, e_3 by $-\beta$, α, σ, I, J, $2\sigma K$ respectively it will be seen that A is equivalent to A_1. The multiplication table of A_1 is

$$I^2 = -\beta\gamma, \qquad J^2 = -\gamma\alpha, \qquad K^2 = -\alpha\beta,$$
$$IJ = -JI = \gamma K, \quad JK = -KJ = \alpha I, \quad KI = -IK = \beta J.$$

Let $\alpha = A\alpha'$, $\beta = B\beta'$, $\sigma = C\sigma'$ where A, B, C are defined as follows. A is the product of all the prime factors A_i of α such that $-2\beta\sigma$ is a quadratic non-residue of every A_i; or $A = 1$ if α contains no such factor. Let B, C be similarly defined by a cyclic interchange of A, B, C; A_i, B_i, C_i; α, β, 2σ.

* *Transactions of the American Mathematical Society*, Vol. 22 (1920), pp. [illegible].
† *Annals of Mathematics*, Vol. 28 (1926-27), pp. 263-270.

The properties of the integral elements of A obtained by Miss Darkow will be enumerated in the new notation. There are odd integers E_i, F_i, H_i prime to β', α', σ' such that

$$1 + 2\alpha\sigma E_i^2 \equiv 0 \bmod \beta', \quad 1 + 2\beta\sigma F_i^2 \equiv 0 \bmod \alpha', \quad 1 + \alpha\beta H_i^2 \equiv 0 \bmod \sigma'.$$

Define the class ijk of algebras A_1 as the class of algebras for which

$$-\beta\sigma \equiv i \bmod 8, \quad \alpha\sigma \equiv j \bmod 8, \quad -\alpha\beta \equiv k \bmod 8.$$

Since $ij \equiv k \bmod 8$, algebras of four types are defined.

A_1 of type	A_1 of class.			
A	155	375	515	735
B	133	357	573	717
C	111	331	551	771
D	177	313	537	753

Define set S' as the totality of elements in the form

$$\Omega = \tfrac{1}{2}\left[\xi - \frac{\eta}{2\beta'\sigma'} I + \frac{\zeta}{2\alpha'\sigma'} J - \frac{\lambda}{\alpha'\beta'} K\right] \tag{1}$$

where ξ, η, ζ, λ are integers which satisfy one congruence of each pair of the equivalent congruences

$$\begin{array}{lcl} \eta \equiv 2A\sigma' E\lambda & \text{or} & \lambda \equiv -\alpha' CE\eta \bmod \beta', \\ \zeta \equiv 2B\sigma' F\lambda & \text{or} & \lambda \equiv -\beta' CF\zeta \bmod \alpha', \\ \zeta \equiv B\alpha' H\eta & \text{or} & \eta \equiv -\beta' AH\zeta \bmod \sigma', \end{array} \tag{2}$$

and also satisfy condition (3) below. E, F, H are arbitrarily chosen E_i, F_i, H_i mentioned above. Let S'', S''', S^{IV}, S^{V}, S^{VI} be defined in the same way except that instead of (3) we employ (4), (5), (6), (7), (8) respectively.

$$\xi \equiv \lambda \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4; \tag{3}$$

$$\begin{array}{lll} \xi \equiv \lambda \equiv 1 \bmod 2, & \eta, \zeta \text{ even}, & \zeta \equiv \eta + 2 \bmod 4, \\ \text{or } \xi \equiv \lambda \equiv 0 \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta \bmod 4; \end{array} \tag{4}$$

$$\xi \equiv \lambda \bmod 2, \quad \eta \equiv \zeta \bmod 4; \tag{5}$$

$$\begin{array}{lll} \xi \equiv \lambda \equiv 0 \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta \bmod 4, \\ \text{or } \xi \equiv \lambda \equiv 1 \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta + 2 \bmod 4, \\ \text{or } \xi \text{ even}, & \eta, \zeta, \lambda \text{ odd}, & \eta \equiv \zeta \bmod 4, \\ \text{or } \lambda \text{ even}, & \xi, \eta, \zeta \text{ odd}, & \eta \equiv \zeta + 2 \bmod 4; \end{array} \tag{6}$$

$$(7)\qquad \begin{array}{llll} & \xi \equiv \lambda \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta \bmod 4, \\ \text{or} & \xi \equiv \lambda \bmod 2, & \eta, \zeta \text{ odd}, & \eta \equiv \zeta + 2 \bmod 4; \end{array}$$

$$(8)\qquad \begin{array}{llll} & \xi \equiv \lambda \equiv 0 \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta \bmod 4, \\ \text{or} & \xi \equiv \lambda \equiv 1 \bmod 2, & \eta, \zeta \text{ even}, & \eta \equiv \zeta + 2 \bmod 4, \\ \text{or} & \xi \text{ even}, & \eta, \zeta, \lambda \text{ odd}, & \eta \equiv \zeta + 2 \bmod 4, \\ \text{or} & \lambda \text{ even}, & \xi, \eta, \zeta \text{ odd}, & \eta \equiv \zeta \bmod 4. \end{array}$$

Let E', E'', E''', E^{IV} be integers, congruent mod β' to E, such that

$$A\sigma'E' \equiv \alpha'CE'' \equiv 3A\sigma'E''' \equiv 3\alpha'CE^{IV} \equiv 1 \bmod 8.$$

Let F', F'', $\cdots$, H^{IV} satisfy the conditions

$$\begin{aligned} &B\sigma'F' \equiv B\alpha'H' \equiv 7\beta'CF'' \equiv 7\beta'AH'' \\ &\equiv 3B\sigma'F''' \equiv 3B\alpha'H''' \equiv 5\beta'CF^{IV} \equiv 5\beta'AH^{IV} \equiv 1 \bmod 8, \end{aligned}$$

where each $F^{(i)}$ is congruent mod α' to F and each $H^{(i)}$ is congruent mod σ' to H.

3. *Construction of the theta functions corresponding to sets $S^{(i)}$ and derivation of the identities.* It may be shown that η, ζ, λ satisfy (2) and (3) if and only if they may be written in form (9) below with r and p subject to the indicated inequalities. Similarly η, ζ, λ satisfy (2) and (4), (5), or (6) if and only if they may be written in the forms (10), (11), or (12) respectively. Let (11′), (12′) be the forms obtained from (11), (12) respectively by replacing E', F', H', E'', F'', H'', by E''', F''', H''', E^{IV}, F^{IV}, H^{IV} respectively. Then it may be shown that η, ζ, λ satisfy (2) and (7) or (8) if and only if they may be written in the forms (11′) or (12′) respectively.

$$(9)\qquad \begin{aligned} &\lambda = 2\alpha'\beta'm_1 + r, \\ &\eta = 4\beta'\sigma'm_2 + 2A\sigma'Er - 2\alpha\beta'Hp, \\ &\zeta = 4\alpha'\sigma'm_3 + 2B\sigma'Fr + 2\alpha'p, \\ &\qquad 0 \leqq r < 2\alpha'\beta', \qquad 0 \leqq p < 2\sigma'. \end{aligned}$$

$$(10)\qquad \begin{aligned} &\lambda = 2\alpha'\beta'm_1 + r, \\ &\eta = 4\beta'\sigma'm_2 + 2(\beta' + AE)\sigma'r - 2\alpha\beta'Hp, \\ &\zeta = 4\alpha'\sigma'm_3 + 2B\sigma'Fr + 2\alpha'p, \\ &\qquad 0 \leqq r < 2\alpha'\beta', \qquad 0 \leqq p < 2\sigma'. \end{aligned}$$

$$(11)\qquad \begin{aligned} &\lambda = 2\alpha'\beta'm_1 + r, \\ &\eta = 4\beta'\sigma'm_2 + 2A\sigma'E'r - \alpha\beta'H'p, \\ &\zeta = 4\alpha'\sigma'm_3 + 2B\sigma'F'r + \alpha'p, \\ &\qquad 0 \leqq r < 2\alpha'\beta', \qquad 0 \leqq p < 4\sigma'. \end{aligned}$$

$$
(12)\qquad \begin{aligned}
&\lambda = 2\alpha'\beta' m_1 + r,\\
&\eta = 4\beta'\sigma' m_2 + 2(\beta' + AE'')\sigma' r - (\alpha H'' - 2\sigma')\beta' p,\\
&\zeta = 4\alpha'\sigma' m_3 + 2B\sigma' F'' r + \alpha' p\\
&\qquad 0 \leqq r < 2\alpha'\beta', \qquad 0 \leqq p < 4\sigma'.
\end{aligned}
$$

Let q be a complex number, $|q| < 1$; $a = A/\alpha'$, $b = B/\beta'$, $c = C/2\sigma'$ and

$$
\begin{aligned}
F(w; r) &= \sum_{m_1} q^{ab\lambda^2/4} e^{ab\lambda w i},\\
G_j(y; r, p) &= \sum_{m_2} q^{bc\eta^2/4} e^{bc\eta y i},\\
H_j(z; r, p) &= \sum_{m_3} q^{ac\zeta^2/4} e^{ac\zeta z i}, \qquad (j = 1, 2 \cdots 6),
\end{aligned}
$$

where the summation extends from $-\infty$ to $+\infty$; r, p are integers; $\lambda = 2\alpha'\beta' m_1 + r$; η, ζ are given by (9), (10), (11), (12), (11′) or (12′) according as $j = 1, 2, 3, 4, 5$, or 6. We may write

$$
F(w; r) = q^{abr^2/4} e^{abrwi} \vartheta_3[AB(w - rki), q^{a\beta}],
$$

where $2k = \log q$, and ϑ_3 is Jacobi's theta function. Hence F is absolutely convergent. The other twelve functions may be similarly expressed in terms of theta functions and hence are absolutely convergent. Let s, p be integers such that $0 \leqq s < \alpha'\beta'$, $0 \leqq p < 2\sigma'$. Then

$$
\begin{aligned}
\psi_{11}(s, p) &\equiv \vartheta_3(x) F(w; 2s) G_1(y; 2s, p) H_1(z; 2s, p)\\
&= \sum q^{(\xi^2 + bc\eta^2 + ac\zeta^2 + ab\lambda^2)/4} e^{(\xi x + bc\eta y + ac\zeta z + ab\lambda w) i}
\end{aligned}
$$

where the summation extends over all quadruples ξ, η, ζ, λ such that

$$
\xi \equiv \lambda \equiv 0 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4,
$$

and such that η, ζ, λ may be written in form (9) with r replaced by $2s$. For every term of ψ_{11} there is a corresponding element of S'. Conversely, if Ω is an element of S', with ξ even, there is a uniquely determined pair of integers s, p; $0 \leqq s < \alpha'\beta'$; $0 \leqq p < 2\sigma'$, such that ψ_{11} contains exactly one term corresponding to Ω. Let

$$
\psi_{12}(s, p) \equiv \vartheta_2(x) F(w; 2s+1) G_1(y; 2s+1, p) H_1(z; 2s+1, p)
$$

and repeat the above argument except that

$$
\xi \equiv \lambda \equiv 1 \bmod 2, \quad \eta, \zeta \text{ even}, \quad \eta \equiv \zeta \bmod 4.
$$

Then

$$
\Phi_1(x, y, z, w) \equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{2\sigma'-1} [\psi_{11} + \psi_{12}] = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}
$$

where, assuming for the present that x, y, z, w are real

$$X = x - \frac{y}{2\beta'\sigma'} I + \frac{z}{2\alpha'\sigma'} J - \frac{w}{\alpha'\beta'} K;$$

$R(\Omega X')$ is the real part of the quaternion $\Omega X'$; and the summation extends without omission or duplication over the elements of S'.

Let

$$\begin{aligned}
\psi_{21}(s, p) &\equiv \vartheta_2(x) F(w; 2s+1) G_2(y; 2s+1, p) H_2(z; 2s+1, p),\\
\psi_{22}(s, p) &\equiv \vartheta_3(x) F(w; 2s) G_2(y; 2s, p) H_2(z; 2s, p),\\
\psi_{31}(s, p) &\equiv \vartheta_2(x) F(w; 2s+1) G_3(y; 2s+1, p) H_3(z; 2s+1, p),\\
\psi_{32}(s, p) &\equiv \vartheta_3(x) F(w; 2s) G_3(y; 2s, p) H_3(z; 2s, p),\\
\psi_{41}(s, t) &\equiv \vartheta_3(x) F(w : 2s) G_4(y; 2s, 2t) H_4(z; 2s, 2t),\\
\psi_{42}(s, t) &\equiv \vartheta_2(x) F(w; 2s+1) G_4(y; 2s+1, 2t) H_4(z; 2s+1, 2t),\\
\psi_{43}(s, t) &\equiv \vartheta_3(x) F(w; 2s+1) G_4(y; 2s+1, 2t+1) H_4(z; 2s+1, 2t+1),\\
\psi_{44}(s, t) &\equiv \vartheta_2(x) F(w; 2s) G_4(y; 2s, 2t+1) H_4(z; 2s, 2t+1),
\end{aligned}$$

and

$$\begin{aligned}
\Phi_2(x, y, z, w) &\equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{2\sigma'-1} [\psi_{21}(s, p) + \psi_{22}(s, p)],\\
\Phi_3(x, y, z, w) &\equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{p=0}^{4\sigma'-1} [\psi_{31}(s, p) + \psi_{32}(s, p)],\\
\Phi_4(x, y, z, w) &\equiv \sum_{s=0}^{\alpha'\beta'-1} \sum_{t=0}^{2\sigma'-1} [\psi_{41}(s, t) + \psi_{42}(s, t) + \psi_{43}(s, t) + \psi_{44}(s, t)].
\end{aligned}$$

Let $\Phi_5(x, y, z, w)$, $\Phi_6(x, y, z, w)$ be the functions obtained from $\Phi_3(x, y, z, w)$, $\Phi_4(x, y, z, w)$ respectively by replacing G_3, H_3, G_4, H_4 by G_5, H_5, G_6, H_6 respectively. Then it may be shown as for Φ_1 and S' that

$$\Phi_j(x, y, z, w) = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}, \qquad (j = 1, 2 \cdots 6)$$

where the summation extends over the elements of $S^{(j)}$. For each of the sets $S^{(j)}$ we thus have a Φ_j such that the terms of Φ_j are in one-to-one correspondence with the elements of $S^{(j)}$. By Miss Darkow's results, the sets indicated below have the properties R, C, U, M according as the algebra is of type A, B, C or D.

Type A, S',
Type B, S'',
Type C, S''', S^{V},
Type D, S^{IV}, S^{VI}.

If we let S be one of the $S^{(j)}$ which has these properties and let Φ be the corresponding Φ_j, then the argument of Latimer's paper beginning, "Let

x_1, y_1, z_1, w_1 be . . . ," and leading to equation (1) may be repeated word for word, except that we set

$$X_1 = x_1 - \frac{y_1}{2\beta'\sigma'} I + \frac{z_1}{2\alpha'\sigma'} J - \frac{w_1}{\alpha'\beta'} K$$

and employ the Φ, S, X, Ω of this paper. We obtain

THEOREM 1. *Let S be one of the $S^{(j)}$ $(j=1,2,\cdots 6)$ which has the properties R, C, U, M and let Φ be the corresponding Φ_j. If U is an element of S of norm unity and if x_1, y_1, z_1, w_1 are defined in terms of x, y, z, w by means of $XU = X_1$, then*

$$\Phi(x, y, z, w) = \Phi(x_1, y_1, z_1, w_1).$$

The theorem is trivial for certain U's, for example 1. Furthermore, there may not be a set with a unit yielding a non-trivial identity. It may be shown that if $\alpha=\beta=1$ and σ be a product of distinct primes in the form $4n+1$, then the algebra contains a set with a unit which yields a non-trivial identity.

4. *An identity based on $\Delta = s^2 + prt^2$.* The sets of elements used hereafter will not in general be integral. Let r and p be positive integers and let s and t be integers such that s is prime to

$$\Delta = s^2 + prt^2.$$

Let $\rho = r^{1/2}$, $\tau = ip^{1/2}$ and let S be the totality of numbers in the form

$$\Omega = \Delta^{-1/2}(\xi\rho + \eta\tau),$$

where ξ and η are integers such that

$$s\xi \equiv pt\eta \mod \Delta.$$

Let S_1 be the totality of numbers in the form

$$\Omega_1 = \xi_1\rho + \eta_1\tau,$$

where ξ_1 and η_1 are ordinary integers.

If $U = \Delta^{-1/2}(s + t\rho\tau)$ and

$$\Omega U = \Omega_1$$

it may be shown that Ω_1 belongs to S_1 if and only if Ω belongs to S.

It may be shown that if

$$\Phi_\mu(x, y) \equiv q^{p\mu^2} e^{2p\mu i(rtx+sy)} \vartheta_3[\Delta rx - prt\mu ki, q^{r\Delta}] \vartheta_3[\Delta py - ps\mu ki, q^{p\Delta}],$$

where $k = \log q$, then

$$\Phi_\mu(x, y) = \sum q^{(r\xi^2+p\eta^2)/\Delta} e^{2(r\xi x+p\eta y)i}$$

where ξ and η range over all integers such that

$$\xi \equiv pt\mu \bmod \Delta, \qquad \eta \equiv s\mu \bmod \Delta.$$

Then

$$\Phi(x, y) \equiv \sum_{\mu=0}^{\Delta-1} \Phi_\mu(x, y) = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')}$$

where $X = \Delta^{1/2}(\rho x + \tau y)$, and the summation extends over all numbers of S.

It may also be shown that

$$\Psi(x_1, y_1) \equiv \vartheta_3(rx_1, q^r)\vartheta_3(py_1, q^p) = \sum_{\Omega_1} {}^{N(\Omega_1)} e^{2R(\Omega_1 X'_1)i}$$

where $X_1 = \rho x_1 + \tau y_1$, and the summation extends over all numbers of S_1.

Let x_1, y_1 be defined in terms of x, y by

$$X_1 = XU,$$

or

$$x_1 = sx - pty,$$
$$y_1 = rtx + sy.$$

By means of $\Omega_1 = \Omega U$ we have a one-to-one correspondence between the terms of Φ and Ψ. For corresponding terms

$$N(\Omega_1) = N(\Omega)N(U) = N(\Omega),$$
$$\Omega_1 X'_1 = (\Omega U)(XU)' = (\Omega U)(U'X') = \Omega X'.$$

Hence corresponding terms are equal and

$$\Psi(x_1, y_1) = \Phi(x, y).$$

If we express this in terms of theta functions and replace $r\Delta x$, $p\Delta y$, rx_1, py_1 by x, y, x_1, y_1 respectively, we obtain the identity below.

THEOREM 2. *If r and p are positive integers and s, t are integers such that s is prime to*

$$\Delta = s^2 + prt^2$$

then

$$\vartheta_3(x_1, q^r)\vartheta_3(y_1, q^p) = \sum_{\mu=0}^{\Delta-1} q^{p\mu^2} e^{2\mu i y_1}\vartheta_3(x - prt\mu ki, q^{r\Delta})\vartheta_3(y - ps\mu ki, q^{p\Delta})$$

$$x_1 = (sx - rty)/\Delta, \qquad y_1 = (ptx + sy)/\Delta.$$

5. *An identity based on* Δ^2 [illegible] be defined as before and let

$$\Delta^2 = s^2 + prt^2$$

[illegible] Δ [illegible]

Let S be the totality of numbers in the form

$$\Omega = \xi\rho + \eta\tau,$$

where ξ, η are integers such that

$$s\xi \equiv pt\eta \bmod \Delta,$$

and let S_1 be the totality of numbers in the form

$$\Omega_1 = \xi_1\rho + \eta_1\tau$$

where ξ_1, η_1 are integers such that

$$s\xi_1 \equiv - pt\eta_1 \bmod \Delta.$$

Let $$U = s + t\rho\tau.$$

If $\Omega U = \Delta\Omega_1$ then Ω_1 belongs to S_1 if and only if Ω belongs to S. If μ is an arbitrarily chosen integer and if

$$\Phi_\mu(x, y) \equiv q^{p\Delta^2\mu^2} e^{2p\mu i(rtx+sy)} \vartheta_3(r\Delta x - \Delta prt\mu ki, q^{r\Delta^2}) \vartheta_3(p\Delta y - p\Delta s\mu ki, q^{p\Delta^2})$$

then

$$\Phi_\mu(x, y) = \sum q^{(r\xi^2+p\eta^2)} e^{2(r\xi x+p\eta y)i}$$

where the summation extends over all pairs of integers ξ, η such that

$$\xi \equiv pt\mu, \qquad \eta \equiv s\mu \bmod \Delta.$$

Therefore

$$\Phi(x, y) \equiv \sum_{\mu=0}^{\Delta-1} \Phi_\mu(x, y) = \sum_{\Omega} q^{N(\Omega)} e^{2R(\Omega X')i}$$

where the summation extends over all the numbers of S and

$$X = x\rho + y\tau.$$

Let

$$\Psi_\mu(x_1, y_1) \equiv q^{p\Delta^2\mu^2} e^{2p\mu i(sy_1-rtx_1)} \vartheta_3(r\Delta x_1 + \Delta prt\mu ki, q^{r\Delta^2}) \vartheta_3(p\Delta y_1 - p\Delta s\mu ki, q^{p\Delta^2})$$

then

$$\Psi_\mu(x_1, y_1) = \sum q^{(r\xi_1^2+p\eta_1^2)} e^{2(r\xi_1 x_1+p\eta_1 y_1)i}$$

and the summation extends over all pairs of integers ξ_1, η_1 such that

$$\xi_1 \equiv - pt\mu, \qquad \eta_1 \equiv s\mu \bmod \Delta.$$

Therefore

$$\Psi(x_1, y_1) \equiv \sum_{\mu=0}^{\Delta-1} \Psi_\mu(x_1, y_1) = \sum_{\Omega_1} q^{N(\Omega_1)} e^{2R(\Omega_1 X'_1)i}$$

where the summation extends over all numbers of S_1 and

$$X_1 = x_1\rho + y_1\tau.$$

The terms of Φ are in one-to-one correspondence with the terms of Ψ and for corresponding terms $N(\Omega) = N(\Omega_1)$. If we set

$$\Delta X_1 = XU,$$

or

$$x_1 = (sx - pty)/\Delta,$$
$$y_1 = (rtx + sy)/\Delta,$$

it follows, as before, that

$$\Phi(x, y) = \Psi(x_1, y_1).$$

If we express this in terms of theta functions and replace q^{Δ^2}, Δrx, Δpy, Δrx_1, Δpy_1 by q, x, y, $-x_1$, y_1 respectively we obtain the identity below.

THEOREM 3. *Let p, r be positive integers and let s, t be integers such that*

$$\Delta^2 = s^2 + prt^2,$$

where Δ is a positive integer prime to s. If

$$F(x, y) \equiv \sum_{\mu=0}^{\Delta-1} q^{p\mu^2} e^{2\mu y_1 i} \vartheta_3(x - prt\mu ki/\Delta, q^r)\vartheta_3(y - ps\mu ki/\Delta, q^p),$$

then

$$F(x, y) = F(x_1, y_1),$$

where

$$x_1 = (-sx + rty)/\Delta, \qquad y_1 = (ptx + sy)/\Delta.$$

EASTERN TEACHERS' COLLEGE,
RICHMOND, KENTUCKY.

ON THE ZEROS OF A POLYNOMIAL AND OF ITS DERIVATIVE.

By HUBERT E. BRAY.

1. *Introduction.* This paper deals with polynomials all of whose roots (zeros) are real.* We subject the roots of the polynomial $P(x)$ to certain simple operations and study the effect of these operations upon the roots of the derivative $P'(x)$. The main object of this study was to find a proof of the theorem (A), below, which is a striking extension of the classical theorem on the coincidence of the respective centroids of the roots of $P(x)$ and $P'(x)$.

The statement of the theorem, communicated to the writer by Doctor J. Shohat, originated with M. T. Popovici.

THEOREM A. *If $P(x)$ is a real polynomial of degree $n \geqq 2$, whose roots $x_1, x_2, \cdots x_n$, are all real and non-negative, and if the roots of $P'(x) = D_x P(x)$ are $\xi_1, \xi_2, \cdots, \xi_{n-1}$ and if m is a positive integer, then*

$$\sum_{i=1}^{n} x_i^m / n \geqq \sum_{i=1}^{n-1} \xi_i^m / (n-1).$$

The equality is valid if, and only if, either $m = 1$ or all the roots are equal.

It is a little surprising that a theorem so simple in statement, whose proof requires fairly elementary methods, should be new. This may be due to the fact that the theorem, so far as the writer could determine, is not immediately deducible from the well-known recurrence formulae of Newton, which would seem at first glance to provide a natural method of proof. The writer proved a few special cases of the theorem ($m = 2, 3, 4, 5$), by direct use of Newton's formulae, but the calculations became quite complicated as m increased and seemed to lead to no general method. However, he was led by this means to a stronger inequality than that of Theorem (A). This inequality is established in general in § 4.

2. *Auxiliary Theorems.*

THEOREM 1. *If $P(x)$ is a real polynomial, all of whose roots are real, then in the domain of real numbers the polynomial*

$$P'^2 - PP''$$

* In what follows we shall refer to the roots of $P(x) = 0$ as roots of $P(x)$, instead of zeros; this is in order to avoid awkwardness of expression in distinguishing between zeros which are equal to zero and those which are not.

is never negative and vanishes at multiple roots of $P(x)$, but at no other points. If x' is a multiple root of $P(x)$, of order k, x' is a multiple root of $P'^2 - PP''$ of order $2k-2$.

The proof is immediate. Write

$$P(x) = (x - x')^k Q(x)$$

where $Q(x') \neq 0$. We have, after differentiating twice,

$$P'^2 - PP'' = (x - x')^{2k}(Q'^2 - QQ'') + k(x - x')^{2k-2} Q^2.$$

Assuming that the theorem is true for the polynomial Q of lower degree it follows immediately for P. It is obviously true when Q is of the form $(x - x_0)^r$ and we may take this as the initial case in order to complete the proof by induction.

The proof of theorem (A) may be simplified in the following manner. Let

$$\delta^{(m)} = \sum_{i=1}^{n} x_i^m / n - \sum_{i=1}^{n-1} \xi_i^m / (n-1)$$

and suppose that all the roots of $P(x)$ and $P'(x)$ are subjected to a positive displacement of magnitude h; then the quantity $\delta^{(m)}$ becomes a function of h, $\delta^{(m)}(h)$; $\delta^{(m)}(0) = \delta^{(m)}$. Replacing x_i by $x_i + h$, ξ_i by $\xi_i + h$ we obtain easily

$$\delta^{(m)}(h) = \delta^{(m)} + \binom{m}{1}\delta^{(m-1)}h + \binom{m}{2}\delta^{(m-2)}h^2 + \cdots + \binom{m}{m-2}\delta^{(2)}h^{m-2},$$

since $\delta^{(1)}$ is zero. It follows immediately, from this formula, that if Theorem A is true in the case where the smallest root of $P(x)$ is zero, then it is true, *a fortiori*, when all the roots are positive. In future, therefore, we may confine our attention to this case and we shall write $P(x) = x^k Q(x)$, where $Q(x)$ is a polynomial of degree $n - k$.

We next consider the effect produced upon the roots of $P'(x)$ when we subject the non-zero roots of $P(x)$ to a positive displacement, the other roots of $P(x)$ remaining equal to zero. To fix the ideas let $Q(x)$ represent a polynomial whose smallest root is equal to zero and write

$$P(x, h) = x^k Q(x - h),$$

[illegible] the smallest positive root of $P(x, h)$ is equal to h. By varying the parameter h the positive roots of $P(x, h)$ are rigidly displaced. We now prove

THEOREM 2. *If*

[illegible]

where k is a positive integer, and if ξ is a positive root of $P'(x,h) = (\partial/\partial x)P(x,h)$, but not a root of $P(x,h)$, then ξ is an increasing function of h such that

$$0 < d\xi/dh < 1, \qquad (h \geq 0).$$

We have

$$P'(x,h) = kx^{k-1}Q(x-h) + x^k Q'(x-h).$$

Consequently,

$$\xi = -kQ(\xi-h)/Q'(\xi-h).$$

Since ξ is positive, and not a root of $Q(x-h)$, $Q(\xi-h)$ and $Q'(\xi-h)$ are different from zero and of opposite signs. Differentiating with regard to h we have

$$\xi' = \frac{d\xi}{dh} = k\,\frac{Q'^2(\xi-h) - Q(\xi-h)Q''(\xi-h)}{Q'^2(\xi-h)}\,(1-\xi').$$

By Theorem 1 the first factor of the right-hand member is positive, since ξ is not a multiple root of $Q(x-h)$. Hence ξ' and $1-\xi'$ are of the same sign and therefore both are positive. Explicitly we obtain, on eliminating k,

$$\xi' = \xi(Q'^2 - QQ'')/[\xi(Q'^2 - QQ'') - QQ'].$$

This quantity is clearly positive, and less than unity, since QQ' is negative; thus the theorem is proved.

3. *Proof of Theorem A.* The proof of Theorem A is effected by showing that Theorem A is implied by Theorem B, below, as follows: Suppose that the different roots of $P(x)$ in ascending order are $0, a_1, a_2, \cdots, a_p$, with respective multiplicities $m_0, m_1, m_2, \cdots, m_p$. Let

$$a_1 = h_1,\ a_2 - a_1 = h_2, \cdots a_p - a_{p-1} = h_p,$$

where $h_1, \cdots, h_p > 0$, and consider the sequence of polynomials, of degree n;

$$P_0 = P,\ P_1,\ P_2, \cdots P_p,$$

which are related in the following manner:

The roots of P_i are obtained from those of P_{i-1} by decreasing all the positive roots of P_{i-1} by the same quantity h_i, the zero roots of P_{i-1} remaining unaltered, $(i = 1, 2, \cdots, p)$. In p stages $P(x)$ is thus transformed into $P_p(x) = x^n$, the case in which Theorem A is evident. Hence the theorem will be established in general if we can show that the quantity $\delta^{(m)}$ formed for $P_{i-1}(x)$ is greater than the corresponding quantity formed for $P_i(x)$. But if we represent $P_i(x)$ by the formula

$$P_i(x) = x^k Q(x)$$

W[illegible]re the smallest root of $Q(x)$ is equal to zero, then

$$P_{i-1}(x) = x^k Q(x - h_i) \qquad (i = 1, 2, \cdots, p).$$

It [illegible]ill be sufficient, therefore, to prove that if

$$P(x, h) = x^k Q(x - h)$$

th[illegible]n $\delta^{(m)}$ formed for the polynomial $P(x, h)$ is an increasing function of h. In other words, Theorem (A) is implied by the

THEOREM B. *If the smallest root of $Q(x)$ is zero and if $h \geq 0$, and*

$$P(x, h) = x^k Q(x - h) \qquad (k \geq 1)$$

and

$$\delta^{(m)}(h) = \sum_1^n x_i^m / n - \sum_1^{n-1} \xi_i^m / (n - 1) \qquad (m \geq 2)$$

where $\{x_i\}$ are the roots of $P(x, h)$ and $\{\xi_i\}$ the roots of $P'(x, h)$, then $\delta^{(m)}(h)$ is an increasing function of h.

We prove that $d\delta^{(m)}(h)/dh$ is positive. On differentiating with regard to h, remembering that dx_i/dh is equal to 1, if x_i is positive, and equal to zero otherwise,

$$d\delta^{(m)}(h)/dh = m[\sum_1^n x_i^{m-1}/n - \sum \xi_i^{m-1} \xi'_i / (n - 1)].$$

Hence

$$d\delta^{(m)}(h)/dh - m\delta^{(m-1)}(h) = m \sum_{i=1}^{n-1} \xi_i^{m-1}(1 - \xi'_i)/(n - 1).$$

N[illegible], if h is positive at least one of the roots x_i is positive and therefore at [illegible] one of the roots ξ_i is positive and unequal to any x_i. Hence, by Theorem [illegible] right hand member of this equation is positive. The proof is now com[illegible]d by induction. Assuming that $\delta^{(m-1)}(h)$ is an increasing function it [illegible] that Theorem (A) is true in this case; that is, $\delta^{(m-1)}(h)$ is non-nega[illegible]d consequently

$$d\delta^{(m)}(h)/dh > 0 \qquad (h > 0).$$

[illegible] is clearly a continuous function it is an increasing function

[illegible] Theorem A. is proved.

[illegible] *Inequalities.* In this section we shall find it [illegible] [illegible]ght change of notation. Let $P_1(x)$ represent the polynomial [illegible] is the smallest root of $P(x)$. We designate the

$$\begin{array}{rll} P(x) & \text{by} & x_1, x_2, \cdots, x_n, \\ P_1(x) & \text{by} & x_2, \cdots, x_n, \\ P'(x) & \text{by} & \xi_2, \xi_3, \cdots, \xi_n, \\ P'_1(x) & \text{by} & \eta_3, \eta_4, \cdots, \eta_n, \end{array}$$

each set of roots in order of increasing magnitude.

Consider the quantity

$$\theta^{(m)} = n\left(\sum_1^n x_i^m - \sum_2^n \xi_i^m\right) - (n-1)\left(\sum_2^n x_i^m - \sum_3^n \eta_i^m\right).$$

We shall prove in this section that $\theta^{(m)}$ is positive except in two cases

(a) when all the roots are equal to zero,
(b) when $x_1 = 0$ and $m = 1$.

If we increase all the roots by an amount h we find, after replacing x_i by $x_i + h$, ξ_i by $\xi_i + h$, etc., the new value of $\theta^{(m)}$,

$$\theta^{(m)}(h) = \theta^{(m)} + \binom{m}{1}\theta^{(m-1)}h + \cdots + \binom{m}{m-2}\theta^{(2)}h^{m-2} + \binom{m}{m-1}\theta^{(1)}h^{m-1} + h^m.$$

This formula shows that $\theta^{(m)}$ is increased by a positive displacement of all the roots. Hence we may confine our attention to the case where x_1, the smallest root of $P(x)$, is zero.

Theorem 3. *If $P(x)$ is of the form $x^kQ(x)$ where k is positive but not necessarily an integer, and $Q(x)$ is a polynomial of degree $\geqq 1$ whose roots are real and positive; if ξ is a positive root of $P'(x) = D_xP(x)$, but not a root of $P(x)$, then ξ is an increasing function of k.*

We have

$$P'(x) = x^kQ'(x) + kx^{k-1}Q(x).$$

The positive roots of $P'(x)$ are roots of the polynomial

$$xQ'(x) + kQ(x),$$

that is

$$\xi Q'(\xi) + kQ(\xi) = 0. \tag{1}$$

Regarding ξ as a function of the parameter k, we have, with regard to k

$$\{(k+1)Q'(\xi) + \xi Q''(\xi)\}\xi' + Q(\xi) = 0$$

and on eliminating k from the last two equations

$$[Q(\xi)Q'(\xi) + \xi\{Q(\xi)Q''(\xi) - Q'^2(\xi)\}]\xi' + Q(\xi) = 0$$

or

$$\xi' = d\xi/dk = Q^2/[\xi(Q'^2 - QQ'') - QQ'].$$

From (1) we see that Q, Q' are different from zero and of opposite signs. And since $Q'^2 - QQ''$ is non-negative, by Theorem 1, it follows that ξ' is positive and the theorem is proved.

N. B. If $k = 0$, that is $P(x) \equiv Q(x)$, then it may happen that P' has no positive root which is not a (multiple) root of P. This will be the case if, and only if, P has but one positive root and this a multiple root.

It is worth while also to remark that if ξ is a positive root of $P'(x)$ and also a (multiple) root of $P(x)$ then $d\xi/dk \equiv 0$ since multiplication by a factor x^k will not affect the positions of multiple roots of $P(x)$.

We utilize this theorem in order to show that if k is an integer ≥ 0, and if

$$P(x) = x^{k+1}Q(x),$$
$$P_1(x) = x^kQ(x),$$

where $P(x)$ is a polynomial of degree $n \geq 3$, and $Q(x)$ is the polynomial of Theorem 3, then

$$\sum_2^n \xi_i{}^m > \sum_3^n \eta_i{}^m$$

where $\xi_2, \cdots, \xi_n$ are the roots of $P'(x)$ and $\eta_3, \cdots, \eta_n$ those of $P'(x)$, the roots being arranged in ascending order of magnitude. This result follows immediately from Theorem 3 if k is positive, since if ξ_i is a positive root of P' but not a root of P, (such a root always exists if $k > 0$), it is greater than the corresponding root η_i of P_1'. But if $k = 0$ then P' has exactly one positive root ξ_2 lying between 0 and the smallest root of $Q(x)$, i. e. $P_1(x)$; each of the other roots ξ_i of P', $(i \geq 3)$, is equal to or exceeds the corresponding root η_i of P' according as ξ_i is a (multiple) root of P or not, by Theorem 3 and the remark following it.

In any case therefore the inequality

$$\sum_{i=2}^n \xi_i{}^m > \sum_{i=3}^n \eta_i{}^m$$

is valid. Introducing the abbreviations

$$p_m = \sum_1^n x_i{}^m - \sum_2^n \xi_i{}^m,$$

$$q_m = \sum_2^n x_i{}^m - \sum_3^n \eta_i{}^m,$$

we may now state the result

$$p_m < q_m,$$

[illegible] is used to prove

THEOREM C. *If $Q(x)$ is a polynomial of degree ≥ 1 whose roots are real and positive, and if $P(x)$ is a polynomial of degree $n \geq 3$ of the form:*

$$P(x) = x^{k+1}Q(x),$$

and

$$P_1(x) = x^k Q(x),$$

where k is an integer ≥ 0; then

$$n\left(\sum_1^n x_i^m - \sum_2^n \xi_i^m\right) > (n-1)\left(\sum_2^n x_i^m - \sum_3^n \eta_i^m\right)$$

where $\xi_i(i=2,\cdots,n)$, $\eta_i(i=3,\cdots,n)$, $x_i(i=1,\cdots,n)$, $x_i(i=2,\cdots,n)$ are the roots of P', P_1', P, P_1, respectively, and m is an integer ≥ 2.

We first observe that $\sum\limits_1^n x_i^m$ is the coefficient of $x^{-(m+1)}$ in the expansion of the rational function $P'(x)/P(x)$, analytic at ∞, in powers of $1/x$. For if we write

$$\sum_1^n 1/(x-x_i) = P'(x)/P(x) = \sum_{i=0}^{\infty} a_i x^{-i}$$

then it follows that

$$\sum_{i=1}^n x_i^m = (1/2\pi i)\int_C [P'(x)/P(x)]x^m dx = a_{m+1}$$

on integrating termwise about a contour C, in the complex plane, which encloses all the roots of $P(x)$. Consequently the quantity

$$p_m = \sum_{i=1}^n x_i^m - \sum_{i=2}^n \xi_i^m$$

is the coefficient of $x^{-(m+1)}$ in the expansion in powers of $1/x$ of the function

$$P'(x)/P(x) - P''(x)/P'(x) \equiv [(P'^2 - PP'')/P^2]/(P'/P).$$

But since

$$P'/P = \sum_1^n (x-x_i)^{-1}$$

then, also, on differentiating

$$(P'^2 - PP'')/P^2 = \sum_1^n (x-x_i)^{-2}.$$

Thus we have to calculate the coefficient of $x^{-(m+1)}$ in the expansion of

$$\sum_1^n (x-x_i)^{-2} \Big/ \sum_1^n (x-x_i)^{-1}.$$

Now this coefficient is the same as the coefficient of y^m in the expansion in powers of y of the function:

$$\sum_{1}^{n} (1 - x_i y)^{-2} / \sum_{1}^{n} (1 - x_i y)^{-1}$$

obtained by substituting $x = 1/y$ in the preceding and dividing by y. Hence

$$\sum_{i=1}^{n} (1 - x_i y)^{-2} / \sum_{i=1}^{n} (1 - x_i y)^{-1} = p_0 + p_1 y + p_2 y^2 + \cdots,$$

or

$$(n + \sum_{r=1}^{\infty} (r+1) s_r y^r) / (n + \sum_{r=1}^{\infty} s_r y^r) = \sum_{r=0}^{\infty} p_r y^r,$$

where $s_r = \sum_{i=1}^{n} x_i^r$. On clearing of fractions and equating coefficients we obtain the recurrence formulae

$$\begin{aligned} &p_0 = 1 \\ &np_1 + p_0 s_1 = 2s_1 \\ &np_2 + p_1 s_1 + p_0 s_2 = 3s_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &np_m + \sum_{1}^{m} s_i p_{m-i} = (m+1) s_m. \end{aligned}$$

or

$$\begin{aligned} &np_1 = s_1 \\ &np_2 + s_1 p_1 = 2s_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &np_m + \sum_{i=1}^{m-1} s_i p_{m-i} = m s_m. \end{aligned}$$

The same analysis provides corresponding recurrence formulae for the quantities $q_m = \sum_{i=2}^{n} x_i^m - \sum_{i=3}^{n} \eta_i^m$. We have immediately, since $x_1 = 0$,

$$(n-1) q_m + \sum_{i=1}^{m-1} s_i q_{m-i} = m s_m.$$

On subtracting the last two equations we have:

$$np_m - (n-1) q_m = \sum_{i=1}^{m-1} s_i (q_{m-i} - p_{m-i}), \qquad (m \geq 2).$$

But we have already proved that $q_i - p_i > 0$ $(i = 1, 2, 3, \cdots)$. Hence

$$\theta = np_m - (n-1) q_m > 0$$

except in the case $m = 1$, where we have $np_1 - (n-1) q_1 = s_1$.

Uniting the result of this theorem with the fact, established at the beginning of this section, that $\theta^{(m)}$ is increased by a rigid displacement of all the roots in the positive direction, we obtain

THEOREM D. *If $P(x)$ is a polynomial of degree $n \geqq 3$ whose roots are real and non-negative, and $P_1(x)$ is the polynomial $P(x)/(x - x_1)$, where x_1 is the smallest root of $P(x)$, if we denote the respective roots of P, P_1, P', P_1', by $(x_1, \cdots, x_n)$, $(x_2, \cdots, x_n)$, $(\xi_2, \cdots, \xi_n)$, $(\eta_3, \cdots, \eta_n)$, then the inequality*

$$\theta^{(m)} = n\left(\sum_1^n x_i^m - \sum_2^n \xi_i^m\right) - (n-1)\left(\sum_2^n x_i^m - \sum_3^n \eta_i^m\right) \geqq 0$$

is valid; moreover $\theta^{(m)} = 0$ if, and only if, either

(a) $x_1 = 0$ *and* $m = 1$,

or

(b) *all the roots are equal to zero.*

Finally we remark that if $x_1 = 0$, so that $P(x) = xP_1(x)$, then, the hypotheses of Theorem C are satisfied and we have, if $m \geqq 2$,

$$\theta^{(m)} = n(n-1)\left\{ \frac{\sum_1^n x_i^m}{n} - \frac{\sum_2^n \xi_i^m}{n-1} - \frac{n-2}{n}\left(\frac{\sum_2^n x_i^m}{n-1} - \frac{\sum_3^n \eta_i^m}{n-2}\right)\right\} > 0$$

provided not all the roots are equal to zero.

This formula evidently provides an immediate proof of Theorem A, by recurrence, for all values of $n \geqq 3$; for it shows that if Theorem (A) is true for the polynomial $P_1(x)$ of degree $n-1$, then it is true for the polynomial $P(x)$ of degree n having its smallest root equal to zero; whence the theorem follows, *a fortiori*, when all the roots are positive. When $n = 2$, Theorem (A) may easily be proved by elementary algebra, and when this is done the proof by recurrence is complete.

THE RICE INSTITUTE,
HOUSTON, TEXAS.

ON THE NUMERICAL CALCULATION OF THE CURRENT IN AN ANTENNA.

By F. H. Murray.

In a preceding paper * the additional field produced by a system of conductors in an impressed electromagnetic field $(\boldsymbol{E}^0 e^{pt}, \boldsymbol{H}^0 e^{pt})$ has been represented in terms of surface integrals, which are equivalent to certain integrals of MacDonald † in the applications to be made. In a calculation which was made of the resonance wavelength of a long wire it appeared that the integral representing $\boldsymbol{H}$ could be taken as the basis of a numerical calculation of the current distribution, giving a fair check with experimental results. The usual formulas of alternating current theory correspond, however, to the representation of $\boldsymbol{E}$, and the problem reduces to the numerical solution of an integro-differential equation. In the present paper the analysis of the problem of a straight wire in an impressed field has been developed and extended to include the theory of the loaded vertical antenna, and the inverted L antenna; the latter is briefly discussed in the special case in which the height is small compared to the length of the antenna.

The theory is based on the method, well-known in the theory of integral equations, of assuming an expansion of the unknown function in a series of mutually orthogonal functions $J_n(s)$ which satisfy the boundary conditions at the open ends of the antenna. If the boundary conditions at the surface of the wire are written down, a set of linear equations can be constructed for the unknown Fourier coefficients of the current. It is shown that in the case of the loaded vertical antenna the functions J_n can be so chosen that the principal diagonal terms in these equations dominate the others; if the antenna wire is of small diameter, a good approximation is obtained with a small number of equations. The assumption often made in practice ‡ that a single current component J_1 gives an adequate approximation can therefore be partially justified analytically. It is shown that when this is possible, the impedance of the loaded antenna may be defined; from this impedance the fundamental period of the antenna and the radiation resistance can be calculated.§

* *American Journal of Mathematics*, Vol. 53, April, 1931, pp. 275-288.

† MacDonald, *Electric Waves* (1903), p. 16. The retarded potentials of Lorentz [illegible] after a suitable transformation.

‡ See [illegible] "On the [illegible] Single Vertical Antenna [illegible] Wavelengths Below the Fundamental," *Proceedings [illegible] Radio Engineers*, Vol. 12 (1924), p. 823.

§ [illegible] "Zur Theorie der Strahlungskopplung von Kurzwellen [illegible]

An extension of the analysis to other types of antenna, constructed of straight wires in the same plane, is made possible by the formulas of Appendix IV; the analysis, however, is not developed here.

1. The fundamental equations are taken in the form

$$4\pi E_x = \int_S\int \{[n, E] \operatorname{curl} v' + E_n \operatorname{div} v' + \kappa[n, H]\, v'\}\, dS$$

$$4\pi E_y = \int_S\int \{[n, E] \operatorname{curl} v'' + E_n \operatorname{div} v'' + \kappa[n, H]\, v''\}\, dS$$

$$4\pi E_z = \int_S\int \{[n, E] \operatorname{curl} v''' + E_n \operatorname{div} v''' + \kappa[n, H]\, v'''\}\, dS.$$

It is assumed that the conductivity of the exterior region is zero, while the dielectric constant and the permeability are both unity; hence, $h = \lambda = -\kappa = i\omega/c$. In these formulas

$$\phi = e^{-h\rho}/\rho,\quad \rho = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2},$$
$$v' = (\phi, 0, 0),\quad v'' = (0, \phi, 0),\quad v''' = (0, 0, \phi)$$

and the field components appearing in the integrand are those of the total field exterior to the conductors.

If the conductor is a segment of a cylinder, while (r, θ, z) form a right-handed coördinate system on the surface, and (l_0, θ_0, z_0) are the coördinates in space, the axis of the cylinder being the z-axis, then

$$n = (\cos\theta, \sin\theta, 0),$$
$$E_x = E_n \cos\theta - E_\theta \sin\theta, \qquad E_n = E_x \cos\theta + E_y \sin\theta,$$
$$E_y = E_n \sin\theta + E_\theta \cos\theta, \qquad E_\theta = -E_x \sin\theta + E_y \cos\theta.$$

Substituting,

$$[n, E] \operatorname{curl} v' = -E_z \cos\theta\, \partial\phi/\partial z - E_\theta \partial\phi/\partial y$$
$$[n, E] \operatorname{curl} v'' = -E_z \sin\theta\, \partial\phi/\partial z + E_\theta \partial\phi/\partial x$$
$$[n, E] \operatorname{curl} v''' = E_z \partial\phi/\partial r.$$

Inserting in the fundamental formulas, if the contributions of the ends can be neglected,

$$
\begin{aligned}
4\pi E_{l_0} &= \int_S\int \{-E_z \cos(\theta - \theta_0)\partial\phi/\partial z \\
&\qquad + E_\theta/l_0 \partial\phi/\partial\theta_0 - E_r \partial\phi/\partial l_0 + \kappa H_z \sin(\theta - \theta_0)\phi\}\, dS \qquad (1.1)\\
4\pi E_{\theta_0} &= \int_S\int \{-E_z \sin(\theta - \theta_0)\partial\phi/\partial z \\
&\qquad - E_\theta \partial\phi/\partial l_0 - E_r/l_0 \partial\phi/\partial\theta_0 - \kappa H_z \cos(\theta - \theta_0)\phi\}\, dS \\
4\pi E_{z_0} &= \int_S\int \{E_z \partial\phi/\partial r + E_r \partial\phi/\partial z + \kappa H_\theta \phi\}\, dS.
\end{aligned}
$$

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nansystemen," *Annalen der Physik*, 5. Folge, 1930, Band 4, Heft 7, p. 829, developed a related method for a different problem.

In cylindrical coördinates,

$$\phi = e^{-h\rho}/\rho,\ \rho = [(z - z_0)^2 + l_0^2 + r^2 - 2l_0 r \cos\ (\theta - \theta_0)]^{1/2}.$$

If $\boldsymbol{E}$ and $\boldsymbol{H}$ are interchanged, while κ is replaced by λ, the representation of $\boldsymbol{H}$ results. In particular, the average value $\bar{\boldsymbol{H}}$ of $\boldsymbol{H}_{\theta_0}$ on a circle of radius l_0, and the average value of $\boldsymbol{E}_z$ become

$$4\pi\, \bar{H}(l_0, z_0) = \int_a^b \{-\bar{H}(r, z) r \int_0^{2\pi} (\partial\phi/\partial l_0)\, d\theta_0$$

$$(1.2) \qquad - \lambda \bar{E}(z) r \int_0^{2\pi} \phi \cos\ (\theta - \theta_0)\, d\theta_0\}\ dz$$

$$4\pi\, \bar{E}(l_0, z_0) = \int_a^b \{\bar{E}(z) r \int_0^{2\pi} (\partial\phi/\partial r)\, d\theta_0$$

$$+ \bar{\boldsymbol{E}}_r(z)(\partial/\partial z) r \int_0^{2\pi} \phi d\theta_0 + \kappa \bar{H}(r, z) r \int_0^{2\pi} \phi d\theta_0\}\ dz$$

the integrals with respect to θ_0 being independent of θ. From a statement above it is known that the average components $\bar{\boldsymbol{E}}_z$, $\bar{\boldsymbol{H}}_\theta$ which appear under the integral signs may be replaced by the corresponding averages of the total field components $\bar{\boldsymbol{E}}_z + \bar{\boldsymbol{E}}_z{}^0$, $\bar{\boldsymbol{H}}_\theta + \bar{\boldsymbol{H}}_\theta{}^0$. Let

$$K(z - z_0, l_0) = r \int_0^{2\pi} (\partial\phi/\partial l_0)\, d\theta_0, \qquad M(z - z_0, l_0) = r \int_0^{2\pi} \phi \cos\ (\theta - \theta_0)\, d\theta_0.$$

The boundary conditions become

$$\boldsymbol{E}_z + \boldsymbol{E}_z{}^0 = \boldsymbol{E}'_z, \qquad \boldsymbol{H}_\theta + \boldsymbol{H}_\theta{}^0 = \boldsymbol{H}'_\theta.$$

For ordinary solid copper or iron conductors of small radius at both low and high frequencies the relation $\boldsymbol{E}'_z = \boldsymbol{C}\boldsymbol{H}'_\theta$ holds very approximately. (Riemann-Weber, II, 2d. Edition, pp. 516-518); at high frequencies

$$C = (1/c)[i\omega\mu/4\pi\sigma\ (\text{e. m. u.})]^{1/2};$$

hence, if

$$\bar{F}(z) = \boldsymbol{E}_z{}^0(z) - C\bar{\boldsymbol{H}}_\theta{}^0(z)$$

the first equation may be written

$$(1.3) \qquad 4\pi\, \bar{H}(l_0, z_0) + \int_a^b \{K(z - z_0, r) + \lambda C M(z - z_0, l_0)\}$$

$$\times\ \bar{H}(r, z)\, dz = \lambda \int_a^b M(z - z_0, l_0) \bar{F}(z)\, dz.$$

This equation may [illegible]
[illegible] calculation it is more convenient in this form.

From the field equation in cylindrical coördinates

$$\lambda(E_r^0 + E_r) = (1/r)\partial(H_z^0 + H_z)/\partial\theta - \partial(H_\theta^0 + H_\theta)/\partial z$$

one obtains by integration

$$\lambda(\bar{E}_r^0 + \bar{E}_r) = -\partial(\bar{H}_\theta^0 + \bar{H}_\theta)/\partial z.$$

In the second equation (1. 2) let the components of the total field be substituted under the integral sign; with the aid of the definitions

$$L(z - z_0, l_0) = (1/2\pi)\int_0^{2\pi}(\partial\phi/\partial r)\,d\theta_0, \quad G(z - z_0, l_0) = (1/2\pi)\int_0^{2\pi}\phi\,d\theta_0$$

this equation becomes

$$(1.4)\quad 4\pi\bar{E}(l_0, z_0)/2\pi r = C\int_a^b \bar{H}'(z)L(z - z_0, l_0)\,dz + (d/\lambda dz_0)\int_a^b (d\bar{H}'(z)/dz)G(z - z_0, l_0)\,dz + \kappa\int_a^b \bar{H}'(z)G(z - z_0, l_0)\,dz.$$

An integro-differential equation for $\bar{H}'$ is obtained from the boundary condition along the wire,

$$\lim_{l_0 \to r} \bar{E}(l_0, z_0) = -\bar{E}^0(r, z_0) + C\,\bar{H}'(z_0).$$

From the fundamental equations, the total current through a section of the cylinder perpendicular to the z-axis satisfies the equation $2\pi r\bar{H}' = 4\pi I$; for a perfect conductor $C = 0$ and the first term on the right of (1. 4) vanishes, while at high radio frequencies it is still very small. Neglecting this term, let

$$P(z_0) = -(1/h)\int_a^b (dI/dz)G(z - z_0)\,dz \qquad (-\kappa = \lambda = h)$$

$$W(z_0) = \int_a^b I(z)G(z - z_0)\,dz$$

(1. 4) becomes

$$(1.41)\qquad \bar{E}(l_0, z_0) = -dP/dz_0 - hW(z_0) \qquad (l_0 \geqq r).$$

2. If the conductors of an antenna system are not all straight, but are constructed from wire of small radius, it will be assumed that with a negligible error the cross-section of each is circular. Assume also that if H_θ is the angular tangential component of the total magnetic field H, only the average value of H_θ about a circumference need be considered. (s, θ) may be taken as the coördinates of any point of any conductor, except junctions of several conductors which will be neglected, and approximately $dS = r\,ds\,d\theta$ on each conductor.

With these approximations, and neglecting the tangential components of the total electric field as before, the reflected field exterior to the conductors is represented in the form

$$E = -\operatorname{grad} P - h\boldsymbol{W},$$

$$P = (1/4\pi)\int_0^{2\pi} d\theta \int E'_r \phi r ds \qquad \phi = e^{-h\rho}/\rho$$

$$\boldsymbol{W} = (1/4\pi)\int_0^{2\pi} d\theta \int [\boldsymbol{n}, \boldsymbol{H}']\,\phi r ds \quad \rho = [(x_1 - x)^2 + (y - y)^2 + (z_1 - z)^2]^{1/2}.$$

P is the retarded scalar potential, and $\boldsymbol{W}$ the retarded vector potential, if the time-factor $exp(pt)$ is omitted. Neglecting $[\boldsymbol{n}, \boldsymbol{H}']_s$,

$$r\int_0^{2\pi} [\boldsymbol{n}, \boldsymbol{H}'] d\theta = r\int_0^{2\pi} \boldsymbol{H}'_\theta d\theta = 4\pi \boldsymbol{I} = 4\pi(idx/ds + jdy/ds + kdz/ds)J.$$

Hence if the variation of $\boldsymbol{H}'_\theta$ over the circumference may be ignored,

$$\boldsymbol{W} = \int \boldsymbol{I} ds \int_0^{2\pi} r\phi d\theta.$$

From the field equations in cylindrical coördinates, (see equations just after (1.3)) the average field components on the surface of the conductor satisfy the equation

$$h(\bar{\boldsymbol{E}}_r + \bar{\boldsymbol{E}}_r^0) = -d(\bar{\boldsymbol{H}}_\theta^0 + \bar{\boldsymbol{H}}_\theta)/ds = -d\bar{\boldsymbol{H}}'_\theta/ds = -(2/r)dJ/ds$$

$$\bar{\boldsymbol{E}}'_r = -(2/hr)\,dJ/ds.$$

In the integral for P let $\boldsymbol{E}'_r$ be replaced by its average over a circumference; then if

$$G(s, s_0) = (1/2\pi)^2 \int_0^{2\pi}\int_0^{2\pi} \phi d\theta d\theta_0 = G(s_0, s)$$

at a point (s_0) the averages of P, $\boldsymbol{W}$ become

$$\bar{P}(s_0) = -(1/h)\int G(s, s_0)(dJ/ds)ds,$$

(2.1)

$$\bar{\boldsymbol{W}}(s_0) = \int G(s, s_0)\boldsymbol{I}(s)ds, \quad \bar{\boldsymbol{W}}_s(s_0) = \int G(s, s_0)J(s)\cos(s, s_0)ds.$$

The average values being understood, the tangential component of E at the [illegible]

(2.2) $$E_s = -d\bar{P}/ds_0 - h\bar{\boldsymbol{W}}_s(s_0).$$

If the antenna system is composed of several unconnected systems, P and $\boldsymbol{W}$ will be represented by sums of integrals each taken over one connected system.

3. *The vertical antenna.* If the earth is assumed to be perfectly conducting, $\boldsymbol{E}_s$ on the antenna is the same as the $\boldsymbol{E}_s$ due to the current in the antenna and its image in the earth's surface, the earth being absent, and twice the loading impedance connecting the antenna and its image, if the loading impedance is at the base. Let this total impedance be an inductance L in electromagnetic units. The variable current J will be assumed to be represented by an expansion in mutually orthogonal functions J_n which vanish at the open ends of the antenna, and satisfy other conditions to be specified later; on the central inductance it is assumed that each J_n is constant. On the antenna,

$$J = \sum_{n=1}^{\infty} a_n J_n(z),$$

$$(3.1)\qquad \begin{aligned} J_n &= e^{\alpha_n(d+z)} - e^{-\alpha_n(d+z)} && -d \leq z \leq -c \\ &= e^{\alpha_n(d-z)} - e^{-\alpha_n(d-z)} && c \leq z \leq d, \quad d - c = l. \end{aligned}$$

The coefficients a_n are to be determined from the boundary conditions along the antenna and the central inductance; on the surface of the coil

$$\boldsymbol{E}_s + \boldsymbol{E}_s{}^0 = Z'_i J(s)$$

and along the antenna

$$\boldsymbol{E}_z + \boldsymbol{E}_z{}^0 = Z_i J(z), \qquad Z_i = (1/cr)\,[\mu i\omega/\pi\sigma]^{1/2}$$

if cZ'_i, cZ_i are the internal self-impedances of the coil and of the antenna, respectively, $\boldsymbol{E}_s{}^0$, $\boldsymbol{E}_z{}^0$ are the average values of the impressed field.

On the antenna it is assumed that

$$\boldsymbol{E}_z{}^0 = \sum e_n J_n(z).$$

The constants α_n will be determined later in such a way that the condition of orthogonality on the antenna

$$\left(\int_{-d}^{-c} + \int_{c}^{d}\right) J_m(z) J_n(z) = \begin{matrix} 0, \; m \neq n \\ N_m, \; m = n \end{matrix}$$

is satisfied, $N_m = -4l(1 - \sin 2\beta_m l/2\beta_m l)$.

To compute the coefficients a_n, let each side of the boundary equation

$$(3.2)\qquad -dP/ds - h\boldsymbol{W}_s = -\boldsymbol{E}_s{}^0 + Z_i J$$

be multiplied by J_m and the products integrated over the coil and the antenna. Integrating the left-hand side over the coil,

$$-\int_{s_1}^{s_2} J_m(dP/ds_0)\,ds_0 = J_m(c)\,[P(s_1) - P(s_2)]$$

$$(3.3)\qquad \int_{s_1}^{s_2} J_m W_s(s_0)\,ds_0 = J_m(c)J(c)\int_{s_1}^{s_2} ds_0 \int_{s_1}^{s_2} ds\, G(s, s_0)\cos(s_0, s)$$
$$= LJ_m(c)J(c)$$

if the axis of the coil is parallel to the antenna. Integrating over the antenna

$$\left(\int_{-d}^{-c} + \int_{c}^{d}\right) J_m(z_0)(-dP/dz_0)\,dz_0 = J_m(c)\,[P(c) - P(-c)]$$
$$+\left(\int_{-d}^{-c} + \int_{c}^{d}\right) P(z_0)(dJ_m/dz_0)\,dz_0$$

since J_m vanishes at $-d$ and d. The integral of the left-hand member becomes

$$-hLJ(c)J_m(c) + \left(\int_{-d}^{-c} + \int_{c}^{d}\right) \{P(z_0)\,dJ_m/dz_0 - hW_z(z_0)J_m(z_0)\}\,dz_0$$

$$(3.4)\qquad = -\Delta E^0 J_m(c) + Z'J(c)J_m(c) + N_m(-e_m + Z_i a_m).$$

Assuming

$$\Delta E^0 = \int_{s_1}^{s_2} \mathbf{E}_s{}^0\,ds, \qquad Z' = \int_{s_1}^{s_2} Z'_i\,ds$$

writing $\sum A_{mn}a_n$ for the integral on the left of (3.4), this becomes

$$(3.5)\qquad \sum_{n=1}^{\infty} \{A_{mn} - hLJ_m(c)J_n(c) - Z'J_m(c)J_n(c) - \delta_n{}^m N_m Z_i\}a_n$$
$$= -\Delta E^0 J_m(c) - N_m e_m$$

$$\delta_n{}^m = \begin{cases} 0, & m \neq n \\ 1, & m = n, \end{cases} \qquad (m = 1, 2, 3, \cdots).$$

Since $h = p/c$, this indicates that the ratio $cA_{mn}/J_m(c)J_n(c)$ has the dimensions of an impedance.

Assume the antenna constant c so small that with a negligible error, $\exp(\alpha_n c) = 1$. Let

$$k_n = h - \alpha_n = i\epsilon_n\,|\,k_n\,|, \qquad k'_n = h + \alpha_n = i\epsilon'_n\,|\,k'_n\,|.$$

Then approximately (see Appendix I), if $\gamma = .5772157$,

$$I_n(l) = \int_0^l G(w)e^{\alpha_n w}\,dw$$
$$= \log(2l/r) - \gamma - \log|k_n l| + Ci(|k_n l|) - \epsilon_n i Si(|k_n l|).$$

Let $h = ik$, and

$$\Delta_n = I_n(2l) - I_n(l).$$

From the analysis of Appendix II it is found that if $\alpha_m + \alpha_n \neq 0$,

$$\begin{aligned} hA_{mn}/2 = &(k^2 - \alpha_m\alpha_n)/(\alpha_m + \alpha_n)\{(I_{-m} + I_{-n})[1 + e^{l(\alpha_m+\alpha_n)}] \\ &-(I_m + I_n)[1 + e^{-l(\alpha_m+\alpha_n)}] + e^{l(\alpha_m-\alpha_n)}(I_{-m} - I_n) \\ &+ e^{-l(\alpha_m-\alpha_n)}(I_{-n} - I_m) - e^{2l\alpha_m}\Delta_{-m} + e^{-2l\alpha_m}\Delta_m - e^{2l\alpha_n}\Delta_{-n} + e^{-2l\alpha_n}\Delta_n\} \\ &+ (k^2 + \alpha_m\alpha_n)/(\alpha_m - \alpha_n)\{(I_m + I_{-n})[1 + e^{-l(\alpha_m-\alpha_n)}] \\ &-(I_{-m} + I_n)[1 + e^{l(\alpha_m-\alpha_n)}] + e^{l(\alpha_m+\alpha_n)}(I_{-n} - I_{-m}) \\ &+ e^{-l(\alpha_m+\alpha_n)}(I_m - I_n) + e^{2l\alpha_m}\Delta_{-m} - e^{-2l\alpha_m}\Delta_m + e^{-2l\alpha_n}\Delta_n - e^{2l\alpha_n}\Delta_{-n}\}. \end{aligned} \tag{3.6}$$

Since the constants α_n are unrestricted above, the coefficients A_{nn} may be calculated by a passage to the limit. With the approximation employed already

$$dI_n(l)/d\alpha_n = (1 - e^{-k_n l})/k_n = I'_n$$
$$dI_{-n}(l)/d\alpha_n = -(1 - e^{-k'_n l})/k'_n = I'_{-n}.$$

Allowing α_m to approach α_n above,

$$\begin{aligned} hA_{nn}/2 = &(k^2 - \alpha_n^2)/\alpha_n\ [I_{-n}(1 + e^{2l\alpha_n}) \\ &- I_n(1 + e^{-2l\alpha_n}) + I_{-n} - I_n + e^{-2l\alpha_n}\Delta_n - e^{2l\alpha_n}\Delta_{-n}] \\ &+ (k^2 + \alpha_n^2)[-2l(I_{-n} + I_n) + 2(I'_n - I'_{-n}) + I'_n e^{-2l\alpha_n} \\ &- I'_{-n}e^{2l\alpha_n} + 2l(\Delta_{-n}e^{2l\alpha_n} + \Delta_n e^{-2l\alpha_n}) + \Delta'_{-n}e^{2l\alpha_n} - \Delta'_n e^{-2l\alpha_n}]. \end{aligned} \tag{3.7}$$

For ordinary antenna wire the functions $\log(2l/r)$, $\log(4l/r)$ are the dominant terms in $I_n(l)$ and $I_n(2l)$, respectively; taking only these terms, and replacing Δ_n by zero, it is seen that the corresponding value of A_{mn} becomes

$$16KL' \sin l\beta_m \sin l\beta_n/h,$$

if $\alpha_n = i\beta_n$, and the constants β_n for every n satisfy the equation

$$\beta_n \cot l\beta_n = K, \qquad L' = \log(2l/r). \tag{3.8}$$

Now $J_n(0) = 2i \sin \beta_n l$, hence the dominant terms on the left of (3.5) become

$$\sin l\beta_m \sin l\beta_n\ [16\,KL' + 4h^2L]/h.$$

These vanish if

$$K = k^2L/4L'$$

except if $m = n$; hence the principal diagonal terms, $m = n$, dominate the others and the equations (3.5) can be solved by the method of successive approximations.

To determine the fundamental period of the loaded antenna, approximately, assume $k = \beta_1$ and consider only the logarithmic terms as before; neglecting Δ_1, Δ_{-1},

$$A_{11} = -16\, i\, L' \cot kl \sin^2 kl$$

and approximately,

$$A_{11} - hL\, J_1(0)^2 = i \sin^2 kl \, [-16\, L' \cot kl + 4kL].$$

This vanishes if

$$\cot kl = kL/4L'.$$

A value of k which satisfies this equation is also one of the set of numbers β_n defined by the transcendental equation (3.8).

If a condenser is in series or in parallel with the inductance, the total retarded potential will have an added term due to its charge; the calculation is similar to the preceding, if the mutual capacity between the condenser plates and any part of the antenna may be neglected.

It is easily verified that the functions J_n determined by equation (3.8) are orthogonal on the antenna.

4. *Theory of the inverted L antenna.* The antenna considered here consists of two arms of equal length l, parallel and with the distance d between the centers of the wires; it is assumed that the length l and the wavelength λ of the fields to be considered are large compared with d, and that the current is sensibly constant on the conductor or conductors joining the two branches. The inverted L antenna over a perfectly conducting earth is equivalent to such a pair of arms, if the earth is assumed absent and the image of the actual antenna is assumed to be present instead.

Let s be the distance increasing from the lower to the upper open end of the antenna, while I has the direction of increasing s at every point, $J(-s) = J(s)$. Employing (2.2) and integrating by parts as before, one obtains the equations (3.5) for the Fourier constants α_n, if

$$hA_{mn} = \int^{l}\int^{l} G(s, s_0)\{k^2 J_m(s_0) J_n(s) \cos(s_0, s) - (dJ_m/ds_0)(dJ_n/ds)\}\, ds\, ds_0$$

the integration being extended over the antenna alone. The same value of [illegible] before. The interaction between the vertical and horizontal parts of the antenna will be neglected, and L is assumed to be the [illegible] inductance plus the self-inductance of the vertical connecting arm.

Let $h = \alpha_1$, $dt_l = l + [d^2 + l^2]^{1/2}$, and on the upper arm

$$J_m = e^{a_m(l-z)} - e^{-a_m(l-z)}, \quad 0 \leq z \leq l.$$

Let

$$(4.2) \quad \begin{aligned} I_1 &= \log(2l/r), \\ \bar{I}_1 &= Ci(dk) - Ci(dk/t_l) - i\,[Si(dk) - Si(dk/t_l)] \\ I_{-1} &= \log(2l/r) - \gamma - \log(2kl) + Ci(2kl) - iSi(2kl) \\ \bar{I}_{-1} &= -Ci(kd) + Ci(kdt_l) - i\,[Si(dkt_l) - Si(dk)]. \end{aligned}$$

Under the above assumptions,

$$A_{11} = (4k^2/h^2)\{(1 + e^{2hl})(I_{-1} - \bar{I}_{-1}) - (1 + e^{-2hl})(I_1 - \bar{I}_1)\}.$$

If $d \ll l$, $kd \ll 1$, then $dt_l \doteq 2l$, $d/t_l \doteq d^2/2l$, and

$$Ci(kd) \doteq \gamma + \log kd, \qquad Ci(dk/t_l) \doteq \gamma + \log(d^2/2l)$$

$$I_{-1} - \bar{I}_{-1} \doteq I_1 - \bar{I}_1 \doteq \log(d/r) = L'.$$

Hence

$$\begin{aligned} R(A_{11}/i) &\doteq -16L' \cot kl \sin^2 kl \\ &= 4L' \cot kl \; J_1^2(0), \quad J_1(0) = 2i \sin kl \end{aligned}$$

and the imaginary part of the coefficient of a_1 in (4.1) is small if

$$(4.3) \qquad \cot kl = kL/4L'.$$

The equation

$$\beta_n \cot \beta_n l = k^2 L/4L'$$

again defines an orthogonal set of functions $J_n(s)$ on the antenna.

If the interaction of the vertical and horizontal parts of the antenna is to be taken into account, it is necessary to employ the formulas of Appendix IV.

5. *Calculation of radiation resistance.* Employing electromagnetic units, $(E_{em} = cE_{es})$, and writing

$$\begin{aligned} Z_0 &= i\omega L + cZ', \\ Z_{mn} &= (-cA_{mn} + \delta_n{}^m N_m c Z_i)/J_m(0) J_n(0) \end{aligned}$$

equations (3.5) become

$$(5.1) \qquad \sum_{n=1}^{\infty} (Z_{mn} + Z_0) a_n J_n(0) = \Delta E^0 + N_m e_m / J_m(0).$$

In many cases a rough approximation to the antenna current can be found by considering only the first equation and neglecting $a_2, a_3, \cdots$. If the antenna is transmitting, e_m is zero, and

$$a_1 = \Delta E^0/(Z_{11} + Z_0)J_1(0), \quad J(0) = a_1 J_1(0).$$

If Z is defined by the equation

$$\Delta E^0 = ZJ(0),$$

then

$$Z = Z_0 + Z_{11}.$$

The orthogonal functions J_n were determined in such a way that the imaginary part of $Z_0 + Z_{11}$ is very nearly zero if k is suitably chosen; if the imaginary part is zero, and the remaining Fourier components of the current are negligible, the antenna behaves as a resistance, of amount $Z_0 + Z_{11}$. The constant $R(cZ_i)$ represents the high-frequency resistance per unit length along the antenna; the radiation resistance is $cR(-A_{11})/J_1(0)^2$. For instance, suppose the unloaded vertical antenna excited by a field in which $k = \pi/2l$; in this case $\alpha_n = n\pi i/2l$, and from (3. 7)

$$A_{11} = -4[I_{-1}(2l) - I_1(2l)].$$

Since $J_1(0) = 2i$,

$$\begin{aligned} -cA_{11}/J_1(0)^2 &= c[I_1(2l) - I_{-1}(2l)] \\ &= c[\gamma + \log(2\pi) - Ci(2\pi) + iSi(2\pi)]. \end{aligned}$$

Hence the reactance of the unloaded antenna vanishes only for a wavelength slightly different from $4l$; it can be shown that if

$$L_0 = 2\log(4l/r) - 1 - [\gamma + \log 2\pi - Ci(2\pi)],$$

and the ratio $4l/r$ is large, the fundamental wavelength is very approximately given by

$$1 - 4l/\lambda = Si(2\pi)/\pi[L_0 - 1 + Si(2\pi)/\pi].$$

As the difference is small, the radiation resistance at resonance is very nearly *

$$c[\gamma + \log 2\pi - Ci(2\pi)].$$

The outward energy flow due to the reflected field is easily expressed in terms of the coefficients a_n and A_{mn}. If $\sum$ is a sphere of large radius enclosing the antenna, assumed isolated in space, the energy crossing $\sum$ per unit time is equal to

$$S = (c/8\pi) \iint_\Sigma ([\mathbb{E}, \bar{\mathbb{H}}]_n + [\bar{\mathbb{E}}, \mathbb{H}]_n)\, dS$$

[illegible]

* Ballantine, *l. c.*, p. 828 (15).

The integral is independent of the shape or position of the surface; evaluating it on the antenna, and assuming $[\boldsymbol{n}, \boldsymbol{H}]/4\pi = I/2\pi r$, as in the preceding paragraphs,

$$S = -(c/2) \int (E_s \bar{J} + \bar{E}_s J)\, ds.$$

Let E_s have the expansion in orthogonal functions

$$E_s = \sum_{n=1}^{\infty} b_m J_m(s).$$

Then

$$S = -(c/2) \sum_{m=1}^{\infty} (b_m \bar{a}_m + \bar{b}_m a_m) N_m. \tag{5.2}$$

From (4.1),

$$N_m b_m = \sum_n a_n \{-hL J_m(0) J_n(0) + A_{mn}\}.$$

Hence if $J_m(0)J_n(0)$ is real,

$$S = -(c/2) \sum_{m,n=1}^{\infty} J_m(0) J_n(0) [\bar{a}_m a_n(-hL) + a_m \bar{a}_n(hL)]$$
$$-(c/2) \sum_{m,n=1}^{\infty} [A_{mn} \bar{a}_m a_n + \bar{A}_{mn} a_m \bar{a}_n].$$

The first sum on the right drops out; if all the current components except the first are neglected,

$$S' = -c \mid a_1 \mid^2 (\bar{A}_{11} + \bar{A}_{11})/2.$$

The current at $s = 0$ is given by $J = a_1 J_1(0)$, hence

$$S' = -cJ^2 R(A_{11})/J_1(0)^2 = J^2 R_1$$

if

$$R_1 = cR(-A_{11})/J_1(0)^2 \tag{5.3}$$

the value found previously.

6. *Explicit formulas for a perfectly conducting earth.* A set of functions $J_n(s)$ which are orthogonal on the antenna with its image are also orthogonal on the upper part alone, while when this half alone is considered, $N'_m = N_m/2$. Also, if the integration is extended only over the upper half in the calculation of A_{mn}, just half the former value is obtained, as a result of the equation $J_n(-s) = J_n(s)$, $s = 0$ denoting as always the grounding point. If Z'_0 denotes the loading inductance in the upper half, $Z'_0 = (1/2)Z_0$ in equation (5.1), and $Z'_{mn} = (1/2)Z_{mn}$. Hence instead of (5.1) one obtains

$$(6.1) \qquad \sum_{n=1}^{\infty} (Z'_{mn} + Z'_0) a_n J_n(0) = \Delta E^0 + N'_m e_m / J_m(0).$$

The radiation resistance, when this can be defined by means of $J_1(s)$, comes out half its former value; the definition of the numbers α_n, β_n is the same as before. The constants e_m in the equation above are now the Fourier constants of the impressed field on the upper half.

7. *The reciprocal theorem.* Let two antennas be present, while the presence of the earth is neglected. Let E'_s be the electric intensity on antenna 1 which would exist if this antenna were absent; $\boldsymbol{E}'$ is the field impressed by acurrent J'' in antenna 2; similarly, $\boldsymbol{E}''$ is the field impressed along antenna 2 by a current J' in antenna 1. The fields $\boldsymbol{E}', \boldsymbol{E}''$ can be represented in the form

$$\boldsymbol{E}' = -\operatorname{grad} \boldsymbol{P}'' - h\boldsymbol{W}'',$$
$$\boldsymbol{E}'' = -\operatorname{grad} \boldsymbol{P}' - h\boldsymbol{W}'.$$

Here

$$\boldsymbol{P}''_{s_1} = -(1/h) \int_{(2)} (dJ''/ds_2) G(s_1, s_2) ds_2,$$

$$\boldsymbol{W}''_{s_1} = \int_{(2)} J'' G(s_1, s_2) \cos(s_1, s_2) ds_2.$$

It follows that

$$\int_{(1)} \boldsymbol{E}'_s J'(s_1) ds_1 = -\int_{(1)}\int_{(2)} J'(s_1)(d/ds_1)[(-dJ''/hds_2) G(s_1, s_2)] ds_1 ds_2$$
$$- h \int_{(1)}\int_{(2)} J'(s_1) J''(s_2) \cos(s_1, s_2) G(s_1, s_2)\, ds_1 ds_2$$
$$= (1/h) \int_{(1)}\int_{(2)} G(s_1, s_2) \{k^2 J' J'' \cos(s_1, s_2)$$
$$- (dJ'/ds_1)(dJ''/ds_2)\} ds_1 ds_2.$$

Hence

$$\int_{(1)} \boldsymbol{E}'_s J'(s_1) ds_1 = \int_{(2)} \boldsymbol{E}''_s J''(s_2) ds_2.$$

Let J' be the *mth* orthogonal function on 1, J'' the *nth* orthogonal function on 2; the integrals above represent Fourier expansion constants on their [illegible] antennas, if the ΔE impressed between the terminals of the loading [illegible] may be neglected. [illegible]

$$\int J'^2_m ds_1 = N'_m, \quad \int_{(2)} J''^2_n ds_2 = N''_n.$$

0

and it may be assumed that on the antenna,

$$E'_s = \sum e'_m J'_m(s_1), \quad e'_m = (1/N'_m)\int_{(1)} E'_s J'_m(s_1)\,ds_1.$$

The integral identity becomes,

$$N'_m e'_{nm} = N''_n e''_{mn}$$

if e'_{nm} is *mth* Fourier constant of the field due to the *nth* orthogonal function of the set defined for antenna 2. The first subscript indicates the origin in each case. The functions on each antenna may be normalized by the condition $N'_m = N''_n \doteq 1$; in this case the simpler equation $e'_{nm} = e''_{mn}$ results.

Appendix I.

The integrals $I_n(l)$ may be calculated as follows. Let δ be a number large compared to r, but such that $\alpha\,\delta$ is very small, $h - \alpha = k = \epsilon i\,|\,k\,|$.

$$I_a(l) = \int_0^l G(w)e^{aw}dw = \left(\int_0^\delta + \int_\delta^l\right) Ge^{aw}dw = I' + I''.$$

$$I' = \int_0^\delta G(w)[1 + \alpha w + \alpha^2 w^2/2! + \cdots]\,dw \doteq \int_0^\delta G(w)\,dw$$

$$= (1/2)\int_{-\delta}^{\delta} G(w)\,dw = I_0(hr)K_0(hr)^* - \int_\delta^\infty e^{-hw}dw/w$$

$$= -\log(hr/2) - \gamma + Ci(d\,|\,h\,|) + i\,[\pi/2 - Si(d\,|\,h\,|)]$$

$$= -\log(r/2) - \log|\,h\,| - \pi i/2 - \gamma + [\gamma + \log\delta\,|\,h\,|] + i\,[\pi/2 - Si(\delta\,|\,h\,|)]$$

$$= \log(2\delta/r) + \delta\,|\,h\,|\,(\cdot\cdot\cdot).$$

$$I'' = -[Ci(\delta\,|\,k\,|) - Ci(l\,|\,k\,|)] - \epsilon i[Si(l\,|\,k\,|) - Si(\delta\,|\,k\,|)].$$

Neglecting powers of $\delta\,|\,k\,|$,

$$Ci(\delta\,|\,k\,|) = \gamma + \log|\,k\,| + \log\delta.$$

$$I_a(l) = \log(2l/r) - \gamma - \log|\,kl\,| + Ci(l\,|\,k\,|) - \epsilon i Si(l\,|\,k\,|).$$

Appendix II.

Let $J_n(z)$ on the verticle antenna be defined by

(1) $$J_n(z) = 2i\sin\beta_n(d + z), \quad a \leq z \leq b \qquad (i\beta_n = \alpha_n)$$

(2) $$= 2i\sin\beta_n(d - z), \quad c \leq z \leq d \qquad (l = d - c).$$

$$a = -d, \qquad b = -c.$$

* Integrating first from $-\infty$ to $+\infty$, $4\pi k_0(hd)$ results from No. 917 of the tables of G. A. Campbell, *Bell System Technical Journal*, October, 1928, p. 687, if d is the distance from a point outside the cylinder to a line on the surface. The integration with respect to θ is easily performed using the addition theorem, Watson, *Theory of Bessel Functions*, p. 361 (8).

If $h = ik$, and the lower and upper parts of the antenna are denoted by (1), (2) respectively,

$$hA_{mn} = \sum_{i,j=1}^{2} \int_{(i)} dz_0 \int_{(j)} dz G(z - z_0) \{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)(dJ_m/dz)\}.$$

Each term of this sum can be expressed linearly in terms of the integrals

$$I_{rs}^{mn} = \int_{(r)} dz_0 \int_{(s)} dz G(z - z_0) e^{\alpha_m z_0 + \alpha_n z} \qquad (r, s = 1, 2).$$

Integrating by parts, assuming $\alpha_m + \alpha_n \neq 0$,

$$I_{11}^{mn} = \int_a^b dz_0 \int_a^b dz G(z - z_0) e^{\alpha_m z_0 + \alpha_n z} = \int_a^b e^{(\alpha_m + \alpha_n) z_0} dz_0 \int_{a-z_0}^{b-z_0} G(w) e^{\alpha_n w} dw$$

$$= [1/(\alpha_m + \alpha_n)]\{e^{b(\alpha_m+\alpha_n)}[I_{-n}(l) + I_{-m}(l)] - e^{a(\alpha_m+\alpha_n)}[I_m(l) + I_n(l)]\}$$

if

$$I_m(l) = \int_0^l G(w) e^{\alpha_m w} dw, \qquad I_{-m}(l) = \int_0^l G(w) e^{-\alpha_m w} dw.$$

Similarly,

$$I_{22}^{mn} = [1/(\alpha_m + \alpha_n)]\{e^{d(\alpha_m+\alpha_n)}[I_{-m}(l) + I_{-n}(l)] - e^{c(\alpha_m+\alpha_n)}[I_m(l) + I_n(l)]\}.$$

Also

$$I_{21}^{nm} = I_{12}^{mn} = [1/(\alpha_m + \alpha_n)]\Big[e^{-(\alpha_m+\alpha_n)c} \int_{2c}^{l+2c} G(w) e^{\alpha_n w} dw$$

$$- e^{-(\alpha_m+\alpha_n)d} \int_{l+2c}^{2l+2c} G(w) e^{\alpha_n w} dw$$

$$+ e^{(\alpha_m+\alpha_n)d} \int_{l+2c}^{2l+2c} G(w) e^{-\alpha_m w} dw - e^{(\alpha_m+\alpha_n)c} \int_{2c}^{l+2c} G(w) e^{-\alpha_m w} dw\Big].$$

It is seen that if $c = 0$,

$$I_{11}^{-m,-n} = I_{22}^{m,n} = e^{l(\alpha_m+\alpha_n)} I_{11}^{mn}, \qquad I_{12}^{-m,-n} = I_{12}^{nm} = I_{21}^{mn}.$$

The formulas (3.6) are obtained on substitution of the integrals above in the expression for A_{mn}, $c = 0$.

APPENDIX III.

Let the superscripts 1, 2 denote the upper and lower arms, respectively; the integral in (4.1) is the sum of four terms

$$A_{mn} = A_{mn}^{11} + A_{mn}^{12} + A_{mn}^{21} + A_{mn}^{22}$$

the first position above and below denoting [illegible]. On the lower half $z = -$[illegible], $0 \leq z \leq l$. Hence

0

$$A^{11}_{mn} = A^{22}_{mn} = (1/h) \int_0^l \int_0^l G(z_0, z)\{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)(dJ_n/dz)\} dz_0$$

$$A^{12}_{mn} = A^{21}_{nm} = (-1/h) \int_0^l \int_0^l G(z_0, z)\{k^2 J_m(z_0) J_n(z) - (dJ_m/dz_0)(dJ_n/dz) dz_0$$

$$\bar{G} = e^{-h\rho}/\rho, \qquad \rho = [(z - z_0)^2 + d^2]^{1/2}.$$

The reduction of the double integrals to single integrals takes place as before; if I^{mn}_{22} has the same meaning as in Appendix II,

$$\begin{aligned} hA^{11}_{mn} = {} & (k^2 - \alpha_m \alpha_n)[e^{-l(\alpha_m+\alpha_n)} I^{mn}_{22} + e^{l(\alpha_m+\alpha_n)} I^{-m,-n}_{22}] \\ & - (k^2 + \alpha_m \alpha_n)[e^{-l(\alpha_m-\alpha_n)} I^{m,-n}_{22} + e^{l(\alpha_m-\alpha_n)} I^{-m,n}_{22}] \end{aligned}$$

and

$$\begin{aligned} -hA^{12}_{mn} = {} & (k^2 - \alpha_m \alpha_n)[e^{-l(\alpha_m+\alpha_n)} K^{mn}_{12} + e^{l(\alpha_m+\alpha_n)} K^{-m,-n}_{12}] \\ & - (k^2 + \alpha_m \alpha_n)[e^{-l(\alpha_m-\alpha_n)} K^{m,-n}_{12} + e^{l(\alpha_m-\alpha_n)} K^{-m,n}_{12}] \end{aligned}$$

if K^{mn}_{12} is obtained from I^{mn}_{12} by replacing G by $\bar{G}$. If $\bar{I}_n$ is obtained from I_n by the same substitution,

$$K^{mn}_{12} = [1/(\alpha_m + \alpha_n)]\{e^{l(\alpha_m+\alpha_n)}(\bar{I}_{-m} + \bar{I}_{-n}) - (\bar{I}_m + \bar{I}_n)\}.$$

The integral

$$\bar{I}_n = \int_0^l e^{\alpha_n w - h[d^2+w^2]^{1/2}} dw/(d^2 + w^2)^{1/2}$$

is easily evaluated by substituting

$$\begin{aligned} dt &= [d^2 + w^2]^{1/2} + w, & [d^2 + w^2]^{1/2} &= (d/2)(t + 1/t) \\ (d/t) &= [d^2 + w^2]^{1/2} - w, & w &= (d/2)(t - 1/t). \end{aligned}$$

Let

$$k = (h - \alpha)/2, \qquad k' = (h + \alpha)/2.$$

Then

$$\bar{I}_n = \int_{t_1}^{t_2} e^{-dkt - dk'/t} dt/t = \sum_{n=0}^{\infty} [(-dk')^n/n!] \int_{t_1}^{t_2} e^{-dkt} dt/t^{n+1}$$

Integrating by parts

$$\begin{aligned} \int_t^\infty e^{-\gamma t} dt/t^n = {} & e^{-\gamma t}[1/nt^n - \gamma/n(n-1)t^{n-1} \\ & + \gamma^2/n(n-1)(n-2)t^{n-2} + \cdots + (-\gamma)^{n-1}/n!t] \\ & + (-\gamma)^n/n! \int_t^\infty e^{-\gamma t} dt/t. \end{aligned}$$

The above expansion may be used if $h + \alpha$ is small; if $h - \alpha$ is small, the integral can be written

$$\bar{I}_n = \int_{1/t_2}^{1/t_1} e^{-dk'u - dk/u} du/u$$

and expanded in the same manner. The second and fourth formulas of (2.4) result from the substitution $h = \pm \alpha_1$.

Appendix IV.

If an antenna is constructed of straight wires in the same plane, the constant A_{mn} may be calculated as a sum of integrals over the same segment, already discussed, and integrals of the form

$$I = \int_{y_1}^{y_2} e^{\beta y} dy \int_{x_1}^{x_2} e^{\alpha x - h\rho} dx/\rho ; \qquad \rho = [x^2 + y^2 - 2cxy]^{1/2}$$
$$c = \cos(x, y).$$

In a first approximation only integrals occur for which $h - \alpha$ or $h + \alpha$, and $h - \beta$ or $h + \beta$ are small compared to h. If $h - \alpha$ is small, the transformation $\alpha' = -\alpha$, $x' = -x$, $c' = -c$ reduces the integral to the form in which $h + \alpha'$ is small, and this will be assumed.

Let t be defined by

$$y(1-c)t = \rho + x - cy$$
$$y(1+c)/t = \rho - (x - cy)$$

Then

$$\rho = (y/2)[(1-c)t + (1+c)/t],$$
$$x - cy = (y/2[(1-c)t - (1+c)/t], \quad \partial x/\partial t = \rho/t.$$

Substituting

$$k = (h-\alpha)(1-c)/2, \quad k' = (h+\alpha)(1+c)/2.$$
$$I = \int_{y_1}^{y_2} e^{y(\beta+ac)} J(y)\, dy,$$

if

$$J(y) = \int_{t_1}^{t_2} e^{-kyt - k'y/t} dt/t = \int_{yt_1}^{yt_2} e^{-ku - k'y^2/u} du/u, \quad t_1 = t(x_1, y).$$

One obtains

$$\partial J/\partial y = J' + J'',$$
$$J' = [e^{-kyt-k'y/t}[(1/yt)(\partial yt/\partial y) + 2k'/t]\,]_{t_1}^{t_2}$$
$$J'' = 2kk'y \int_{yt_1}^{yt_2} e^{-ku - k'y^2/u} du/u.$$

neglecting terms in k'^2. Integrating by parts,

$$I = [J(y) e^{y(\beta+\alpha c)}/(\beta + \alpha c)]_{y=y_1}^{y=y_2} - [1/(\beta + \alpha c)] \int_{y_1}^{y_2} e^{y(\beta+ac)} (J' + J'') dy.$$

Let

$$f(y) = e^{y(\beta+\alpha c)-kyt-k'y/t}$$

$$R(x) = \int_{y_1}^{y_2} f(y)\,(\partial yt/\partial y)\,dy/yt, \qquad S(x) = \int_{y_1}^{y_2} f(y)\,(\partial yt/\partial y)\,dy/t,$$

$$T(x) = \int_{x_1}^{x_2} f(y)\,dy/t.$$

$$V(y) = 2kk'e^{y(\beta+\alpha c)}[y/(\beta+\alpha c) - 1/(\beta+\alpha c)^2].$$

Then if $\beta + \alpha c \neq 0$,

$$I'' = \int_{y_1}^{y_2} J'' e^{y(\beta+\alpha c)} dy = [V(y) \int_{yt_1}^{yt_2} e^{-ku}(\cdot du/u)]_{y=y_1}^{y=y_2}$$
$$- [2kk'/(\beta+\alpha c)]\,\{S(x_2) - S(x_1) - [R(x_2) - R(x_1)]/(\beta+\alpha c)\} + k'^2(\cdots)$$

and to terms in k'^2

$$I = [J(y)e^{y(\beta+\alpha c)}/(\beta+\alpha c)]_{y_1}^{y_2} - [1/(\beta+\alpha c)]$$
$$\times \{R(x_2) - R(x_1) + 2k'[T(x_2) - T(x_1)] + I''\}.$$

The integrals R, S, T can be evaluated by the same method. From the definition of t,

$$(\partial yt/\partial y)/t = 1 - x/\rho.$$

Let the variable u be defined by

$$x(1-c)u = \rho + y - cx \qquad \rho = (x/2)[(1-c)u + (1+c)/u]$$
$$x(1+c)/u = \rho - (y - cx). \quad y - cx = (x/2)[(1-c)u - (1+c)/u]$$
$$\partial y/\partial u = \rho/u.$$

Let

$$\gamma = (1+c)/(1-c), \quad m = (h-\beta)(1-c)/2, \quad m' = (h+\beta)(1+c)/2.$$

It results that

$$y = x(1-c)(u-1)(u+\gamma)/2u, \quad 1/t = (u-1)/(u+\gamma).$$
$$y(\beta+\alpha c) - kyt - k'y/t = x(\alpha+\beta c) - mxu - m'x/u.$$

Substituting,

$$R(x) = e^{x(\alpha+\beta c)} \int_{u_1}^{u_2} e^{-mxu-m'x/u}[-1/u + 2/(u+\gamma)]\,du$$

$$S(x) = e^{x(\alpha+\beta c)} \int_{u_1}^{u_2} e^{-mxu-m'x/u}[x(1-c)(1+\gamma/u^2)/2 - x/u]\,du$$

$$T(x) = [x(1-c)e^{x(\alpha+\beta c)}/2] \int_{u_1}^{u_2} e^{-mxu-m'x/u}$$
$$[1 + (\gamma+1)/\gamma u - 1/u^2 - (\gamma+1)^2/\gamma(u+\gamma)]\,du.$$

For small m' the series expansion of Appendix III can be employed.

ON THE GROUPS GENERATED BY TWO OPERATORS OF ORDERS TWO AND THREE WHOSE PRODUCT IS OF ORDER EIGHT.

By H. R. Brahana.

1. *Hypotheses and definition of H.* A group G whose generators, S and T, satisfy the relations:

$$S^3 = T^2 = (ST)^8 = 1, \tag{1}$$

has a commutator subgroup K generated by $\sigma_1 = TS^2TS$ and $\sigma_2 = TSTS^2$ which, since $(ST)^8 = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\ S^2 = 1$, contains S. If K contains T also then $\{S, T\}$ is perfect and hence is isomorphic with some simple group of composite order. The knowledge that a group is perfect reduces very greatly the number of additional facts that must be ascertained before the group can be identified because the number of simple groups of low order is small. We shall be interested here in the cases where G is not perfect.

When G is not perfect, K is of index 2 and contains all operators which are squares in G. $(ST)^4$ written in terms of the σ's and S is $\sigma_2^{-1}\sigma_1\sigma_2 \cdot S$. Then $(ST)^4S^2$ is conjugate to σ_1. Hence* we have the

Theorem. *If S and T satisfy conditions* (1) *and if in addition the order of TS^2TS is of the form $6k \pm 1$, then the commutator subgroup of $\{S, T\}$ contains a perfect group.*

The conjugates of T_1 which we shall use to denote $(ST)^4$ generate an invariant subgroup H^1 which is in K. The quotient group G/H^1 is of order 2, 6, or 24, its generators satisfying conditions which determine the octahedral group. We shall consider cases where G/H^1 is of order 24. Since this group is octahedral the operator TS^2TS must have an order which is a multiple of 3. If this order were exactly 3 it is easy to prove that the order of $\{S, T\}$ is 24 and that the order of (ST) is 4 instead of 8. The lowest order of σ_1, if (ST) is actually to be of order 8, is 6. We propose to consider this case. We are then dealing with groups generated by two operators satisfying (1), isomorphic with the octahedral group, in which TS^2TS is of order 6.

Under the conditions above the group $\{S, (ST)^4\}$ is generated by two operators of orders 3 and 2 whose product is of order 6. Then $\{S, T_1\}$ con-

* "Certain Perfect Groups, etc.," *American Journal of Mathematics*, Vol. 50 (1928), p. 347.

tains an invariant Abelian subgroup* of one-sixth its order, its commutator subgroup, which is generated by one or two operators. This Abelian group is generated by (T_1T_2) and (T_1T_3) where $T_2 = S^{-1}T_1S$ and $T^3 = S^{-1}T_2S$.

The group $\{T_1, T_2, T_3\}$ is generalized dihedral and may or may not be invariant under G, depending on whether or not T transforms T_3 into some operator of $\{T_1, T_2, T_3\}$. If $\{T_1, T_2, T_3\}$ is invariant then the quotient group corresponding to it is of order 24. If it is not invariant then it is one of a set of conjugate subgroups which are transformed according to a group generated by operators satisfying the same conditions as generators of G.

In case $\{T_1, T_2, T_3\}$ is invariant we have the abelian invariant subgroup $\{T_1T_2, T_1T_3\}$ which we denote by H. Then none of the operators S, T, and T_1 is permutable with every operator of the Abelian group and hence $\{S, T\}$ corresponds to a subgroup of order 48 in the group of isomorphisms of H. This subgroup is generated by operators S_i and T_i which satisfy (1) and for which $(S_iT_i)^4$ is an invariant operator. H cannot be cyclic since its group of isomorphisms contains G_{48}. This last fact requires that $\{S, T_1\}$ be among those groups described by Miller (*loc. cit.*) which have non-cyclic commutator subgroups. A further condition on H is imposed by the fact that its group of isomorphisms contains G_{48}. The Abelian groups of order p^2 and type 1, 1 are of particular interest because there is much known of their groups of isomorphisms that is applicable here.

2. *H of order p^2.* Let us suppose that H is of order p^2 and type 1, 1. There is a $1 - 1$ correspondence between the operators of the group of isomorphisms of H and matrices $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ whose determinants are different from zero, mod p. Let us arrange the operators of I in cosets with respect to the central C which consists of operators of the form $\begin{pmatrix}\alpha & 0 \\ 0 & \alpha\end{pmatrix}$ and let us arrange the operators of the central C so that the first two are $\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}$. The determinants of matrices in C are all squares. The matrices whose determinants are squares form a group whose order is half the order of I. We may then require that the first half of the cosets contain only operators whose determinants are squares. We may arrange the matrices in each coset of this first half so that the first is of determinant unity for if it were another square, b^2, the set would contain a matrix whose determinant $b^2\alpha_i^2$ for some α_i would be unity. Then the second matrix is also of deter-

* Miller, "On the Groups Generated, etc.," *Quarterly Journal*, Vol. 33 (1901), p. 76.

minant unity. There are no matrices of determinant unity except those of the first two rows of the first half of the cosets. The remaining operators of I may be obtained by multiplying the matrices whose determinants are squares by $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ where α is an arbitrary not-square. The group made up of the matrices of the first two rows of the first half of the cosets is $SLH(2, p)$; its quotient group with respect to the two operators in C is $LF(2, p)$.

Now G transforms H according to a group of order 48 which has an invariant operator of order 2. Hence G_{48} will be isomorphic with a subgroup of order 24 of the quotient group of I with respect to C. Since both I/C and the G_{24} contain a subgroup of half its order then the G_{12} of G_{24} is in the invariant subgroup composed of operators which correspond to squares in I/C. G_{48} determines therefore a G_{12} in $LF(2, p)$. The G_{48} can determine a G_{24} in $LF(2, p)$ only if p is of the form $8h \pm 1$.

We require not only that G determine a G_{24} in I/C but also that G be generated by S and T of orders 3 and 2 respectively. Then the determinant of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ corresponding to T must be 1 or -1. If this determinant were 1, T could not be of order 2; T might correspond to an operator of order 2 of $LF(2, p)$ but would itself be of order 4.* If G_{24} is in $LF(2, p)$ p is of the form $8h \pm 1$; then since the determinant of T is -1, -1 must be a square, mod p, which requires that p be of the form $4k \pm 1$. Hence,

If G is isomorphic with a G_{24} of $LF(2, p)$ then p is of the form $8h + 1$.

If on the other hand the G_{24} of I/C is not in $LF(2, p)$ then -1 must be a not-square, mod p, and p must be of the form $4k - 1$. Therefore,

There exists no group G of the type in question containing an invariant subgroup H of order p^2 and type 1, 1 where p is of the form $8h - 3$.

Let us consider next an operator S of order 3 which we denote by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ whose determinant is Δ. Then $\Delta^3 \equiv 1$, mod p, and we have the following conditions* on α, β, γ, and δ:

$$\begin{aligned} \alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta &\equiv 1, \\ \alpha\beta\gamma + 2\beta\gamma\delta + \delta^3 &\equiv 1, \\ \beta(\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma) &\equiv 0, \\ \gamma(\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma) &\equiv 0. \end{aligned}$$

The last two require that $(\alpha + \delta)^2 - \Delta \equiv 0$, unless both β and γ are zero.

* Klein-Fricke, *Theorie der elliptischen Modulfunktionen*, p. 393.

* The congruences that follow are all mod p.

We have seen that the determinant of T must be -1; then since (ST) must be of order 8 and such that $(ST)^2$ corresponds to an operator of order 2 in $LF(2, p)$ its determinant $(-\Delta)^2$ must be 1. Since we have both $\Delta^2=1$ and $\Delta^3=1$, it follows that $\Delta=1$. So whenever not both β and γ are zero we have $\alpha+\delta\equiv\pm 1$. If we select the positive sign the above conditions on α, β, γ, and δ become:

$$\alpha^3+\alpha\beta\gamma+\beta\gamma(\alpha+\delta)\equiv 1,$$
$$\alpha\delta-\beta\gamma\equiv 1, \qquad \alpha+\delta\equiv 1.$$

The first reduces as follows, in view of the other two:

$$\alpha^3+\alpha\beta\gamma+\beta\gamma\equiv 1, \qquad \alpha^3+(\alpha+1)(\alpha\delta-1)\equiv 1,$$
$$\alpha^3+\alpha^2\delta+\alpha\delta-\alpha-1\equiv 1, \qquad \alpha^2+\alpha\delta-\alpha-1\equiv 1, \qquad \alpha-\alpha-1\equiv 1,$$

which is impossible when $p\neq 2$. Hence we may assume that $\alpha+\delta\equiv -1$ when not both β and γ are zero. Since all the subgroups of order 3 of I/C are conjugate and since for every p there exist operators of order 3 for which not both β and γ are zero, it will not be necessary to consider that alternative. We may therefore assume that our G_{48} contains the particular $S_i=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$ We next seek a T_i such that $\{S_i, T_i\}$ is a G_{48} in I.

Let T_i be $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ where α, β, γ, and δ satisfy the relations

$$\alpha+\delta\equiv 0, \qquad \alpha\delta-\beta\gamma\equiv -1,$$

which we determined above. We now consider (S_iT_i) and arrive at further conditions on α, β, γ, and δ

$$(S_iT_i)=\begin{pmatrix}-\gamma & -\delta\\ \alpha-\gamma & \beta-\delta\end{pmatrix}.$$

The determinant of (S_iT_i) is -1.

$$(S_iT_i)^2=\begin{pmatrix}\gamma^2-\alpha\delta+\gamma\delta & \gamma\delta-\beta\delta+\delta^2\\ -\alpha\gamma+\gamma^2+(\alpha-\gamma)(\beta-\delta) & -\alpha\delta+\gamma\delta+(\beta-\delta)^2\end{pmatrix}=\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}.$$

If this is to be an operator of order 4 and to correspond to an operator of order 2 in $LF(2, p)$ we must have $\alpha'+\delta'\equiv 0$. This condition, in terms of α, β, γ, and δ is

$$\gamma^2-2\alpha\delta+2\gamma\delta+\beta^2-2\beta\delta+\delta^2\equiv 0,$$

which may be written as

$$\gamma^2+\beta^2+\delta^2-2\beta\delta+2\gamma\delta-2\beta\gamma-2(\alpha\delta-\beta\gamma)\equiv 0,$$

which is

$$(\beta-\gamma-\delta)^2\equiv 2(\alpha\delta-\beta\gamma)\equiv -2.$$

This last relation states among other things that -2 is a square: this implies that 2 and -1 are simultaneously squares or not-squares. Now -1 is a square if and only if p is of the form $4h+1$, and 2 is a square if and only if p is of the form $8k \pm 1$. Hence, the last condition can be satisfied if p is of the form $8h+1$ or $8h+3$.

Conversely, when p is of the form $8h+1$ or $8h+3$, then -2 is a square, and I contains the operator $T_i = \begin{pmatrix} 1 & -1 \pm (-2)^{1/2} \\ 0 & -1 \end{pmatrix}$ of order 2. I also contains $S_i = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. (S_iT_i) is $\begin{pmatrix} 0 & 1 \\ 1 & \pm(-2)^{1/2} \end{pmatrix}$ and $(S_iT_i)^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which is of order two and invariant. Therefore S_i and T_i satisfy the relations $S_i^3 = T_i^2 = (S_iT_i)^8 = 1$. Hence, $\{S_i, T_i\}$ is a subgroup of order 48 of the group of isomorphisms of H. We thus have the

THEOREM. *A necessary and sufficient condition that an Abelian group of order p^2 and type 1, 1 have in its group of isomorphisms a group of order 48 isomorphic with the octahedral group and generated by an operator of order two and one of order three is that p be of the form $8h+1$ or $8h+3$.*

The existence of the G_{48} establishes the existence of a group of order $48p^2$. We have now to determine whether this group of order $48p^2$ can be generated by two operators of orders two and three, respectively.

Let us consider G to be written in cosets with respect to H. Then these cosets will correspond to operators of G_{48}, and we shall designate the coset which corresponds to S_i or T_i by H_{S_i} or H_{T_i}. The operators of H_{T_i} are all of even order since T_i is of order 2, and T_i is an operator of the quotient group. If we take any such operator, the cyclic group generated by it is contained half in H and half in H_{T_i}. Since p is an odd prime, *the operator of order two is in H_{T_i}.* Denote one such operator of order two by T.

Likewise, if $p \neq 3$, then *H_{S_i} contains an operator of order three.* For any operator of H_{S_i} has an order a multiple of 3 and generates a cyclic group which lies in H, H_{S_i}, and $H_{S_i^2}$; the operators of order 3 could not be in H. If $p = 3$ the group of isomorphisms of H is G_{48} and it is known that the G of order $48 \cdot 3^2$ can be generated by two operators of orders two and three [illegible]. [illegible] denote one such operator by S. Then ST must be of order [illegible] and every operator in H_{ST} has its fourth power [illegible]. All the op[illegible] order two because one [illegible] must be of order two and the group generat[illegible]

[illegible] Society, Vol. 3 [illegible]

is generalized dihedral. Thus we have shown so far that for p of the form $8h+1$ or $8h+3$ there exists a group $\{S, T\}$ contained in G and that S and T satisfy (1). We have yet to show that there exists an $\{S, T\}$ satisfying the same conditions and containing H.

If $\{S, T\}$ contains one operator of H it contains H because the invariant subgroup of index 48 is non-cyclic. So we shall assume that $\{S, T\}$ is a proper subgroup of G. It must then be G_{48}. Then $T_1 = (ST)^4$ will be invariant in $\{S, T\}$. Now since $TM_2T = M_2^{-1}$ the operator M_2T which is in G will be of order 2. Let us denote this operator by T' and consider the group $\{S, T'\}$. If we use the relations

$$\begin{aligned} TM_1T &= M_1M_2^{k} \qquad \text{and} \qquad & S^{-1}M_1S &= M_2^{-1} \\ TM_2T &= M_2^{-1} & S^{-1}M_2S &= M_1M_2^{-1}, \end{aligned}$$

where $k = -1 \pm (-2)^{1/2}$, to simplify the relation

$$(ST')^4 = SM_2TSM_2TSM_2TSM_2T$$

we arrive finally at

$$T_1' = (ST')^4 = (ST)^4 \cdot M_1^{-(k^2+3k+4)} \cdot M_2^{-(k^3+4k^2+8k+7)}.$$

On substituting the value of k we get

$$T'_1 = (ST)^4 M_1^{\mp(-2)^{1/2}} \cdot M_2^{\mp(-2)^{1/2}} = T_1 \cdot M_1^{\mp(-2)^{1/2}} \cdot M^{\mp(-2)^{1/2}}.$$

Then if we let $T'_2 = S^{-1}T'_1S$, we have $T'_1T'_2 = M_2^{\pm 3(-2)^{1/2}}$. Since this last operator is not the identity, the group $\{S, T'\}$ must be G. Therefore,

For every prime of the form $8h+1$ *or* $8h+3$ *there exists a group* $\{S, T\}$ *which contains the Abelian group of order* p^2 *and type* 1, 1 *invariantly and transforms its operator according to the group of order* 48 *whose generators as well as* S *and* T *satisfy* (1) $S^3 = T^2 = (ST)^8 = 1$.

3. *H of order p^n.* Now let us suppose that H is of order p^n where $n > 2$. H will have two generators since its group of isomorphisms I contains G_{48}. Then H must contain a characteristic subgroup H_1 of order p^2 and type 1, 1. This subgroup H_1 will be transformed according to a group isomorphic with G_{48}; moreover the invariant operator of G_{48} is the operator which transforms every operator of H into its inverse and therefore transforms every operator of H_1 into its inverse. Hence H_1 must be transformed according to a group of order 48, 24, or 6, which is non-cyclic and contains an invariant operator of order two. Since, moreover, this group is generated by operators of orders 2 and 3 whose product is of order 8, it must be G_{48}. Therefore, the group

of isomorphisms I_1 of H_1 must contain G_{48}. This fact combined with the theorem of § 2 gives the

THEOREM. *If H is of order p^n, then p must be of the form $8h+1$ or $8h+3$.*

The G_{48} of I_1 is isomorphic with some G_{24} in the group $LF(2, p)$ which is determined by H_1. Since these G_{24}'s are all conjugate we may fix attention on one of them. If the invariants of H are not equal then H will contain a characteristic subgroup of order p, which will be in H_1. The subgroups of order p of H_1 may be designated by matrices $(1\ \alpha)$ determined by generators $M_1M_2^{\alpha}$ where M_1 and M_2 are generators of H_1. Then taking the S_i and T_i of § 2 we seek conditions on p that there exist a subgroup of order p in H_1 which is invariant under G_{48}.*

The subgroup $(1\ \alpha)$ is transformed by S_i into $(\alpha\ \ -(1+\alpha))$. If this subgroup were invariant under S_i we should have the congruence

$$\alpha^2+\alpha+1\equiv 0, \bmod p. \tag{1}$$

The operator T_i transforms the subgroup $(1\ \alpha)$ into $(1\ \ -1\pm(-2)^{1/2}-\alpha)$. This is $(1\ \alpha)$ only if $\alpha\equiv -1\pm(-2)^{1/2}-\alpha$, or

$$\alpha\equiv(-1\pm(-2)^{1/2})2, \bmod p. \tag{2}$$

Substituting the value of α from (2) in (1) we get $1/4\equiv 0$, mod p, which is impossible.

It follows therefore that G_{48} leaves no subgroup of order p of H invariant and that H can have no characteristic subgroup of order p. Hence,

If H is of order p^n, p must be of the form $8h+1$ or $8h+3$, n must be an even number $2m$, and H must be of type m, m.

We inquire whether the above necessary conditions on H are sufficient to insure the existence of a group G of order $48p^{2m}$, and of the type in question.

If H is abelian of order p^{2m} and type m, m its group of isomorphisms I is of order $p^{4m-3}(p-1)(p^2-1)$.† I transforms the characteristic subgroup H_1 of order p^2 and type $1, 1$ according to its group of isomorphisms I_1 of order $p(p-1)(p^2-1)$, for there is a $1-1$ correspondence between operators of I_1 and matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ of non-zero determinants whose elements are

* The subgroup generated by M [illegible] into another subgroup.

† Ranum, "Groups of Classes of Congruent Matrices, etc.," *Transactions of the American Mathematical Society*, Vol. 8 (1907), p. 87.

integers reduced modulo p, and every such matrix determines an operator of I, viz.

$$M_1 \sim M_1^{\alpha} M_2^{\beta}, \qquad M_2 \sim M_1^{\gamma} M_2^{\delta}$$

where M_1 and M_2 are generators of H. I is therefore isomorphic with I_1 and the corresponding invariant subgroup of H is of order p^{4m-4}. It consists of all matrices $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$ where a, b, c, and d are residues mod p^m which are congruent to zero, mod p.

When p is of the form $8h+1$ or $8h+3$, I_1 contains a G_{48} and I contains a group isomorphic with it. We shall show that I contains a group *simply* isomorphic with it.

I contains the operators $S_i = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $T_i = \begin{pmatrix} 1 & -1+(-2)^{1/2} \\ 0 & -1 \end{pmatrix}$ * where the elements of the matrices are residues mod p^m, for -2 is a square mod p^m whenever it is a square mod p.† These operators are of orders 3 and 2 respectively and $(S_iT_i)^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The group $\{S_i, T_i\}$ must then be G_{48}, and we have accordingly established the existence of a group G of order $48p^{2m}$ which contains H invariantly and transforms it according to G_{48}. We have yet to show that G may be generated by operators S and T which satisfy conditions (1) of § 1.

Let the operators of G be arranged in cosets with respect to H, and let H_{Q_i} denote the coset which corresponds to the operator Q_i of the G_{48} in I. The operators of $H_{(S_iT_i)^4}$ are all of even order and one of them must be of order 2 since H contains no operator of even order. Then it follows that all the operators of $H_{(S_iT_i)^4}$ are of order 2. From this it follows that all the operators of $H_{(S_iT_i)}$ are of order 8, and all the operators of $H_{(S_iT_i)^2}$ are of order 4. The operators of H_{T_i} are all of even order and therefore one of them is of order 2. Let this operator of order 2 in H_{T_i} be denoted by T. If $p \neq 3$, then H_{S_i} contains an operator of order 3. The argument of § 2 in which the M_1 and M_2 are now taken to be the operators of order p^m which generate H proves that *G of order $48p^{2m}$* is generated by S and T of orders 3 and 2 whose product is of order 8.

If $p = 3$, then it is conceivable that every operator of H_{S_i} has for a third power some operator not the identity of H. Such an operator must of course be invariant under S_i. Every operator of H may be written as $M_1^{\alpha} M_2^{\beta}$, and this is transformed by S_i into $M_1^{\beta} M_2^{-2\alpha}$. If $M_1^{\alpha} M_2^{\beta}$ is invariant under

* We shall hereafter denote the corresponding operators of I by S_i, and T_i.

† Tchebichef, *loc. cit.*, p. 93.

S_i we must have $\beta \equiv \alpha$ and $\beta \equiv -2\alpha$, mod p^m. From this it follows that $3\alpha \equiv 0$, mod 3^m, and α is a multiple of 3^{m-1}. Hence the only operators of H invariant under S_i are of order 3 and the operators of H_{S_i} must be of orders 3 or 9. Moreover, since $(M_1{}^{\alpha}M_2{}^{\beta}S)^3 = S^3$, where S is any operator of H_{S_i}, it follows that every operator of H_{S_i} is of order 3 or every operator is of order 9. We shall show that they cannot be of order 9.

The group G of order $48p^{2m}$ contains an invariant subgroup of order $24p^{2m}$, which in turn contains an invariant subgroup of order $8p^{2m}$; these subgroups consist of operators in the cosets corresponding to the group $\{S_i, (S_iT_i)^2\}$ and the group generated by the conjugates of $(S_iT_i)^2$, respectively. The former group is generated by an operator S of H_{S_i} and the second group. By a theorem proved in *On the groups which contain etc.*[*] the third power of S must be in the central of the group corresponding to $\{S_i, (S_iT_i)^2\}$. We have already seen that S^3 must be $M_1{}^{a}M_2{}^{a}$. $(S_iT_i)^2 = \begin{pmatrix} (-2)^{1/2} & -1 \\ 1 & (-2)^{1/2} \end{pmatrix}$ which transforms $M_1{}^{a}M_2{}^{a}$ into $M_1{}^{a(1+(-2)^{1/2})}M_2{}^{a(-1+(-2)^{1/2})}$. Therefore, S must be of order 3.

Incidentally we have proved the

THEOREM. *The central of G of order $48p^{2m}$ is the identity.*

We are now assured of the existence in G of S and T which satisfy the relations $S^3 = T^2 = (ST)^8 = 1$. We have still to show that S and T may be chosen so that $\{S, T\}$ is G. This will follow provided we can show that $\{S, T\}$ contains one operator of highest order of H. We suppose that $T_1'T_2' = (S'T)^4 \cdot S'^{-1}(S'T)^4S = R$, where S' is in H_{S_i} and R is some operator not of highest order in H. Then if we take $S = M_2S'$ and find $T_1T_2 = (ST)^4 \cdot S^{-1}(ST)^4S$ we get $T_1T_2 = M_1{}^{-2}M_2{}^{4-3(-2)^{1/2}} \cdot R$. Since -2 is prime to p and R is not of highest order, T_1T_2 is of highest order. Hence, G is generated by S and T. We have thus arrived at the

THEOREM. *If p is any prime of the form $8h+1$ or $8h+3$ and m is any positive integer, there exists a group of order $48p^{2m}$ which contains invariantly the abelian group of order p^{2m} and type m, m and transforms its operators according to the G_{48} isomorphic with the octahedral group, and G is generated by operators of orders 2 and 3 whose product is of order 8.*

Conversely,

[illegible] group $\{S, T\}$ [illegible] which satisfy the relations

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[*] *American Journal of Mathematics*, Vol. 52 (1930), p. 915.

$S^3 = T^2 = (ST)^8 = 1$, *which is also isomorphic with the octahedral group, whose commutator* TS^2TS *is of order* 6, *and which contains an abelian invariant subgroup of index* 48 *of two invariants, and of order a power of a prime, is such that* H *is of order* p^{2m} *and type* m, m *and* p *is of the form* $8h + 1$ *or* $8h + 3$.

4. *H of any order.* Now let us suppose the order of H is $p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ where the p's are distinct primes. The group of isomorphisms of H is the direct product of the groups of isomorphisms of its Sylow subgroups of orders $p_1^{n_1}, p_2^{n_2}, \cdots p_k^{n_3}$. Since I contains G_{48} which is not a direct product some Sylow subgroup of order $p_i^{n_i}$ must have G_{48} in its group of isomorphisms. Hence some p_i is of the form $8h + 1$ or $8h + 3$ and n_i is an even number. Every Sylow subgroup of order $p_i^{n_i}$ is transformed according to a group Γ_i isomorphic with G_{48} and, unless all the operators of the Sylow subgroup are of order 2, Γ_i contains an invariant operator of order 2. This operator, which transforms every operator of H into its inverse, is the fourth power of an operator in Γ_i and hence Γ_i contains operators of order 8. Therefore Γ_i is G_{48} itself. From this it follows that every one of the p's which is different from 2 is of the form $8h + 1$ or $8h + 3$, and the n's are all even numbers. If p_1 were 2 the operator $(ST)^4$ which transforms every operator of H into its inverse might transform every operator of the Sylow subgroup of order $p_1^{n_1}$ into itself, in which case the Sylow subgroup would be of type $1, 1, 1, \cdots$ and hence would be of order 4. If the Sylow subgroup is not of type 1, 1 then Γ_1 is G_{48}. Moreover, the operators of order 4 would be transformed according to G_{48}. This requires that both invariants be at least as great as 2, and that the Sylow subgroup contain a characteristic subgroup of order 16 and type 2, 2 whose group of isomorphisms contains G_{48}. This is impossible.*

Let us consider the possibility of H being the four-group. Then T_1T_2 is of order 2. The group $\{M_1, M_2, T_1\}$ is abelian of order 8 and type 1, 1, 1 and is invariant. It is transformed according to a group of order 24 in its group of isomorphisms, or a group of lower order isomorphic with it. The group will then be in the holomorph of G_8, and can be written on 8 letters. The operators of G_8 are transformed according to an intransitive group on seven letters, since $\{M_1, M_2\}$ is invariant. The four operators of G_8 not in $\{M_1, M_2\}$ must then be transformed according to G_{14}. These four operators must contain the fourth powers of all the operators of order 8 in $\{S, T\}$, and

* The group of isomorphisms of the group of order 16 can be written on 12 letters for the operators of order 4. The fourth power of an operator of order 8 would be of degree 8, but the operator which transforms everything into its inverse must be of degree 12.

therefore one of them must be invariant under (ST). However, if G_{24} is written on four letters the operator (ST) is of degree four. Therefore such a group is impossible.

The only other possibility, if p_1 is 2, is that H is of order 2. Then $T_1T_2 = T_1T_3 = T_2T_3$ and T_1, T_2, and T_3 are the same. In this case, however, *H is identity*. Hence we have the

THEOREM: *A necessary and sufficient condition that there exist a group G generated by S and T satisfying the conditions.*

(1) $S^3 = T^2 = (ST)^8 = 1,$

(2) $(TS^2TS)^6 = 1,$

(3) *the commutator subgroup H of* $\{S, (ST)^4\}$ *is invariant under G,*

is that the order of H be $p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$, *where the n's are even numbers and the p's are of the form* $8h + 1$ or $8h + 3$.

5. *H non-invariant.* We have seen that the group $\{S, T_1\}$ contains an abelian subgroup H. The groups which we have studied are those for which H is invariant under G. Let us suppose now that H is not invariant under G, and denote its conjugates by $H_1 (= H), H_2 \cdots, H_n$. If H contains a subgroup which is invariant under G this subgroup must satisfy the conditions determined for H in the preceding pages. Moreover, the quotient group of G with respect to this invariant subgroup contains n subgroups H'_i corresponding to the H's, which are not invariant and contain no invariant subgroup. Hence the quotient group can be represented on symbols for the n subgroups H_i. Therefore the group G of lowest order for which H is not invariant can be represented on symbols for the H's. Then the group $\bar{H}$ generated by the H's cannot be abelian, for it would leave each of the H's invariant. Then since (ST) does not leave H_1 fixed, and since T_1 does leave H_1 fixed but permutes the H's according to an operator of order 2, it follows that n must be at least 10. It is easy to prove by similar considerations that n must be at least 12. Hence

If H is not invariant under G, then $\bar{H}$ *is not abelian, and* H_1 *is one of at least* 12 *conjugates.*

Since practically nothing is known concerning non-abelian groups whose groups of isomorphisms contain subgroups isomorphic with G_{24} the consideration of H non-invariant promises to be a large undertaking and to involve considerations that have not entered in the preceding pages.

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ON A NEW NORMAL FORM OF THE GENERAL CUBIC SURFACE.

By Arnold Emch.

1. *Introduction.* Mapping processes are not uncommonly used to give information on, or to facilitate the investigation of certain geometric problems. I refer for example to Segre's cubic variety in S_4 and its relation to the Kummer quartic surface, or to the Veronese surface based upon the system of conics in a plane with applications to Steiner's Roman surface, the theory of plane quartics, etc.

Such problems arise for example also in the mapping of projective spaces S_r upon rational hypersurfaces in S_{r+1}. Associating the variables (x) and (y) to these spaces respectively, the mapping may be accomplished by

$$\begin{gathered}\rho y_1 = x_2 x_3 \cdots x_{r+1}, \quad \rho y_2 = x_1 x_3 \cdots x_{r+1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho y_{r+1} = x_1 x_2 \cdots x_r, \quad \rho y_{r+2} = x_1^r x_2^r + \cdots x^r_{r+1}.\end{gathered} \tag{1}$$

By eliminating the x's it is found that S_r is mapped upon the hypersurface S_r':

$$y_2^r y_3^r \cdots y_{r+1}^r + y_1^r y_3^r \cdots y_{r+1}^r + \cdots y_1^r y_2^r \cdots y_r^r - (y_1 y_2 \cdots y_{r+1})^{r-1} y_{r+2} = 0 \tag{2}$$

of order r^2 in $S_{r+1}(y)$. It is easily recognized that the mapping of S_r upon S_r' is birational, moreover that spaces with an even or odd number of dimensions are respectively mapped upon hypersurfaces of even and odd order. Restricting the problem to an Euclidean space by putting $y_{r+2} = 0$, that spaces have maps of finite or infinite extent according as they are of even or odd dimensions. This fact is related to the unilateralness or bilateralness of projective S_{2k} or S_{2k+1} respectively.

As a more detailed study of the transformation (1) will appear elsewhere, I shall restrict myself to some interesting applications.

One that is already well known is the mapping of a plane S_2 upon a Steiner surface S_2':

$$y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2 - y_1 y_2 y_3 y_4 = 0, \tag{3}$$

or conversely. To the plane sections $\Sigma a_i y_i = 0$, $(i = 1, 2, 3, 4)$ of S_2' corresponds in S_2 an ∞^3 system of conics

$$a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 + a_4 (x_1^2 + x_2^2 + x_3^2) = 0. \tag{4}$$

To the ∞^2 sections by tangent-planes corresponds in S_2 a net of couples of lines among which are four double lines corresponding to the tropes of S_2'. The degenerate order conics of this net are apolar to the ∞^1 class-conics inscribed in the quadrilateral of these double lines. The latter are moreover the invariant lines in the quadratic line transformations $\rho u_1' = u_2 u_3$, $\rho u_2' = u_3 u_1$, $\rho u_3' = u_1 u_2$ in S_2. These few indications suffice to show the importance of this mapping process.

In this paper I shall consider the mapping of S_3 upon the nonic hypersurface S_3':

$$(5) \qquad y_2^3 y_3^3 y_4^3 + y_1^3 y_3^3 y_4^3 + y_1^3 y_2^3 y_4^3 + y_1^3 y_2^3 y_3^3 - y_1^2 y_2^2 y_3^2 y_4^2 y_5 = 0,$$

and its connection with the theory of cubic surfaces.

2. *The cubic surface as the map of a hyperplane in S_4.* It is obviously not difficult to establish corresponding figures in S_3 and S_3' by means of the transformation (1) when $r = 3$, i. e.,

$$(6) \qquad \begin{gathered} \rho y_1 = x_2 x_3 x_4, \quad \rho y_2 = x_1 x_3 x_4, \quad \rho y_3 = x_1 x_2 x_4, \quad \rho y_4 = x_1 x_2 x_3, \\ \rho y_5 = x_1^3 + x_2^3 + x_3^3 + x_4^3. \end{gathered}$$

Thus, to a line $l: \delta x_i = a_i + \lambda b_i$, $(i = 1, 2, 3, 4)$ in S_3 corresponds in S_3' the space-cubic

$$\tau y_i = K_0^{(i)} \lambda^3 + K_1^{(i)} \lambda^2 + K_2^{(i)} \lambda + K_3^{(i)}, \qquad (i = 1, \cdots, 5),$$

which lies on a certain hyperplane $\Sigma c_i y_i = 0$, and which is uniquely determined by the line l. A plane in S_3 may be represented parametrically by

$$\rho x_1 = \lambda_1 c_1, \ \rho x_2 = \lambda_2 c_2, \ \rho x_3 = \lambda_3 c_3, \ \rho x_4 = \lambda_1 + \lambda_2 + \lambda_3,$$

or explicitly by

$$(7) \qquad c_2 c_3 x_1 + c_3 c_1 x_2 + c_1 c_2 c_3 - c_1 c_2 c_3 x_4 = 0.$$

To it corresponds in parametric representation

$$(8) \qquad \begin{gathered} \delta y_1 = c_2 c_3 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3), \quad \delta y_2 = c_3 c_1 \lambda_3 \lambda_1 (\lambda_1 + \lambda_2 + \lambda_3), \\ \delta y_3 = c_1 c_2 \lambda_1 \lambda_2 (\lambda_1 + \lambda_2 + \lambda_3), \quad \delta y_4 = c_1 c_2 c_3 \lambda_1 \lambda_2 \lambda_3, \\ \delta y_5 = c_1^3 \lambda_1^3 + c_2^3 \lambda_2^3 + c_3^3 \lambda_3^3 + (\lambda_1 + \lambda_2 + \lambda_3)^3, \end{gathered}$$

or also the intersection of the hypercone

$$(9) \qquad c_2 c_3 y_2 y_3 y_4 + c_3 c_1 y_1 y_3 y_4 + c_1 c_2 y_1 y_2 y_4 - c_1 c_2 c_3 y_1 y_2 y_3 = 0$$

with the hypernonic (5). This is a surface of order three since (9) and (5) intersect in a surface of order 27. But (5) has the six planes $(x_i = 0, x_k = 0)$.

$i \neq k$, $i, k = 1, 2, 3, 4$, as fourfold planes and (9) has these as single planes in S_4, so that the residual surface of intersection is indeed of order $27 - 4 \cdot 6 = 3$.

Conversely, to the intersection of the hyperplane

$$(10) \qquad c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5 = 0$$

with (5) corresponds in S_3 the cubic surface F_3

$$(11) \quad c_1x_2x_3x_4 + c_2x_1x_3x_4 + c_3x_1x_2x_4 + c_4x_1x_2x_3 + c_5(x_1^3 + x_2^3 + x_3^3 + x_4^3) = 0$$

depending evidently upon four effective constants.

Now the 19 effective constants of a general cubic surface may be reduced to 4 effective constants by collineation in S_3, which thus indicates, as far as counting of constants is concerned, that *the general cubic surface may be written in the form* (11). That this is true will be proved subsequently.

3. *The configuration* Δ_{18} *connected with the normal form* (11). The reduction to this normal form depends upon the properties of a configuration of 18 points Δ_{18} which is common to all cubics of the ∞^4 system (11). That this configuration is of a very special type appears from the fact that ordinarily 18 generic points determine a pencil of cubics, while here there are 5 linearly independent cubics on the Δ_{18}. The intersections of this system with the coördinate planes are syzygetic pencils. For example on $x_4 = 0$ we get

$$(12) \qquad x_1^3 + x_2^3 + x_3^3 + (c_4/c_5)x_1x_2x_3 = 0,$$

with its fixed flexes on the three sides A_2A_3, A_3A_1, A_1A_2, and the three real flexes on $x_1 + x_2 + x_3 = 0$. The same situation exists on the three remaining coördinate planes. The flexes on A_1A_2 for example are obtained by putting $x_3 = 0$, $x_4 = 0$, so that $x_1^3 + x_2^3 = 0$ determines on A_1A_2 the flexes

$$(1, -1, 0, 0); \quad (1, -w, 0, 0); \quad (1, -w^2, 0, 0); \quad w^3 = 1.$$

In a similar manner are found three flexes on each of the remaining 5 edges A_iA_k. We thus obtain a configuration Δ_{18} of 18 such flexes through which all cubics of the system (11) pass. This configuration is invariant under the Abelian collineation group G_{27}.

$$x_1' = x_1, \quad x_2' = w^\alpha x_2, \quad x_3' = w^\beta x_3, \quad x_4' = w^\gamma x_4,$$

$\alpha, \beta, \gamma = 0, 1, 2$. But as together the 4 syzygetic pencils are invariant under the symmetric group G_{24} on x_1, x_2, x_3, x_4, *the entire configuration* Δ_{18} *is invariant under a collineation group* $G_{27 \cdot 24} = G_{648}$ *of order* 648.

To establish the properties of Δ_{18}, consider the plane $x_1 + x_2 + x_3$

$+ x_4 = 0$ which contains the four lines cut out by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$. These intersect in 6 points $(1, -1, 0, 0)$, . . . which are flexes of the configuration of 18 points. Each of these lines, for example $x_1 + x_2 + x_3 + x_4 = 0$, $x_2 + x_3 + x_4 = 0$ determines a pencil of planes

$$x_1 + (\lambda + 1)x_2 + (\lambda + 1)x_3 + (\lambda + 1)x_4 = 0.$$

For $\lambda = 0$ we get the unit plane. For $\lambda + 1 = w$ we have,

$$x_1 + wx_2 + wx_3 + wx_4 = 0,$$

in which lie the six points $(0, 1, -1, 0)$; $(0, 0, 1, -1)$; $(0, 1, 0, -1)$; $(1, -w^2, 0, 0)$; $(1, 0, -w^2, 0)$; $(1, 0, 0, -w^2)$, which belong to the Δ_{18} and which are the six vertices of a quadrilateral of Maclaurin lines (3 collinear flexes). For $1 + \lambda = w^2$ we obtain the plane $x_1 + w^2x_2 + w^2x_3 + w^2x_4 = 0$ in which lies likewise such a quadrilateral. Thus through every Maclaurin line not identical with an edge A_iA_k there are three planes each containing a quadrilateral formed by Maclaurin lines from the Δ_{18}. There are $(12 - 3)4 = 36$ such lines, hence 3×36 such planes. But as each of these planes is on 4 Maclaurin lines, there are altogether $3 \times 36 : 4 = 27$ such planes. These are obtained from $\Sigma x_i = 0$ by the substitutions of the G_{27}. The result may be stated in the

THEOREM. *The 18 points of the configuration Δ_{18} lie by 6 vertices of a quadrilateral in 27 planes, not including the coördinate planes. Through each Maclaurin line different from an A_iA_k there are 3 planes each containing such a quadrilateral. The entire Δ_{18} is invariant under a collineation group of order* 648.

That there is a finite number of Δ_{18} configurations on the general cubic surface will be proved in the next section.

4. *Reduction of the general cubic surface to the Δ_{18} normal form.* It is necessary to first point out an important property of the configuration $\Delta_{9,12}$ of the nine flexes of a general cubic and its twelve Maclaurin lines. By collineation such a cubic may always be reduced to the Hessian normal form

$$x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1x_2x_3 = 0. \tag{13}$$

The $\Delta_{9,12}$ is independent of the parameter λ, from which follows that these [illegible] general cubics are projectively equivalent. This can also be proved directly [illegible]:

The real flex tangents of (13) at $(1, -1, 0)$; $(1, 0, -1)$; $(0, 1,$ [illegible]

$$(14)\qquad \begin{aligned} \rho x_1' &= \lambda x_1 + x_2 + x_3 = 0, \\ \rho x_2' &= x_1 + \lambda x_2 x_3 = 0, \\ \rho x_3' &= x_1 + x_2 + \lambda x_3 = 0. \end{aligned}$$

If these are chosen as the sides of a new coördinate triangle, (13) assumes the form

$$(15)\qquad (x_1' + x_2' + x_3')^3 - 3\frac{(\lambda+2)^3}{\lambda^2+\lambda+1}\,(x_1' + x_2' + x_3') = 0.$$

(14) may be interpreted as a perspective with (1, 1, 1) as a center and $x_1 + x_2 + x_3 = 0$ as the pointwise invariant axis. Now choose arbitrarily three lines

$$(16)\qquad \begin{aligned} \rho x_1' &= \lambda_1(x_1 + x_2 + x_3) + \mu x_1 = 0, \\ \rho x_2' &= \lambda_2(x_1 + x_2 + x_3) + \mu x_2 = 0, \\ \rho x_3' &= \lambda_3(x_1 + x_2 + x_3) + \mu x_2 = 0, \end{aligned}$$

as flex-tangents of a new cubic at the points $(1, -1)$; $(1, 0, -1)$; $(0, 1, -1)$ respectively, which will be of the form

$$(17)\qquad (x_1 + x_2 + x_3)^3 - 3\gamma[\lambda_1(x_1 + x_2 + x_3) + \mu x_1] \times [\lambda_2(x_1 + x_2 + x_3) + \mu x_2]\ [\lambda_3(x_1 + x_2 + x_3) + \mu x_3] = 0.$$

(16) may again be interpreted as a perspective with $x_1 + x_2 + x_3 = 0$ as an axis and $(\lambda_1, \lambda_2, \lambda_3)$ as a center. From (16)

$$\rho(x_1' + x_2' + x_3') = (x_1 + x_2 + x_3)(\lambda_1 + \lambda_2 + \lambda_3 + \mu),$$

so that by (16) (17) is transformed into

$$(18)\qquad (x_1' + x_2' + x_3')^3 - 3\gamma(\lambda_1 + \lambda_2 + \lambda_3 + \mu)^3 x_1' x_2' x_3',$$

or into one of the cubics of the pencil (15). The three real flexes of (15) are identical with those of (13). Moreover two of the vertices of all flex-triangles of (15): $(1, w, w^2)$ and $(1, w^2, w)$ coincide with those of (13). Hence the

THEOREM. *All cubics of the pencil*

$$x_1 + x_2 + x_3)^3 - 3m x_1 x_2 x_3 = 0$$

have perspective flex-configuration $\Delta_{9,12}$. *Moreover two cubics with real flexes at* $(0, 1, -1)$; $(1, 0, -1)$; $(1, -1, 0)$ *have perspective flex-configurations with* $x_1 + x_2 + x_3 = 0$ *as the axis of perspective.*

This proposition can evidently be extended to the case where the axis of perspective is any of the 12 Maclaurin lines.

Now assume two plane cubics $C_3^{(1)}$ and $C_3^{(2)}$ in S_3 with the two real flex-triangles Δ_1 and Δ_2 respectively, situated in two planes p_1 and p_2 which intersect in a line l. Let Δ_1 and Δ_2 have l as a common side and on it three common flex-points W_1, W_2, W_3 (one real, two imaginary), so that the flex-tangents of $C_3^{(1)}$ and $C_3^{(2)}$ at these points determine the entire flex-configurations $\Delta_{9,12}$ for both curves. According to the foregoing theorem these are perspective with l as the axis of perspective. Choosing Δ_1 as $A_1A_2A_3$ and Δ_2 as $A_1A_2A_4$ of a coördinate tetrahedron, then F_3 has the form

$$x_1^3x_2^3x_3^3 - 3a_4x_1x_2x_3 - 3x_4(a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + a_{14}x_1x_4 \\ + a_3x_1x_2 + a_1x_2x_3 + a_2x_1x_3 + a_{24}x_2x_4 + a_{34}x_3x_4) = 0.$$

For $x_3 = 0$ this must reduce to $x_1^3 + x_2^3 + x_4^3 - 3a_3x_1x_2x_4 = 0$, which is only true when $a_{11} = 0$, $a_{22} = 0$, $a_{33} = 0$, $a_{44} = 1/3$, $a_{14} = 0$, $a_{24} = 0$, $a_{34} = 0$. Thus F_3 has the form

$$(19) \quad x_1^3 + x_2^3 + x_3^3 + x_4^4 - 3(a_1x_2x_3x_4 + a_2x_1x_3x_4 + a_3x_1x_2x_4 + a_4x_1x_2x_3) = 0.$$

The existence of two plane cubics $C_3^{(1)}$ and $C_3^{(2)}$, as described above, on a cubic surface thus necessarily leads to the form (19).

We next determine the manifold of cubic surfaces F_3 which may be constructed in this fashion. There are ∞^1 planes p_1 on a line l in S_3. The line l may be chosen as one side of a real flex-triangle Δ_1. The two other sides of Δ_1 and the line of the 3 real flexes may be chosen in ∞^6 ways. In another plane p_2 on l we can choose a perspective flex-configuration in ∞^3 ways. The planes p_1 and p_2 or l can be chosen in ∞^2 ways, hence the two perspective flex-configuration $(\Delta_1)(\Delta_2)$ in $\infty^{6+3+2} = \infty^{11}$ ways. On each of these there are ∞^4 Δ_{18} cubic surfaces (19), hence on l ∞^{15} such surfaces. As l may be chosen in ∞^4 ways, the manifold of Δ_{18} cubics is therefore ∞^{19}, i. e., identical with that of cubics F_3 in S_3. This may be verified as follows: There are $\infty^{3.5} = \infty^{15}$ pentahedrons (tetrahedron + unit point), hence $\infty^{15} \cdot \infty^4 = \infty^{19}$ Δ_{18} cubics F_3.

The unit-plane cuts F_3 in a cubic on which lies a hexagon formed by the vertices of a quadrilateral cut out by the coördinate planes. On each side of this quadrilateral lie three flexes such that the flex-tangents lie in the corresponding coördinate plane. The existence of this figure again insures the form (19) of the cubic surface. A generic plane p cuts the cubic surface in a plane cubic C_3 in which may be inscribed ∞^1 quadrilateral hexagons [illegible] where [illegible] is the intersection of two sides of a quadrilateral [illegible]
Consider the asymptotic tangents $t_1^{(1)}, t_1^{(2)}$; $t_2^{(1)}, t_2^{(2)}$; $t_3^{(1)}, t_3^{(2)}$ to the

surface at the points A_{i1}, A_{i2}, A_{i3} on the line l_i. Through l_i and $t_{i1}^{(1)}$ pass a plane h, then the condition that $t_{i2}^{(1)}$ shall lie on h absorbs one constant. As $t_{i1}^{(1)}$ and $t_{i2}^{(1)}$ are two flex-tangents of the plane cubic cut out by h, one of the third pair, say $t_{i3}^{(1)}$, will necessarily become a flex-tangent of this cubic. Thus the condition that $t_{i1}^{(1)}$, $t_{i2}^{(1)}$, $t_{i3}^{(1)}$ shall lie in a plane requires one constant. The same argument may be repeated for each of the three remaining lines of the quadrilateral. Thus a quadrilateral (like in the unit-plane of the Δ_{18} cubic surface) requires four conditions and as there are ∞^4 such plane inscribed quadrilaterals on a cubic surface, there can exist only a finite number of Δ_{18} configurations on a general cubic surface. The result may be stated in the

THEOREM: *The manifold of cubic surfaces on all Δ_{18} configurations of S_3 is identical with the manifold of general cubic surfaces in S_3. From this follows that every general cubic surface may be represented by an equation in which the terms $x_i^2x_k$ are missing.*

In this representation the cubic monoids are included as will be seen in the next section.

5. *The 27 lines on the cubic surface.* The tetrahedron on Δ_{18} cuts F_3 in four plane cubics C_1, C_2, C_3, C_4, so that each of the 27 lines on F_3 is a quadrisecant of this system of four cubics. In the theory of ruled surfaces the following theorem is known: Let C_1, C_2, C_3 be three curves of order m_1, m_2, m_3 respectively, and let C_1 and C_2 have s_3, C_2 and C_3 have s_1, C_3 and C_1 have s_2 points in common, then the trisecants which each cut C_1, C_2, C_3 once, generate a ruled surface R of order $n = 2m_1m_2m_3 - (s_1m_1 + s_2m_2 + s_3m_3)$ with C_1, C_2, C_3 as $(m_2m_3 - s_1)$—, $(m_3m_1 - s_2)$—, $(m_1m_2 - s_3)$—fold curves respectively. If we apply this to the three plane cubics C_1, C_2, C_3 on Δ_{18}, we obtain a ruled surface R of order $2\cdot3\cdot3\cdot3 - (3\cdot3 + 3\cdot3 + 3\cdot3) = 27$ with each C_1, C_2, C_3 as a 6-fold curve. C_1, C_2, C_3 cut C_4 in 9 points which in counting the proper intersections of R with C_4 must be subtracted as sixfold points. Hence R cuts C_4 in $3{:}27 - 6.9 = 27$ points outside of the flexes of C_4. From this follows *that the four curves $C_1C_2C_3C_4$ admit of 27 quadrisecants which therefore lie on F_3. This again proves the generic cubic surface on Δ_{18} as a general cubic.*

The foregoing procedure may be applied in precisely the same manner to the four plane cubics cut out on a general F_3 by the faces of a generic tetrahedron and constitutes thus a novel prove for the existence of 27 lines on a cubic.

6. *Cubic Monoids.* To a generic point $P(a_1, a_2, a_3, a_4)$ in S_3 corre-

sponds on S_3' by (6) a point $P'(a_2a_3a_4,\ a_1a_3a_4, \cdots)$. The tangent hyperplane at this point is easily found as

$$
\begin{aligned}
(20)\qquad & a_1(a_2^3+a_3^3+a_4^3-2a_1^3)y_1+a_2(a_1^3+a_3^3+a_4^3-2a_2^3)y_2\\
&+a_3(a_1^3+a_2^3+a_4^3-2a_3^3)y_3\\
&+a_4(a_1^3+a_2^3+a_3^3-2a_4^3)y_4-a_1a_2a_3a_4y_5=0.
\end{aligned}
$$

To this corresponds inversely in S_3 the cubic surface

$$
\begin{aligned}
(21)\quad M_3=\ & a_1(a_2^3+a_3^3+a_4^3-2a_1^3)x_2x_3x_4+a_2(a_1^3+a_3^3+a_4^3-2a_2^3)x_1x_3x_4\\
&+a_3(a_1^3+a_2^3+a_4^3-2a_3^3)x_1x_2x_4\\
&+a_4(a_1^3+a_2^3+a_3^3-2a_4^3)x_1x_2x_3-a_1a_2a_3a_4(x_1^3+x_2^3+x_3^3+x_4^3)=0,
\end{aligned}
$$

which has a double point at $P(a_1, a_2, a_3, a_4)$ and is therefore a cubic monoid. M_3 depends upon three effective constants. Every point (a) in S_3 determines such a monoid uniquely and is the double point of M_3. We have thus the

THEOREM: *The generic points of S_3 as double points determine the ∞^3 cubic monoids on the configuration* Δ_{18} uniquely.

If in M_3 we leave (x) fixed and let (a) vary we have obviously the

THEOREM: *The locus of the double points of all cubic monoids on Δ_{18} through a fixed generic point is a quartic surface.*

To the intersection of a hyperplane $\Sigma c_i y_i = 0$ with S_3' corresponds in S_3 the cubic surface F_3 in a normal form on Δ_{18}. Among these are evidently the monoids M_3. The discriminant D of F_3 is a homogeneous polynomial of degree 32 in the c's, which puts a condition on the c's. From (20) it is evident that this discriminant may be expressed parametrically by

$$
(22)\qquad
\begin{aligned}
\rho c_1 &= a_1(a_2^3+a_3^3+a_4^3-2a_1^3)\\
\rho c_2 &= a_2(a_1^3+a_3^3+a_4^3-2a_2^3)\\
\rho c_3 &= a_3(a_1^3+a_2^3+a_4^3-2a_3^3)\\
\rho c_4 &= a_4(a_1^3+a_2^3+a_3^3-2a_4^3)\\
\rho c_5 &= -a_1a_2a_3a_4
\end{aligned}
$$

as a rational hypersurface of order 32 in a projective hyperspace $S_4(c_1, c_2, c_3, c_4, c_5)$.

This may be verified by making use of the pentahedral normal form

$$
(23)\qquad \Sigma c_i x_i^3 = 0,\ \Sigma x_i = 0,\ i = 1, 2, 3, 4, 5.
$$

with the discriminant

* Salmon-Fiedler, *Analytische Geometrie des Raumes*, Vol. 2 (1874), pp. 367-368.

$$(24) \qquad \pm\sqrt{c_2c_3c_4c_5} \pm \sqrt{c_1c_3c_4c_5} \pm \cdot\cdot \pm \sqrt{c_1c_2c_3c_4} = 0,$$

which may be expressed parametrically by

$$\begin{aligned}
\rho c_1 &= \lambda_2{}^2\lambda_3{}^2(\lambda_1 + \lambda_2 + \lambda_3 + 1)^2,\\
\rho c_2 &= \lambda_3{}^2\lambda_1{}^2(\lambda_1 + \lambda_2 + \lambda_3 + 1)^2,\\
\rho c_3 &= \lambda_1{}^2\lambda_2{}^2(\lambda_1 + \lambda_2 + \lambda_3 + 1)^2,\\
\rho c_4 &= \lambda_1{}^2\lambda_2{}^2\lambda_3{}^2(\lambda_1 + \lambda_2 + \lambda_3 + 1)^2,\\
\rho c_5 &= \lambda_1{}^2\lambda_2{}^2\lambda_3{}^2,
\end{aligned}$$

These results may be stated in the

THEOREM: *The discriminant of the cubic surface in the* Δ_{18} *—, or also the pentahedral normal form, may be represented by a rational hypersurface of order* 32 *in a projective* S_4.

UNIVERSITY OF ILLINOIS.

ON PLANE CREMONA TRIADIC CHARACTERISTICS.

By Charles C. Torrance.

The problem of finding plane Cremona transformations having only three groups of basis points, that is, transformations with triadic characteristics, has been attacked by Ruffini,[1] Montesano,[2] and Farnum.[3] In this paper twelve infinite sequences of two-parameter triadics are obtained, all of which have for one of their groups a single point of highest multiplicity. These triadics result directly from the development of a new procedure[4,5,6] whereby all geometric characteristics may be found. The principal feature of this procedure is its perfect regularity and simple explicit formulation.

1. It is a well-known theorem that if P, Q, and R are basis points of multiplicities $p \geqq 0$, $q \geqq 0$, and $r \geqq 0$ of a plane Cremona transformation T_n of order n and also are the basis points of a quadratic inversion T_2 with which T_n is compounded, the resultant transformation $T_{n'}$ of order n' has the same number of basis points at T_n of the same multiplicities except that it loses the points P, Q, and R, and gains three basis points P', Q', and R' of multiplicities $p' = p + e$, $q' = q + e$, and $r' = r + e$, while $n' = n + e$, where $e = n - (p + q + r)$. It will be convenient to regard the three points gained by $T_{n'}$ as the transforms of the three points lost by T_n.

2. We give the symbol P a particular connotation. If a transformation is to be successively compounded with T_2's and one of its basis points is called P, it is intended that P and each of its transforms are to be successively used as common basis points. If T_n is compounded with a T_2 having P, Q_1, and

1 "Sulla risoluzione delle trasformazioni Cremoniane," *Memorie Istituto Bologna*, Series 3, Vol. 8 (1877), pp. 457-525.

2 "I gruppi Cremoniani di numeri," *Atti Accademia Napoli*, Series 2, Vol. 15 (1914), No. 7.

3 "On triadic Cremona nets of plane curves," *American Journal of Mathematics*, Vol. 50 (1928), p. 357.

4 Montesano, "Su le rete omaloidiche di curve," *Rendiconti Accademia Napoli*, Series 3, Vol. 11 (1905), pp. 259-303.

5 Montesano, "Su i quadri caratteristici delle corrispondenze birazionali piane." *Rendiconti Accademia Napoli*, Series 3, Vol. 21 (1915), pp. 30-38, 69-79, 113-119.

6 Montesano, "Su alcuni problemi fondamentali nella teoria delle [illegible] Cremoniane," *Rendiconti Accademia Napoli*, Series 3, Vol. 34 (1928), pp. 42-50, 89-97, 123-136.

R_1 as common basis points, $e = (n - p) - (q_1 + r_1)$, $n' = n + e$, $p' = p + e$, and $n' - p' = n - p$. If the resultant transformation $T_{n'}$ is then compounded with a T_2 having P', Q_2, and R_2 as common basis points, $e' = (n' - p') - (q_2 + r_2) = (n - p) - (q_2 + r_2)$. A sufficient condition that $e' = e$ is that $q_2 = q_1$ and $r_2 = r_1$. Hence, in a succession of compoundings involving a point P and its transforms, $(n - p)$ is invariant, and e is invariant if Q and R are each repeatedly taken with invariant multiplicity.

3. The characteristic (1) is known to be geometric. Let P be the single point of its first group. Compound it with a T_2 having its other two basis points general [i. e., of multiplicity zero in (1)]. The resultant is (2) since $e = x_{01}$. If this compounding is repeated, (2) is altered in the same way as was (1). The resultant of x_{11} of these compoundings is (3). In general, compounding with T_2's each having one basis point on P and two general will be referred to as compounding in the first way.

$$
\begin{array}{llllll}
(1) & n=[x_{01}+1]; & 1^{1}, & 1^{x_{01}}, & & (2x_{01}-1)^{1}. \\
(2) & n=[x_{01}+1+x_{01}]; & 1^{1+x_{01}}, & (1+2)^{x_{01}}, & & (2x_{01}-1)^{1}. \\
(3) & n=[x_{01}+1+x_{01}x_{11}]; & 1^{1+x_{01}x_{11}}, & (1+2x_{11})^{x_{01}}, & & (2x_{01}-1)^{1}. \\
(4) & n=[x_{01}+1+x_{01}x_{11}-1]; & 1^{1+x_{01}x_{11}-1}, & (1+2x_{11}-1)^{x_{01}}, & 1^{x_{01}-1}, & (2x_{01}-1-1)^{1}. \\
(5) & n=[x_{01}+1+x_{01}x_{11}-x_{12}]; & 1^{1+x_{01}x_{11}-x_{12}}, & (1+2x_{11}-x_{12})^{x_{01}}, & x_{12}{}^{x_{01}-1}, & (2x_{01}-1-x_{12})
\end{array}
$$

4. Compound (3) with a T_2 having one basis point on the last transform of P, one on a point of the second group, and one on a point of the third. The resultant is (4) since $e = -1$. If this compounding is repeated, (4) is altered in the same way as was (3). The resultant of x_{12} of these compoundings is (5). In general, compounding with T_2's each having one basis point on P, a second on a point of the second group, and the third on a point of any other group will be referred to as compounding in the second way.

5. The procedure for obtaining all geometric characteristics may be described in general terms as follows: suppose the characteristic

$$
(6) \qquad n;\ \ 1^{z_{k1}},\ \ (1 + y_{k2})^{z_{k2}},\ \ y_{k3}^{z_{k3}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}
$$

to be geometric, with $n = z_{k1} + z_{k2}$, where the y's and z's are polynomials in x_{rs}, $r = 0, 1, \cdots, k$, and $s = 1, 2, \cdots, 2^r$. An illustration of such a characteristic is (5). Take for P a point of multiplicity z_{k2} (involving a change of P) and write (6) in the form (7) so that P constitutes the first group.

$$
(7) \qquad n;\ \ 1^{z_{k2}},\ \ 1^{z_{k1}},\ y_{k2}^{z_{k2}},\ y_{k3}^{z_{k3}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}.
$$

$$(8) \qquad n' = [n + z_{k1}x_{k+1,1}]; \quad 1^{z_{k2}+z_{k1}x_{k+1,1}}, \quad (1 + 2x_{k+1,1})^{z_{k1}}, \quad y_{k2}^{z_{k2}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}$$

$$(9) \qquad n' = [n + z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}]; \quad 1^{z_{k2}+z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}},$$

$$(1 + 2x_{k+1,1} - \sum_{i=2}^{2^{k+1}} x_{k+1,i})^{z_{k1}}, \; x_{k+1,2^{k+1}}^{z_{k1}-z_{k2^{k+1}}}, \cdots, x_{k+1,i}^{z_{k1}-z_{ki}}, \cdots,$$

$$x_{k+1,2}^{z_{k1}-z_{k2}}, \; (y_{k2} - x_{k+1,2})^{z_{k2}}, \cdots, (y_{ki} - x_{k+1,i})^{z_{ki}}, \cdots,$$

$$(y_{k2^{k+1}} - x_{k+1,2^{k+1}})^{z_{k2^{k+1}}}.$$

Compound (7) in the first way $x_{k+1,1}$ times, where $e = z_{k1}$, getting (8). Compound (8) in the second way $x_{k+1,i}$ times with the third basis point of the compounding T_2 on a point of multiplicity z_{ki}, i having successively all integer values from 2 to 2^{k+1}, getting (9), since $e = -z_{ki}$. As (9) is of the same form as (6) the procedure is inductive. The set of compoundings used to obtain (9) from (6) will be referred to as one step of the inductive procedure, and (6) itself will be referred to as the general characteristic resulting from k steps of this procedure. It should be noted that each general characteristic includes all preceding ones.

The only condition on the x's is that all the base numbers (i. e., the expressions giving the number of points in each group) must be non-negative, and if this condition is satisfied all characteristics obtained by this procedure are geometric, since (1) is geometric.

6. Theorem. *Every plane Cremona geometric characteristic can be obtained as the result of sufficiently many steps of the above procedure* [*starting from* (1)] *and proper evaluation of the x's.*

This theorem will be proved by showing that the inverse of the procedure will reduce a given characteristic to a particular case of (1). Let the characteristic of a given transformation T be n; $y_h^{\,h}$, $y_{h-1}^{\,h-1}$, $\cdots$, $y_1^{\,1}$, where h denotes the highest multiplicity in T. (In the course of this proof h will also denote the highest multiplicity in any particular transform of T.) Take one of the points of multiplicity h as P, and let the multiplicity of this particular point be denoted by p. The reason for the notation is that $(n - p)$ is always invariant, but $(n - h)$ is not.

7. If there exists in T a point Q (other than P) of multiplicity q, [illegible] one on Q, and one general. In the notation of paragraph 1, $e > 0$, [illegible] p, and $e' < (n - p)/2$, so that this compounding is in the

inverse of the second way. For simplicity we drop the primes in considering this and succeeding transforms of T. In this way all points Q may be removed, so that there remain only P of multiplicity $p = h$, points of multiplicity $(n - p)$, and points of multiplicity not greater than $(n - p)/2$. Before making further compoundings we use

8. Noether's Extended Theorem. In any geometric characteristic the sum of the first, third, and fifth highest multiplicities is greater than the order. It follows that there are at least three points of multiplicity greater than $(n - h)/2$. Hence, if $p = h$ and there is no point Q, there are at least two points (other than P in the event that $p = n - p$) of multiplicity $(n - p)$, and $p \geqq n - p$.

9. Compound T in its present form with a T_2 having one basis point on P and two on points A and B of multiplicity $(n - p)$. In obvious notation $e = -(n - p)$ and $a' = b' = 0$, so that this compounding is in the inverse of the first way. Let this compounding be repeated as many times as possible, that is, until $p < n - p$.

If now y_{n-p} is positive, $n - p = h > p = n - h$, and the group consists of but a single point. Since $h + p = n$, p is the second highest multiplicity, and T has been reduced to a transformation T' of the form (6) with P of the proper multiplicity, so that, if the procedure produces T' it also produces T.

10. If, however, y_{n-p} is zero, then $p < h \leqq (n - p)/2$, since T now contains no point Q. Compound T in its present form with a T_2 having its basis points on P, a point H of multiplicity h, and a general point K. Since $h' = n - p > p' \geqq k' = e \geqq (n - p)/2$ this compounding is in the inverse of the second way. But P is now a point of second highest multiplicity and $y_{n-p} = 1$. Hence in this case also T has been reduced to a transformation T' of the form (6).

11. As this last compounding is the inverse of the second way its use here is unwarranted. But the difficulty may be overcome without affecting T'. Because of the invariance of $(n - p)$ the effect on T of each compounding is independent of the others, so that the order of the compoundings may be changed without altering T' and a compounding may be omitted without impairing the effect on T of any of the others. Let K' be the point of multiplicity k' involved in the last compounding. If K' was present in the original T as a point Q, the continued product of the compounding which removed it, the last compounding of paragraph 9, and the compounding of

paragraph 10, is the identity. Hence they may all be omitted without affecting T'. If K' was not present as a point Q, the compounding introducing it is necessary, but may be performed in paragraph 7.

12. Now take the single point of multiplicity $(n-p)$ as a new P and repeat the above process, noting that the new value of $(n-p)$ is less than the old one. By sufficiently many repetitions of this process it is possible to reduce the value of $(n-p)$ to one, and when this has been done the result must necessarily be a particular case of (1).[7]

13. This proof was conducted in such a way as to show that any characteristic may be obtained with P a point of highest multiplicity, not only in it, but throughout the process of getting it. Hence, in constructing T from (1) by the inverse of the above reduction, all x_{r1}, $r=0, 1, 2, \cdots, k$ (where k is the required number of steps in the inductive procedure) are positive. The vanishing of an x_{r1} by no means invalidates the procedure; it leads only to duplicates. We shall assume in the sequel that all such x's are positive.

14. If we now impose the condition that all but three of the base numbers vanish in the general characteristic resulting from k steps of the inductive procedure, $k>1$, it results that all the x's can be evaluated by six different sets of formulas in terms of x_{01} and x_{11} taken as independent parameters and in no other way. If, for successive values of k, these values of the x's are substituted in the general characteristics, triadics 1, 2, 3, 4, 5, and 6 result. (In these triadics we have written merely x_0 for x_{01} and x_1 for x_{11}.) We omit the proofs of these statements because of typographical difficulties.

Other formulas for evaluating the x's so that triadics result may be obtained by making two or more of the multiplicities equal in the general characteristics. For example, the condition $x_{01}=2$ makes every other pair of multiplicities equal, so that four, or even five base numbers may be positive without involving more than three different multiplicities. However, this particular condition leads only to special cases of the triadics given below, together with some half-dozen isolated triadics.

15. Let $\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ be the two-way sequence of polynomials satisfying the relations

$$(A_e) \qquad u_{i+1}+u_{i-1}=x_i u_i, \quad i \text{ even},$$

$$(A_o) \qquad u_{i+1}+u_{i-1}=u_i, \quad i \text{ odd},$$

to the condition that

$$u_0=0, \quad u_1=1.$$

[7] Hudson, *Cremona Transformations*, p. 98.

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It follows immediately that all the u's are polynomials in the one variable x_1 and that $u_i = -u_{-i}$, since $u_{-1} = -1$. These polynomials are connected by many relations, among which the following are used in the sequel:

$$(B_e) \qquad u_{i\pm 2} + x_1 u_i - 2u_{i\pm 1} = u_{i\mp 2}, \quad i \text{ even, signs dependent.}$$

$$(B_o) \qquad u_{i\pm 2} + x_1 u_i - 2x_1 u_{i\pm 1} = u_{i\mp 2}, \quad i \text{ odd, signs dependent.}$$

$$(C1) \qquad u_{i+(j+1)} u_{i-(j+1)} = x_1 u_{i+j} u_{i-j} - u_{2j+1}, \quad (i \pm j) \text{ even.}$$

$$(C2) \qquad x_1 u_{i+(j+1)} u_{i-(j+1)} = u_{i+j} u_{i-j} - u_{2j+1}, \quad (i \pm j) \text{ odd.}$$

$$(C3) \qquad u_{i+(j+1)} u_{i-(j+1)-1} = u_{i+j} u_{i-j-1} - u_{2j+2}, \quad \text{all } (i \pm j).$$

$$(C4) \qquad u_{i+(j+2)} u_{i-(j+2)} = u_{i+j} u_{i-j} - u_{2j+2}, \quad (i \pm j) \text{ even.}$$

$$(C5) \qquad u_{i+(j+2)} u_{i-(j+2)} = u_{i+j} u_{i-j} - x_1 u_{2j+2}, \quad (i \pm j) \text{ odd.}$$

$$(C6) \qquad u_{i+(j+2)} u_{i-(j+2)-1} = u_{i+j} u_{i-j-1} - u_{2j+3}, \quad \text{all } (i \pm j).$$

Relations (B) are proven directly by substitution from relations (A). Relation $(C1)$ is proven by induction first for the case $j = 0$. Then relations $(C2)$ and $(C3)$ are proven directly for the case $j = 0$ by substitution from relations (A) and $(C1)$. Finally relations $(C1)$, $(C2)$, and $(C3)$ are proven for general j by showing that if they all hold for all proper i when $j = s$, then they all hold for all proper i when $j = s + 1$. Relations $(C4)$, $(C5)$, and $(C6)$ are proven directly by substitution from relations (A) and the other relations (C). Details of the proofs are omitted because they offer no difficulty.

As a digression we may point out an interesting property of these u-polynomials. Transform u_i into u'_i by the substitution $x_1 = x'_1 + 2$. The roots of the equation $u'_i = 0$, where i is positive and even, are equal to $2 \cos (2n\pi)/i$, $n = 1, 2, \cdots, (i-2)/2$.

16. The following triadics can be shown to be geometric and their conjugates can be determined by Montesano's method [6]:

( )	[illegible] $= [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-x_1u_ka_k}$,	$(2x_0-1)^{x_1u_ka_k}$,	$(2x_1)^{u_{k+1}a_k+1}$,	k even.
	[illegible] $= [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-u_ka_k}$	$(2x_0-1)^{u_ka_k}$,	$(2x_1)^{u_{k+1}a_k+1}$,	k odd.
( )	[illegible] $= [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-x_1u_kb_k}$,	$(2x_0-1)^{x_1u_kb_k}$,	$(2x_1)^{u_{k-1}b_k+1}$,	k even.
	[illegible] $= [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-u_kb_k}$,	$(2x_0-1)^{u_kb_k}$,	$(2x_1)^{u_{k-1}b_k+1}$,	k odd.
( )	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-x_1u_ka_k}$,	$(2x_0-1)^{x_1u_ka_k}$,	$(2x_1)^{u_{k-1}a_k}$,	k even.
	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-u_ka_k}$,	$(2x_0-1)^{u_ka_k}$,	$(2x_1)^{u_{k-1}a_k}$,	k odd.
( )	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-x_1u_kb_k}$,	$(2x_0-1)^{x_1u_kb_k}$,	$(2x_1)^{u_{k+1}b_k}$,	k even.
	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-u_kb_k}$,	$(2x_0-1)^{u_kb_k}$,	$(2x_1)^{u_{k+1}b_k}$,	k odd.
( )	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-x_1u_kc_k}$,	$(2x_0-1-2x_1)^{x_1u_kc_k}$,	$(2x_1)^{u_{k+1}c_k}$,	k even.
	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-u_kc_k}$,	$(2x_0-1-2x_1)^{u_kc_k}$,	$(2x_1)^{u_{k+1}c_k}$,	k odd.
( )	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-x_1u_kd_k}$,	$(2x_0-1-2x_1)^{x_1u_kd_k}$,	$(2x_1)^{u_{k-1}d_k}$,	k even
	[illegible] $= [\,$[illegible]$\, + 1]$;	$1^{(n-1)-u_kd_k}$,	$(2x_0-1-2x_1)^{u_kd_k}$,	$(2x_1)^{u_{k-1}d_k}$,	k odd.
(1)	$n = [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-x_1u_kc_k}$,	$(2x_0-1-2x_1)^{x_1u_kc_k}$,	$(2x_1)^{u_{k-1}c_k+1}$,	k even.
	$n = [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-u_kc_k}$,	$(2x_0-1-2x_1)^{u_kc_k}$,	$(2x_1)^{u_{k-1}c_k+1}$,	k odd.
[illegible]	$n = [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-x_1u_kd_k}$,	$(2x_0-1-2x_1)^{x_1u_kd_k}$,	$(2x_1)^{u_{k+1}d_k+1}$,	k even.
	$n = [\,$[illegible]$\, + x_1 + 1]$;	$1^{(n-1)-u_kd_k}$,	$(2x_0-1-2x_1)^{u_kd_k}$,	$(2x_1)^{u_{k+1}d_k+1}$,	k odd.
(5)	$n = [\,$[illegible]$\, + x_0(x_1-x_0+1)+1]$;	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0}$,	$(2x_0-1)^{u_{k+2}e_{k-1}+x_1-x_0+1}$,	$(2x_1-2x_0+1)^{u_ke_{k-1}+1}$,	k even.
	$n = [e_k^2 + 1]$;	$1^{(n-1)-u_ke_k}$,	$(2x_0-1)^{u_{k+1}e_k}$,	$(2x_1-2x_0+1)^{u_{k-1}e_k}$,	k odd.
5′	$n = [\,$[illegible]$\, + x_0(x_1-x_0+1)+1]$;	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0}$,	$(2x_0-1)^{u_{k-2}f_{k+1}+x_1-x_0+1}$,	$(2x_1-2x_0+1)^{u_kf_{k+1}+1}$,	k even.
	$n = [f_k^2 + 1]$;	$1^{(n-1)-u_kf_k}$,	$(2x_0-1)^{u_{k-1}f_k}$,	$(2x_1-2x_0+1)^{u_{k+1}f_k}$,	k odd.
6)	$n = [\,$[illegible]$\, + (x_1-x_0)^2+1]$;	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0}$,	$(2x_0-1)^{u_ke_{k-1}}$,	$(2x_1-2x_0+1)^{u_{k+2}e_{k-1}+x_1-x_0}$,	k even.
	$n = [\,$[illegible]$\, - x_1 + 1]$;	$1^{(n-1)-u_ke_k}$,	$(2x_0-1)^{u_{k-1}e_k+1}$,	$(2x_1-2x_0+1)^{u_{k-1}e_k-1}$,	k odd.
6′	$n = [\,$[illegible]$\, + (x_1-x_0)^2+1]$;	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0}$,	$(2x_0-1)^{u_kf_{k+1}}$,	$(2x_1-2x_0+1)^{u_{k-2}f_{k+1}+x_1-x_0}$,	k even.
	$n = [\,$[illegible]$\, - x_1 + 1]$;	$1^{(n-1)-u_kf_k}$,	$(2x_0-1)^{u_{k+1}f_k+1}$,	$(2x_1-2x_0+1)^{u_{k-1}f_k+1}$,	k odd.

$a_i = x_0$[illegible]$\, u_{i-2}$, all i. $\quad c_i = (x_0-2)u_i - u_{i-2}$, all i. $\quad e_i = x_0(u_{i+1}-u_{i-1}) + (u_{i-1}+u_{i-2})$, i odd.

$b_i = x_0$[illegible]$\, u_{i+2}$, all i. $\quad d_i = (x_0-2)u_i - u_{i+2}$, all i. $\quad f_i = x_0(u_{i-1}-u_{i+1}) + (u_{i+1}+u_{i+2})$, i odd.

17. A necessary and sufficient condition that a characteristic be geometric is that there exist a set of T_2's with which it may be successively compounded so that the resultant is the linear characteristic. If the characteristic is geometric its conjugate may be found by compounding the linear characteristic (written so as to contain a basis point of multiplicity zero to correspond to each basis point of the given characteristic) with the same set of T_2's taken in the same order. Triadics $1c$, $4c$, and $6c$ may be derived in this way as the conjugates of triadics 1, 4, and 6, while all the others in the preceding table are self-conjugate.

Unfortunately the algebra involved in the investigations of these triadics is long and complicated, so that we shall merely sketch the outline of the procedure, using triadic 1 to illustrate it. Let the linear characteristic be written under, and in the same form as, triadic 1. Let P be the single point of the first group (see paragraph 2). Compound with T_2's having two basis points in the second group so that this group is reduced to a group of simple points. (One T_2 will have only one basis point in the second group and one general.) Reduce the multiplicity of the third group by compounding with T_2's having two basis points in it. If the multiplicity of the new third group is repeatedly reduced in the same way it is found that the formulas for all of the multiplicities change according to a certain simple law, so that the formulas for the multiplicities resulting from an arbitrary number of such reductions can be calculated directly. The number of such reductions needed is approximately k. When the triadic has been reduced in this way to the simplest possible form it is finally compounded with a set of T_2's which obliterates the group of simple points that has been carried along, and the result is the linear characteristic. In the meantime the same sets of T_2's have operated on the original linear characteristic to produce triadic $1c$. Algebraic simplifications in this procedure are made with the aid of relations (B) and (C).

18. The conjugates of triadics 1, 4, and 6 may be formally obtained by substituting $-k$ for k in these triadics when they are written explicitly in terms of the u's. Triadics $2'$, $3'$, and $5'$ were obtained by this same formal operation when applied to triadics 2, 3, and 5 respectively.

19. In paragraph 14 the case $k=1$ was not considered. Triadics may be obtained from (5), paragraph 3, either by making one of the base numbers vanish or by making two of the multiplicities equal. Triadics 7, 8, 9, 10, and 11 result respectively from the conditions $2x_{01}-1-x_{12}=0$, $x_{12}=0$, $1+2x_{11}-x_{12}=0$, $x_{01}-1=1$, and $1+x_{01}x_{11}-x_{12}=x_{01}$. Their conjugates are easily derived by Montesano's method. We write merely x_0 for x_{01}, x_1 for x_{11}, and x_2 for x_{12}.

[illegible] $[\,\cdot\, x_0 - \; x_0 + 2]$; $1^{x_1x_0-2x_0+2}$, $(2x_1 - \,\cdot\, 2x_0 + 2)^{\cdot\cdot}$, $(2x_0 \quad 1)^{\cdot\,1}$.

[illegible] $[\,\cdot\, x_0 - \; x_0 + 2]$; $1^{x_1x_0-x_1-x_0+1}$, $(2x_1 - \; 2x_0 + 2)^1$. $(2x_0 \quad 1)^1$.

[illegible] $[\,\cdot\, x_0 + x_0 + 1]$; $1^{x_1x_0+1}$, $(2x_1 + 1)^{x_0}$, $(2x_0 \quad 1)^1$.

[illegible] $[\,\cdot\, x_0 + x_0 + 1]$; $1^{x_1x_0-x_1+x_0}$, $(2x_1 + 1)^1$, $(2x_0 \quad 1)^{\cdot\,1}$.

[illegible] $[\,\cdot\, x_0 - 2x_1 + x_0]$; $1^{x_1x_0-2x_1}$, $(2x_1 + 1)^{x_0\,1}$, $(2x_0 \quad 2x_1 \quad 2\,)^1$.

[illegible] $[\,\cdot\, x_1 - \; 2x_1 + x_0]$; $1^{x_1x_0-3x_1+x_0-1}$, $(2x_1 + 1)^{x_1}$, $(2x_0 \quad 2x_1 \quad 2)$.

[illegible] $[\,\cdot\, x_1 \; - x_2 + 3]$; $1^{2x_1-x_2+1}$, $(1 + 2x_1 - \; x_2)^2$, $x_2^{\,1}$, $(3 \quad x_2)^1$.

[illegible] $[\,\cdot\, x_1 - x_2 + 3]$; $1^{x_1-x_2+2}$, $(1 + 2x_1 - \; x_2)^1$, $x_2^{\,\cdot}$, $(3 \quad x_2)^{\cdot\cdot\,1}$.

[illegible] $[\,\cdot\, x_0]$; 1^{x_0}, $(2x_1 + x_0 - - x_1x_0)^{x_0}$, $(1 + x_1x_0 - \; x_0)^{\cdot\,1}$, $(3x_0 \quad x_1x \quad 2\,)$.

[illegible] $[\,\cdot\, x_0]$; $1^{2x_0-x_1-1}$, $(2x_1 + x_0 - - x_1x_0)^1$, $(1 + x_1x_0 - \; x_0)^{\cdot}$, $(3x_0 \quad x_1x_0 \quad 2$

A DETERMINATION OF THE DOMAINS OF INTEGRITY OF THE COMPLETE RATIONAL MATRIC ALGEBRA OF ORDER 4.*

By E. J. Finan.

1. *Introduction.* The set of all two-rowed square matrices with rational elements constitutes the complete matric algebra of order 4. We shall call a *domain of integrity* (or merely a *domain*) of this algebra any subset which (1) is of order 4, (2) contains the identity matrix, and (3) is closed under multiplication, the constants of multiplication being rational integers. Every such set has a basis † having the above three properties.

If E_1, E_2, E_3, and E_4 is a basis for such a domain, it is well known that E'_1, E'_2, E'_3, and E'_4 also constitute a basis where

$$E'_i = \sum_{j=1}^{4} t_{ij} E_j \qquad (i = 1, 2, 3, 4),$$

the t_{ij} being rational integers such that $| t_{ij} | = \pm 1$, and that every basis of the domain is so obtainable. We shall call this change of basis a transformation (1).

It is evident that if M is any non-singular two-rowed square matrix with rational elements the domain with basis E'_1, E'_2, E'_3, and E'_4 is isomorphic with the domain whose basis is

$$E'_i = ME_iM^{-1} \qquad (i = 1, 2, 3, 4).$$

If two domains are so related we shall call them *equivalent*. We shall call the process of obtaining an equivalent domain a transformation (2).

Since transformations (1) and (2) are both associative and commutative, we may consider any sequence of transformations as a transformation (1) followed by a transformation (2). We shall use this result later to show that two domains with the same discriminant may be non-equivalent.

Since the constants of multiplication remain unchanged, the discriminant is invariant under transformation (2). Also it is invariant under a transformation (1).‡

* Presented to the Society, April 19, 1930.

† Dickson, *Algebras and Their Arithmetics*, Chicago (1923). Page 161.

‡ MacDuffee, *Annals of Mathematics*, Vol. 2, No. 2: 2nd series.

There is an infinite number of non-equivalent domains, but it is shown in this paper that for a given value of the discriminant there is only a finite number. In § 2 we obtain two canonical forms for the basis of a domain. By applying to these canonical forms arbitrary transformations (1) and (2) all sets of matrices are obtained which constitute bases for domains of integrity. Finally the number of non-equivalent domains is calculated for small values of the discriminant and a canonical form is obtained for each.

This problem was first attacked by Du Pasquier,* who obtained after a more laborious reduction a reduced form which included all domains, unfortunately with much repetition. His attempt to classify domains by means of the greatest common denominator in his canonical form was not successful, since this function is not an invariant.

2. Theorem 1. *Every domain of integrity is equivalent under transformations* (1) *and* (2) *to one of the domains.*

Case I

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \quad \begin{Vmatrix} 0 & 1 \\ m & 0 \end{Vmatrix} \quad \begin{Vmatrix} 0 & 0 \\ la & a \end{Vmatrix} \quad \begin{Vmatrix} 0 & 0 \\ ka & 0 \end{Vmatrix}$$

Case II

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \quad \begin{Vmatrix} 0 & 1 \\ m & 1 \end{Vmatrix} \quad \begin{Vmatrix} 0 & 0 \\ la & a \end{Vmatrix} \quad \begin{Vmatrix} 0 & 0 \\ ka & 0 \end{Vmatrix}$$

in which a, k, l, m *are rational integers,* $0 \leq l < k$, $0 \leq m < ka$ *and in Case I* $l^2 - m \equiv 0$, *mod.* k *while in Case II* $l^2 + l - m \equiv 0$, *mod.* k. *Conversely, every set of matrices of this form, satisfying these conditions, constitutes a basis of some domain.*

By a theorem due to Du Pasquier in the article mentioned above, we may assume that the basal matrices have integral elements.

Let this basis be E_1, E_2, E_3, and E_4 in which E_1 is the second order identity matrix. E_1 does not change throughout the reduction.

Evidently the operation of replacing a single E_j $(j = 2, 3, 4)$ by $E_j - xE_k$ $(k = 1, 2, 3, 4;\ j \neq k)$ is a transformation (1). We shall call this kind of transformation (1) a transformation (1-A).

By a transformation (1-A) with $j = 2$ and $k = 1$ we can put E_2 into the form

$$\begin{Vmatrix} 0 & b \\ c & d \end{Vmatrix}.$$

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* *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich* (1909), page 137.

There is an infinite number of non-equivalent domains, but it is shown in this paper that for a given value of the discriminant there is only a finite number. In § 2 we obtain two canonical forms for the basis of a domain. By applying to these canonical forms arbitrary transformations (1) and (2) all sets of matrices are obtained which constitute bases for domains of integrity. Finally the number of non-equivalent domains is calculated for small values of the discriminant and a canonical form is obtained for each.

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in which a, k, l, m are rational integers, $0 \leq l < k$, $0 \leq m < ka$ and in Case I $l^2 - m \equiv 0$, mod. k while in Case II $l^2 + l - m \equiv 0$, mod. k. Conversely, every set of matrices of this form, satisfying these conditions, constitutes a basis of some domain.

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* *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich* (1909), page 137.

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A transformation (2) with

$$M = \begin{Vmatrix} 1 & 0 \\ 0 & b \end{Vmatrix}$$

then replaces E_2 by a matrix of the same form with $b = 1$. Another transformation (2) with

$$M = \begin{Vmatrix} 1 & 0 \\ -n & 1 \end{Vmatrix}$$

followed by a transformation (1-A) with $j = 2$, $k = 1$ and $x = n$ replaces E_2 by

$$\begin{Vmatrix} 0 & 1 \\ e & d - 2n \end{Vmatrix}.$$

By suitable choice of n we can make the lower right element in E_2 0 or 1 according as d is even or odd. We shall now divide the discussion into two cases.

Case I. d is even. E_2 is now in the form

$$\begin{Vmatrix} 0 & 1 \\ e & 0 \end{Vmatrix}$$

By a transformation (1-A) we can subtract multiples of E_1 and E_2 from E_3 and E_4 and give them the form

$$E_3 = \begin{Vmatrix} 0 & 0 \\ f & g \end{Vmatrix} \qquad E_4 = \begin{Vmatrix} 0 & 0 \\ p & q \end{Vmatrix}.$$

There exist relatively prime integers, λ and μ, such that $\lambda g + \mu q = 0$. Also there exist integers, ξ and η, such that $\mu\xi - \lambda\eta = 1$. The transformation (1) with matrix

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \xi & \eta \\ 0 & 0 & \lambda & \mu \end{Vmatrix}$$

leaves E_1 and E_2 unchanged, replaces E_3 by a matrix of the same form and replaces E_4 by

$$\begin{Vmatrix} 0 & 0 \\ r & 0 \end{Vmatrix}.$$

If we now express the products E_4E_3 and E_4E_2 as linear combinations of E_1, E_2, E_3 and E_4 with integral coefficients, we get certain relations between

r, f, g and e. After some simplification these amount to $r = kg$, $f = lg$ and $l^2 \equiv e$, mod. k. Hence the basal matrices may be written

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 \\ m & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ la & a \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ ka & 0 \end{Vmatrix},$$

in which $l^2 \equiv m$, mod. k and $ka \neq 0$. The latter condition follows from the linear independence of the basal matrices.

We shall now show that it is possible by transformation (1) to make all of the elements positive or zero. If ka is negative, apply transformation (1) with T equal to the identity matrix with the exception that $t_{44} = -1$. If a is negative, use a similar transformation with $t_{33} = -1$. Also, with ka positive, we can make $0 \leqq m < ka$ and $0 \leqq l < k$ by repeatedly adding E_4 to E_2 and E_3 respectively. None of these transformations changes the conditions on the elements of the E's which are given above, for the 1 in E_2 is not changed, and none of the zeros is lost in any of the E's. Hence we have arrived at the canonical form given in the theorem.

Case II. d is odd. The procedure is similar to Case I. We shall not include it here. The conditions on the elements of the E's are given in the statement of the theorem.

To show that any four matrices satisfying the conditions given in the theorem constitute a basis for some domain, it is sufficient to construct the multiplication table of the matrices. This shows that the constants of multiplication are integers. The other conditions for a domain are evidently satisfied.

3. *Conditions for equivalence of two domains.* In our investigation we shall need the discriminant matrix of the domain. This is given by the formula [*]

$$D_{rs} = \sum_{ji} c_{rij} c_{sji}$$

in which the c's are the 64 constants of multiplication. Using the basis given in Theorem 1, the discriminant matrix for Case (1) is

$$\begin{Vmatrix} 4 & 0 & 2a & 0 \\ 0 & 4m & 2la & 2ka \\ 2a & 2la & 2a^2 & 0 \\ 0 & 2ka & 0 & 0 \end{Vmatrix}.$$

As mentioned in § 1 $|D| = -16k^2a^4$ is an invariant under transfor-

[*] MacDuffee, *loc. cit.*

mations (1) and (2). From Theorem 1 it is evident that there is only a finite number of non-equivalent domains for a given value of $|D|$. Hence using a constant ka^2, which is evidently the simplest function of the invariant, we can make a determination of the non-equivalent domains with a given discriminant. ($|D|$ is called the discriminant of the domain.) Although we have no formula which gives the number of non-equivalent domains for a given value of ka^2, we shall give 6 auxiliary theorems which are useful in determining this number. The rest of this paper will be devoted to the determination of the number of non-equivalent domains for values of ka^2 from 1 to 31 inclusive. In many cases the calculations are long; hence we are including only one typical calculation. The results are given in tabulated form at the end of the paper.

If two domains are equivalent* under transformations (1) and (2), it is necessary that there exist a matrix T of determinant ± 1 such that

$$T D \bar{T} = D'$$

where D and D' are the discriminant matrices of the two domains and $\bar{T} = T$ *transpose*. Also, since each domain contains the principal unit, it is still quite general if we demand that the first row of T be all zeros except the first element, which is unity.

The last equation evidently gives certain relations between the elements of T, D, and D'. There are 3 sets of such relations that we shall need; one when D and D' both come from domains of type (I); one when both D and D' come from domains of type (II); and one when D is of type (I) and D' is of type (II). However, we shall include only those equations which must hold when both domains are of type (I). There are 9 independent ones and we shall refer to them as A-1, A-2, etc. The theorems which follow from the other sets of equations will be stated without proof. In the following equations the unprimed letters are elements of the basal matrices whose discriminant matrix is D; the primed letters belong to the basal matrices whose discriminant matrix is D'. These A-conditions which must hold when two domains of type (I) are equivalent are

A-1 $2t_{21} + at_{23} = 0,$
A-2 $2t_{41} + at_{43} = 0,$
A-3 $2t_{31} + at_{33} = a',$
A-4 $t^2_{21} + mt^2_{22} + lat_{22}t_{23} + kat_{22}t_{24} = m',$

* MacDuffee, *loc. cit.*

A-5 $\quad -t^2_{31} + mt^2_{32} + lat_{32}t_{33} + kat_{32}t_{34} - at_{33}t_{31} = 0,$

A-6 $\quad t^2_{41} + mt^2_{42} + lat_{42}t_{43} + kat_{42}t_{44} = 0,$

A-7 $\quad -at_{33}t_{21} + 2mt_{32}t_{22} + lat_{33}t_{22} + kat_{34}t_{22} + lat_{32}t_{23} + kat_{32}t_{24} = l'a',$

A-8 $\quad -at_{41}t_{23} + 2mt_{42}t_{22} + lat_{43}t_{22} + kat_{44}t_{22} + lat_{42}t_{23} + kat_{42}t_{24} = k'a',$

A-9 $\quad -at_{41}t_{33} + 2mt_{42}t_{32} + lat_{43}t_{32} + kat_{44}t_{32} + lat_{42}t_{33} + kat_{42}t_{34} = 0.$

4. *Auxiliary theorems.* The following theorems are useful in determining the number of non-equivalent domains with a given discriminant.

THEOREM 2. *If $2l \equiv 0$, mod. a, the domain is equivalent to some domain in canonical form in which $l = 0$.*

To prove this it is sufficient to give the transformation which makes l zero without changing the canonical form given in Theorem 1. The following transformation works for both cases.

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \dfrac{-2l}{a} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} -1 & 0 \\ -l & -1 \end{Vmatrix}.$$

This transformation leaves k and a invariant. If m' is such that the domain is not in canonical form, this can be corrected by adding or subtracting ka a sufficient number of times.

THEOREM 3. *Any domain of type (II) with a odd is equivalent to a domain of type (I).*

The following transformation will put the domain of type (II) into one of type (I).

$$T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ \dfrac{1+a}{-2} & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and

$$M = \begin{Vmatrix} 1 & 0 \\ \dfrac{1-a}{2} & -1 \end{Vmatrix}.$$

THEOREM 4. *If ka^2 contains no square factor other than unity, all domains are equivalent.*

By Theorem 3 we need consider only domains of type (I), since a is evidently 1. By Theorem 2 we may take l as zero. Hence

$$m \equiv 0, \text{ mod. } k.$$

But by subtracting or adding k we can evidently make m zero. Hence m and l are both zero and a is 1. Hence there is only one such domain for such a value of $ka.^2$

THEOREM 5. *Two domains of type I are non-equivalent if a is even in one and odd in the other.*

If a' is odd A-3 is impossible unless a is even. This proves the theorem.

THEOREM 6. *A given domain of type (II) is not equivalent to a domain of type (I) with a even.*

THEOREM 7. *Two domains of type (II) in which the a's are even are not equivalent unless the m's are both even or both odd.*

5. *Non-equivalent domains when $ka^2 = 18$.* We shall now make a determination of the number of non-equivalent domains for which $ka^2 = 18$. This is a typical case. Since 1 and 9 are the only square factors in 18 it is, by Theorem 3, sufficient to consider only domains of type (I). The following seven domains are the only ones satisfying the conditions given in Theorem 1.

	k	a	l	m
D	18	1	0	0
D	2	3	0	0
D	2	3	0	2
D	2	3	0	4
D	2	3	1	1
D	2	3	1	3
D	2	3	1	5

The domains D_1 and D_2 are non-equivalent, for if we apply a transformation (1) to D_2, we get

$$\text{(A-5)} \qquad -t^2_{31} + 6t_{32}t_{34} - 3t_{33}t_{31} = 0,$$

$$\text{(A-1)} \qquad 2t_{31} + 3t_{33} = 1.$$

The first of the above equations demands that $t_{31} \equiv 0$, mod. 3, and this is impossible in the second one.

Also $D_{[illegible]}$ and $D_{[illegible]}$ are non-equivalent, for if we apply [illegible] to D_1, A-4 becomes

$$[illegible] + 18t[illegible] = 2.$$

But t [illegible] 2, mod. 3 has no roots. Hence $D_{[illegible]}$ and $D_{[illegible]}$ are non-equivalent.

Now $D_{[illegible]}$ and $D_{[illegible]}$ are non-equivalent, for if we apply a transformation (1) to $D_{[illegible]}$ we have from A-1 and A-4 respectively:

$$\begin{aligned} 2t_{21} + 3t_{[illegible]} &\;[illegible]\; 0, \\ t^2_{[illegible]} + 6t_{[illegible]}t_{[illegible]} &\;[illegible]\; 2. \end{aligned}$$

which have no common solutions.

The transformations

$$T = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 6 & -3 & -1 & 4 \\ 2 & -1 & -1 & 1 \\ -6 & 6 & 4 & -5 \end{matrix} \right\| ,$$

and

$$M = \left\| \begin{matrix} -1 & 1 \\ 6 & -3 \end{matrix} \right\|$$

put D_4 into D_1. Also the transformations

$$T = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| ,$$

and

$$M = \left\| \begin{matrix} -1 & 0 \\ 3 & -1 \end{matrix} \right\| ,$$

put D_2, D_3, and D_4 into domains which are evidently equivalent to D_6, D_7, and D_5 respectively.

This concludes the determination. There are evidently just three non-equivalent domains. These are D_1, D_2, and D_3 which are listed above.

[illegible] *canonical form for all non-equivalent domains with* $ka^2 \leq 31$. We [illegible]

[illegible]

ka^2	k	a	l	m	Type
1	1	1	0	0	I
4	4	1	0	0	I
4	1	2	0	0	I
4	1	2	0	1	II
8	8	1	0	0	I
8	2	2	0	0	I
8	2	2	0	2	I
9	9	1	0	0	I
9	1	3	0	0	I
9	1	3	0	2	I
12	12	1	0	0	I
12	3	2	0	3	I
12	3	2	0	3	II
16	16	1	0	0	I
16	4	2	0	0	I
16	4	2	0	4	I
16	1	4	0	0	I
16	1	4	0	2	I
16	1	4	0	3	I
16	1	4	0	1	II
18	18	1	0	0	I
18	2	3	0	0	I
18	2	3	0	2	I
20	20	1	0	0	I
20	5	2	0	0	I
20	5	2	0	5	II
24	24	1	0	0	I
24	6	2	0	0	I
24	6	2	0	6	I
25	25	1	0	0	I
25	1	5	0	0	I
25	1	5	0	2	I
27	27	1	0	0	I
27	3	3	0	0	I
27	3	3	0	3	I
27	3	3	0	6	I
27	3	3	1	1	I
28	28	1	0	0	I
28	7	2	0	0	I
28	7	2	0	7	II

Summary of above results.

Value of ka^2	1	4	8	9	12	16	18	20	24	25	27	28
Number of Domains . . .	1	3	3	3	3	7	3	3	3	3	5	3

OHIO STATE UNIVERSITY.

PENCILS OF HYPERSURFACES.*

By Temple Rice Hollcroft.

1. *Introduction.* A general pencil f of hypersurfaces in r-space S_r is defined by the equation

$$(1) \qquad f \equiv \lambda_1 u_1 + \lambda_2 u_2 = 0,$$

in which u_1 and u_2 are given, non-singular hypersurfaces of S_r and $\lambda \equiv \lambda_1/\lambda_2$ is the parameter.

Pencils of plane curves were first studied systematically by Cremona.† Certain properties of pencils of surfaces of S_3 have been found by various mathematicians, but in most cases only as special properties of a k-parameter linear system for $k = 1$. Doehlemann ‡ devotes a short section of his article to properties peculiar to pencils.

The purpose of this paper is to determine the characteristics of a general pencil of hypersurfaces in r dimensions.

2. *The basis manifold.* The basis manifold M of the pencil f is of dimension $r - 2$ and order n^2. It is the complete intersection of u_1 and u_2 or of any two, three, $\cdots$, infinity hypersurfaces of f.

The i-th class m_i of a manifold M_α of dimension α in S_r, $\alpha \leqq r - 1$, is the class of the section of M_α made by an arbitrary $S_{r-\alpha+i}$; that is, the number of $S_{r-\alpha+i-1}$ through an arbitrary $S_{r-\alpha+i-2}$ of $S_{r-\alpha+i}$ that are tangent to the section of M_α made by $S_{r-\alpha+i}$. For a general hypersurface of order m_0, $m_i = m_0(m_0 - 1)^i$.

The basis manifold M of dimension $r - 2$ is the complete intersection of two hypersurfaces in S_r. The i-th class of M is the i-th class of the manifold (hypersurface of S_{i+1}) that is the projection on S_{i+1} from an arbitrary point of S_{i+2} of the section of M made by S_{i+2}. The basis manifold M has, therefore, $r - 2$ distinct classes, m_i, $i = 1, 2, \cdots, r - 2$, which are respectively the classes of the sections of M made by an arbitrary S_{i+2}.

Under the above definition, the order of a manifold may be considered as

* Presented to the American Mathematical Society, September 9, 1930.

† L. Cremona, "Einleitung in eine geometrische Theorie der Ebenen Curven." [illegible]

‡ K. Doehlemann, "Lineare Systeme von Curven und Flächen," *Mathematische Annalen*, Vol. 41 (1893), pp. 560-563.

the class m_0; that is for M, the class of the section of M by an arbitrary plane which is n^2.

The class m_1 of M is the class of a 3-space section of M. This is the rank of a space curve of order n^2 (or class of the plane curve projection of this space curve from a point of S_3) which is the complete intersection of two surfaces of S_3 of order n. Then $m_1 = n^2(n^2-1) - n^2(n-1)^2 = 2n^2(n-1)$.

The class m_2 of M is the class of a 4-space surface (or class of the 3-space surface into which this surface projects from a point of S_4) which is the complete intersection of two hypersurfaces of order n of S_4. Since such a surface has no improper double points, its class

$$m_2 = 3t + \tfrac{1}{2}m_0\,[3m_1 - (m_0 - 1)(m_0 + 2)],^*$$

wherein the order $m_0 = n^2$, the class of a 3-space section $m_1 = 2n^2(n-1)$, the number of apparent triple points $t = n^2(n-1)^2(n-2)^2/6$. Hence $m_2 = 3n^2(n-1)^2$.

Comparing the above three values of m_i for $i = 0, 1, 2$, it is evident that the values of the classes m_i of M are

$$m_i = (i+1)n^2(n-1)^i, \qquad (i = 1, 2, 3, \cdots, r-2).$$

Using the formula for the equivalence † of a manifold of given dimension, order and classes which occurs simply on r hypersurfaces of given order in S_r, the equivalence of the basis manifold M on r hypersurfaces of order N in S_r is found to be

$$E_M = n^2\,[{}_rC_2N^{r-2} - 2{}_rC_3nN^{r-3} + 3{}_rC_4n^2N^{r-4} - \cdots + (-1)^{r-1}(r-2){}_rC_1n^{r-3}N + (-1)^r(r-1)n^{r-2}].\ddagger$$

For $r = 2$ and all values of N, E_M has the value n^2, that is, the equivalence of n^2 points common to any two curves is n^2. Also, E_M has the value n^r for $n = N$, that is, in S_r, M is the complete intersection of r hypersurfaces of order n.

* F. Severi, "Intorno ai punti impropri di una superficie generale dello spazio a quattro dimensioni e suoi punti tripli apparenti," *Rendiconti del Circolo Matematico di Palermo*, Vol. 15 (1901), pp. 34-36.

† F. Severi, "Sulle intersezioni delle varieta algebriche e sopra i loro caratteri e singolarità proiettive," *Memorie delle Reale Accademia delle Scienze di Torino*, Ser. 2, Vol. 52 (1903), p. 115.

‡ The expression resulting directly from substitution in the formula is in powers of $N-1$ with increasingly long coefficients. On expanding the binomials $(N-1)^i$ and combining like terms, the above comparatively simple form is obtained.

3. *Hypersurfaces and systems of hypersurfaces associated with the pencil.* The k-polar hypersurfaces of a pencil of hypersurfaces form systems, linear in the parameter λ and of degree k in the r independent coördinates of a general point of S_r.

The Hessian of a pencil of hypersurfaces of order n is a one-parameter family of hypersurfaces of order $(r+1)(n-2)$ in whose equation the parameter λ occurs to the degree $r+1$. The Hessian does not contain M.

One point P given on the Hessian determines the Hessian by means of an equation of degree $r+1$ in λ. Then a point given on the Hessian of a pencil of hypersurfaces determines $r+1$ hypersurfaces of the pencil and the $r+1$ Hessians associated with them. From the definitions of the Hessian as a locus, we have, therefore: A given point P considered either as a contact of two first polars, as a node* on a first polar, or as a pole whose $(n-2)$-polar is a quadric hypercone (all polars with respect to the pencil of hypersurfaces) determines $r+1$ hypersurfaces of the pencil whose $r+1$ respective Hessians all pass through P.

Eliminating λ from the equations of the pencil and the Hessian, the equation of a fixed hypersurface J of order $2(r+1)(n-1)$ is obtained. This hypersurface contains M $(r+1)$-fold and is the locus of the parabolic manifolds of hypersurfaces of the pencil.

The envelope of the Hessians of a pencil of hypersurfaces of order n is a unique hypersurface of order $(2r+1)(n-2)$ which does not contain M.

The above results for the first Hessian can easily be extended to any k-Hessian, $1 \leq k \leq n-2$,† which includes the Steinerians of the pencil.

The k-polar hypersurfaces of a fixed point P with respect to the hypersurfaces of the pencil themselves form a pencil of hypersurfaces of order $n-k$ with a basis manifold of order $(n-k)^2$ and dimension $r-2$. These pencils are all projective to the original pencil and to each other for the same or different positions of P, since all are linear one-parameter systems with the same parameter λ. A multiple infinity of fixed hypersurfaces associated with the pencil can be determined as loci of the intersections of corresponding hypersurfaces of any two of these projective pencils.

The most important of these fixed hypersurfaces is the locus of intersections of corresponding hypersurfaces of the original pencil and the first polar pencil of a given point P. This hypersurface ϕ_P is of order $2n-1$. It is the locus of contacts of all tangent hyperplanes from P to hypersurfaces

* [illegible] used to mean a multiple point of order [illegible] an entirely general tangent quadric hypercone.

† T. R. Hollcroft, "The Generalized Hessian," *Quarterly Journal of Mathematics*, Vol. 50 (1927), pp. 362-372.

of the pencil. It contains the basis manifold M and passes through the nodes of hypersurfaces of the pencil.

For all points of S_r, the hypersurfaces ϕ_P form an ∞^r system of surfaces of order $2n-1$. This system has M as a basis manifold and the nodes of the pencil as simple basis points. The Jacobian of this system is of order $2(r+1)(n-1)$, contains M $(r+1)$-fold and has nodes at the nodes of the pencil.

This Jacobian hypersurface J is identical with the hypersurface J found above by eliminating λ from the equations of the pencil and the Hessian. This is easily seen by writing out the equation of J according to both definitions. This equation is a determinant, equated to zero, of order $r+1$, whose principal diagonal contains the elements

$$u_2\partial^2 u_1/\partial x_i^2 - u_1\partial^2 u_2/\partial x_i^2, \qquad (i=1,2,\cdots,r+1).$$

4. *Nodes of the pencil.* Algebraically, the determination of the number H of hypersurfaces of the pencil that have nodes consists merely in finding the order of the discriminant of a quantic of order n in r independent variables. This discriminant is of order $(r+1)(n-1)^r$. Hence

$$H=(r+1)(n-1)^r.$$

This result may also be obtained synthetically. Consider r hypersurface ϕ_{P_i}, defined in the preceding section, determined respectively by r distinct arbitrary points P_i of S_r.* Each of these r hypersurfaces passes through M, the H nodes of the pencil and the $r(n-1)^{r-1}$ contacts of hypersurfaces of the pencil with the S_{r-1} determined by the r points P_i (see section 5). Substituting $N=2n-1$ in the formula for E_M derived in section 2, there results, on expanding the binomials and simplifying,

$$E_M{}^{\phi}=n^2[{}_rC_2(n-1)^{r-2}+{}_rC_3 n(n-1)^{r-3}+\cdots+rn^{r-3}(n-1)+n^{r-2}].$$

Hence

$$H=(2n-1)^r-r(n-1)^{r-1}-E_M{}^{\phi}=(r+1)(n-1)^r.$$

Another method used by Cremona (*loc. cit.*, pp. 122-123) can be generalized. Consider the first polar pencils p_i respectively of $r+1$ distinct points P_i and the r hypersurfaces of order $2(n-1)$ generated by the r pairs of projective pencils $p_1p_2, p_1p_3, \cdots, p_1p_{r+1}$. These r hypersurfaces all pass through the nodes of the pencil and the basis manifold of p_1. This basis manifold is of order $(n-1)^2$ and its equivalence E' on r hypersurfaces of

* This method applied to pencils of plane curves was used by Guccia, *Rendiconti del Circolo Matematico di Palermo*, Vol. 9 (1895), p. 15.

or $2(n-1)$ is found by replacing N by $2(n-1)$ and n by $n-1$ in the rmula for E_M. This gives

$$E' = (n-1)^r[{}_rC_2 2^{r-2} - 2{}_rC_3 2^{r-3} + \cdots + (-1)^r(r-1)] \equiv (n-1)^r(2^r - r - 1).$$

ence

$$H = [2(n-1)]^r - E' = (r+1)(n-1)^r.$$

5. *The number of hypersurfaces of a pencil tangent to a given manifold.* e shall first find the number of hypersurfaces of a pencil tangent to a given n-singular hypersurface u of order m.

The solution of this problem is equivalent to finding the order of the ct-invariant T of u and a general hypersurface of the pencil. Since all efficients of both u and the pencil are constants except λ, T is a function λ. The roots of $T=0$ are values of λ defining hypersurfaces of the pencil ngent to u.

To find the order of the tact-invariant T, we shall first consider the net hypersurfaces in S_r,

$$\lambda u + \lambda_1 u_1 + \lambda_2 u_2 = 0,$$

which $u=0$, $u_1=0$, $u_2=0$ are hypersurfaces of orders m, n, p respecvely. The locus of contacts of the hypersurfaces of the net is the Jacobian J, hypercurve whose order will be found. Let $p=n$. Consider the two hyperrfaces $u=0$ and $f \equiv \lambda_1 u_1 + \lambda_2 u_2 = 0$ of orders m and n respectively benging to the net. The contacts of these two hypersurfaces will lie on J and ill, therefore, lie at the intersections of u and J. The number of interctions of u and J will, therefore, be the order of the tact-invariant of u id f in the coefficients of f.

The equations of J are most easily derived from the following definition J: The Jacobian of three hypersurfaces of S_r $u=0$, $u_1=0$, $u_2=0$ of ders m, n, p respectively is the locus of points whose polar hyperplanes with spect to the three hypersurfaces have an S_{r-2} in common.

From this definition, the equations of J are given by the matrix

$$\begin{Vmatrix} u' & u'' & u''' & \cdots & u^{(r+1)} \\ u'_1 & u''_1 & u'''_1 & \cdots & u_1^{(r+1)} \\ u'_2 & u''_2 & u'''_2 & \cdots & u_2^{(r+1)} \end{Vmatrix}$$

which the superscript i denotes the partial derivative with respect to x_i. e equations of J consist of $r-1$ independent determinants of this matrix, [illegible] to zero. Each of these $r-1$ equations [illegible] face of order $m+n+p-3$. The hypercurve J is the portion of the tersection of a set of $r-1$ independent hypersurfaces that is common to all [illegible]

Salmon * has derived a method for obtaining the order of the syste common to all the determinants contained in any given matrix. Using hi notation, we may write the above matrix, which contains 3 rows and r — columns, in the form

$$\begin{Vmatrix} \alpha & \alpha & \alpha & \cdots & \alpha \\ \beta & \beta & \beta & \cdots & \beta \\ \gamma & \gamma & \gamma & \cdots & \gamma \end{Vmatrix}.$$

The order of the system common to all the determinants of this matrix is

$$\sum_0^{r-1} \alpha^i\beta^j\gamma^k, \qquad (i+j+k=r-1);$$

that is, the sum of the $r(r+1)/2$ products $\alpha^i\beta^j\gamma^k$ in which i, j, k each range from 0 to $r-1$ subject to the restriction $i+j+k=r-1$.

Since $\alpha=m-1$, $\beta=n-1$, $\gamma=p-1$, the order of J is

$$\begin{aligned}(m-1)^{r-1}&+(m-1)^{r-2}[(n-1)+(p-1)]\\ &+(m-1)^{r-3}[(n-1)^2+(n-1)(p-1)+(p-1)^2]\\ &+\cdots+(m-1)[(n-1)^{r-2}+(n-1)^{r-3}(p-1)+\cdots+(p-1)^{r}\\ &+(n-1)^{r-1}+(n-1)^{r-2}(p-1)+\cdots+(p-1)^{r-1}.\dagger\end{aligned}$$

For $p=n$, the order of J becomes

$$\begin{aligned}(m-1)^{r-1}&+2(m-1)^{r-2}(n-1)+3(m-1)^{r-3}(n-1)^2\\ &+\cdots+(r-1)(m-1)(n-1)^{r-2}+r(n-1)^{r-1}.\end{aligned}$$

The number of points common to u and J which is the order of the tac invariant T of f and u in the coefficients of f is, therefore,

$$\begin{aligned}m[(m-1)^{r-1}&+2(m-1)^{r-2}(n-1)\\ &+\cdots+(r-1)(m-1)(n-1)^{r-2}+r(n-1)^{r-1}].\end{aligned}$$

This is, moreover, the number of hypersurfaces of the pencil which are ta gent to a given non-singular hypersurface of order m.

The above result is easily generalized to obtain the number of hype surfaces of the pencil that are tangent to a given manifold M_a of order and dimension $\alpha \leq r-1$.

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* G. Salmon, *Lessons Introductory to the Modern Higher Algebra*, 4th editio pp. 286-290.

† Cf. T. R. Hollcroft, "Nets of Manifolds in i Dimensions," *Annali di Mat matica*, Ser. 4, Vol. 5 (1927-28), pp. 261-267. The work of section 2 in finding t order of J and the tact-invariant is a special case of the above for $m=n=p$.

$_{\alpha+1}$, given a hypersurface M_α of order m. The contacts of members pencil with M_α will all lie in $S_{\alpha+1}$, that is, they will be the contacts and a general pencil of manifolds of order n and dimension α, the [illegible] of the pencil of S_r by $S_{\alpha+1}$. This is the same as the original problem $r = \alpha + 1$. Then the number of hypersurfaces of a pencil of order n S_r that are tangent to a given non-singular manifold of order m, dimension $r - 1$, which is contained in an $S_{\alpha+1}$, is

$$N_\alpha = m\,[(m-1)^\alpha + 2(m-1)^{\alpha-1}(n-1) + \cdots + (m-1)(n-1)^{\alpha-1} + (\alpha+1)(n-1)^\alpha].$$

For $m = 1$, formula (2) becomes $(\alpha+1)(n-1)^\alpha$, the number of ersurfaces of the pencil tangent to a linear manifold S_α.

Formula (2) may be interpreted as follows: The pencil of hypersurfaces rder n of S_r intersects M_α in a pencil of manifolds of order mn and ension $\alpha - 1$ lying in M_α. The number of manifolds of this pencil in M_α have a node is N_α. This is evident because a node of a manifold in M_α rs at a contact of M_α with a hypersurface of the pencil.

A hyperspace curve C of order m_0 intersects the pencil of hypersurfaces single infinity of groups of $m_0 n$ points, each group of which is such that one point determines the remaining $m_0 n - 1$. If C is of genus p, this lution on C contains $2(m_0 n + p - 1)$ coincidences.* There are, then, $= 2(m_0 n + p - 1)$ hypersurfaces of the pencil that are tangent to a e of order m_0, genus p and belonging to any space.

This formula shows the effect only of nodes, apparent or actual, since dditional property of a cusp does not appear in a formula involving only order and genus. From either the above formula or the succeeding one, is reduced two by each additional node or apparent double point.

Let m_1 be the rank of C. Then $p = m_1 - 2(m_0 - 1)$ and the above ula becomes

$$N'_1 \equiv m_1 + 2m_0(n-1)$$

e N'_1 is the number of hypersurfaces of the pencil tangent to a hyper- [illegible] C of order m_0 and rank m_1. It is now seen that each cusp reduces [illegible] a general s-fold point of C reduces N'_1 by $s(s-1)$, since [illegible]

[illegible] S [illegible] consider a general manifold M_α [illegible] $m_i = m_0(m_0 - 1)$, ($i = 1, 2, 3, \cdots, \alpha$), such that α [illegible] r [illegible] 1. [illegible]

[illegible] Vol. II (1902), p. 153, line 3.

[illegible] of any space.

$S_{\alpha+1}$, given a hypersurface M_α of order m. The contacts of members ...pencil with M_α will all lie in $S_{\alpha+1}$, that is, they will be the contacts ... $_\alpha$ and a general pencil of manifolds of order n and dimension α, the ...n of the pencil of S_r by $S_{\alpha+1}$. This is the same as the original problem $= \alpha + 1$. Then the number of hypersurfaces of a pencil of order n that are tangent to a given non-singular manifold of order m, dimension ... 1, which is contained in an $S_{\alpha+1}$, is

$$N_\alpha = m\,[\,(m-1)^\alpha + 2\,(m-1)^{\alpha-1}(n-1) + \cdots + (m-1)\,(n-1)^{\alpha-1} + (\alpha+1)\,(n-1)^\alpha].$$

For $m = 1$, formula (2) becomes $(\alpha+1)\,(n-1)^\alpha$, the number of ...ersurfaces of the pencil tangent to a linear manifold S_α.

Formula (2) may be interpreted as follows: The pencil of hypersurfaces ...order n of S_r intersects M_α in a pencil of manifolds of order mn and ...ension $\alpha - 1$ lying in M_α. The number of manifolds of this pencil in M_α ... have a node is N_α. This is evident because a node of a manifold in M_α ...rs at a contact of M_α with a hypersurface of the pencil.

A hyperspace curve C of order m_0 intersects the pencil of hypersurfaces ... single infinity of groups of $m_0 n$ points, each group of which is such that ...one point determines the remaining $m_0 n - 1$. If C is of genus p, this ...lution on C contains $2(m_0 n + p - 1)$ coincidences.* There are, then, $= 2(m_0 n + p - 1)$ hypersurfaces of the pencil that are tangent to a ...e of order m_0, genus p and belonging to any space.

This formula shows the effect only of nodes, apparent or actual, since ...additional property of a cusp does not appear in a formula involving only ...order and genus. From either the above formula or the succeeding one, ... is reduced two by each additional node or apparent double point.

Let m_1 be the rank of C. Then $p = m_1 - 2(m_0 - 1)$ and the above ...ula becomes

$$N'_1 \equiv m_1 + 2m_0(n-1)$$

...e N'_1 is the number of hypersurfaces of the pencil tangent to a hyper-...e curve C of order m_0 and rank m_1. It is now seen that each cusp reduces ...by three. Also a general s-fold point of C reduces N'_1 by $s(s-1)$, since ...reduction occurs in the rank.

In $S_{\alpha+1}$, consider a general manifold M_α of dimension α, order m, and ... 1), $(i = 1, 2, 3, \cdots, \alpha)$, such that $\alpha < r - 1$. Sub...

[illegible]

stituting m_i for the respective products $m_0(m_0-1)^i$ in formula (2) results: The number of hypersurfaces of a pencil of order n in S_r t a manifold M_α of dimension α, order m_0 and classes $m_1, m_2, m_3, \cdots$,

$$(3)\qquad N'_\alpha \equiv m_\alpha + 2m_{\alpha-1}(n-1) + 3m_{\alpha-2}(n-1)^2 + \cdots + \alpha m_1(n-1)^{\alpha-1} + (\alpha+1)m_0(n-1)^\alpha.$$

This is the general formula of which formula (3_1) above for the hyp space curve C is a special case for $\alpha=1$. Formula (3_1) can be derived fr (2) by setting $\alpha=1$ and $m_1=m_0(m_0-1)$, the value of m_1 for a gene plane curve. This is the same process by which (3) was derived from for any α. Formula (3_1), however, was derived independently for a hyp space curve of given order and class and therefore holds for a curve of space.

Because of this and other evidences and analogies (e.g., to equival formulas), it would appear that formula (3) holds for manifolds of order n, classes $m_i \leqq m_0(m_0-1)^i$, and dimension α, defined by in se ions of $r-\alpha$ linear or non-linear hypersurfaces of S_r, but this has been proved.

WELLS COLLEGE.

CORRECTIONS.

J. L. DORROH, *Some metric properties of descriptive planes.*

p. 403, l. 5 change "segment H_{jk}" to "segment of H_{jk}".

p. 407, l. 13 change "$| f(Z,V)-f(Z,x |$" to "$| f(Z,V)-f(Z,x$

p. 407, l. 29 change "Y_e'" to "Y_c^*".

p. 415, l. 12 change "L" to "$\bar{L}$".